J. Plasma Physics (2003), *vol.* 69, *part* 6, *pp.* 541–550. © 2003 Cambridge University Press
DOI: 10.1017/S0022377803002423 Printed in the United Kingdom

Geometry of magnetic field lines approaching a current sheet

MANUEL NÚÑEZ

Departamento de Análisis Matemático, Universidad de Valladolid,
47 005 Valladolid, Spain
(mnjmhd@am.uva.es)

(Received 10 January 2003 and in revised form 14 April 2003)

Abstract. The evolution of a magnetic field line in two dimensions near a neutral sheet is analysed. It is found that the general features of this evolution are rather independent of any particular model, provided that the magnetic field is small and the current density does not vanish. The time of arrival of a field line to the neutral sheet as well as its breaking and reconnection are proved to be finite and to satisfy a simple formula whose main parameter is the resistivity, which may be a spatial function. The shape of the evolving field lines satisfies a differential equation whose solution in some simple cases is shown to agree with certain classical reconnection configurations. Hyperresistivity is found to be more often a hindrance than a positive contribution to the reconnection process.

1. Introduction

Magnetic reconnection is universally accepted as the mechanism by which large amounts of energy stored in a magnetic field may be released suddenly as kinetic energy, producing phenomena such as solar flares. The original models describing a geometry where reconnection occurs were steady and two-dimensional: Sweet [1] and Parker [2,3] considered two masses of plasma colliding head-on with magnetic fields parallel to each other and possessing opposite directions. In the interface the field vanishes (therefore it is called a neutral sheet) and a localized current develops: reconnection occurs there, albeit at a slower rate than the one observed in astrophysical phenomena. This is essentially due to the fact that the plasma cannot escape rapidly enough from the reconnection zone and the mechanism is throttled. The Sweet–Parker configuration was refined and improved by Syrovatskii [4–6]. To avoid the throttling effect in the Sweet–Parker configuration, Petschek proposed a model [7] where the interface is a small segment; MHD shock waves, where energy conversion takes place, propagate from it. Petschek's model was later expanded to other configurations [8, 9], not exempt of criticism [10]. There is still considerable controversy on the essential soundness of the Petschek model: while Biskamp [11, 12] considers it physically flawed, Priest, Jardine and Forbes [13–18] see some promise in it. All these early models are somewhat unrealistic due to the unavoidable simplifications necessary to study them analytically: in addition to being steady-state, diffusivity is limited to a small region around the neutral sheet. The present consensus seems to be that the one-fluid MHD description fails near the sheet: additional resistive effects may help the reconnection rate, and mechanisms

https://doi.org/10.1017/S0022377803002423 Published online by Cambridge University Press

to provide it have been suggested. They usually involve an anomalous or turbulent resistivity, although there is no general agreement on the details [19–24].

We intend to study theoretically, in dimension two, how the geometry of a magnetic field line evolves under the MHD model when approaching a neutral sheet. While numerical studies as well as several classical models predict field lines becoming parallel to the neutral line when pushed by the plasma flow and being expelled laterally, after reconnection, with an hyperbola-like shape, this cannot be taken for granted. We will only assume that at the points of the projection in the neutral line of our field line the current density does not vanish (which precludes field lines orthogonal to the neutral one, and justifies the name of current sheet for it), that the magnitudes vary more rapidly across the neutral line than parallel to it and that the flow velocity remains bounded everywhere. With these reasonable hypotheses the time taken by any point of the field line to reach the neutral one may be found explicitly, and the evolving shapes taken by the line are the solutions of an ordinary differential equation, depending mostly on the resistivity function, whose integration is easy and illuminating in certain simple cases. The addition of an hyperresistivity, whatever its physical cause, is shown to be in many cases disastrous: it tends to expel field lines away from the neutral one, thus precluding reconnection.

2. Evolution of a field line

Let us assume for analytical simplicity that the neutral sheet (possibly degenerating to a single point) is located at the horizontal axis: let it be $\{(x,0) : x \in [-L,L]\}$. With additional algebraic complications, the same analysis may be performed for any smooth curve in the plane. Let the field line be given by the graph of the function f, so that at an instant t it is $\{(x,f(x,t)) : x \in I\}$, for some subinterval I of $[-L,L]$. This precludes field lines orthogonal at one point to the neutral sheet, since there they fail to be the graph of a function of x. We will see that if any of these points is close to the neutral sheet, the current density at this coordinate is small. We will also take the plasma as incompressible; while this does not hold for reconnection in astrophysical plasmas, it is unlikely that in the close vicinity of the sheet the compressibility affects in a decisive way the reconnection process. In a simply connected domain, the field $\mathbf{B}$ is the curl of a potential A, $\mathbf{B} = (-\partial A/\partial y, \partial A/\partial x)$. A is not unique; we may choose its gauge so that it satisfies the 'uncurled' induction equation

$$\frac{\partial A}{\partial t} = \eta \nabla^2 A - \mathbf{u} \cdot \nabla A. \tag{1}$$

$\mathbf{u}$ denotes the flow velocity, η the resistivity and ∇^2 the Laplacian. Field lines are those whose tangent is $\mathbf{B}$, i.e. orthogonal to ∇A; hence they are the level sets $A = \text{const}$. It remains to see what we mean by the evolution of a magnetic field line, which is not a material object which may be followed in time. We will take it to mean the study of how a certain level set $A = A_0$, A_0 fixed, varies in time. The interpretation is supported by the fact that, in the ideal case $\eta = 0$, field lines are transported by the flow as material points: since the induction equation means in this case that the Lagrangian derivative of A vanishes, level sets $A = A_0$ are transported by the flow, so that these represent the same field line as a function of time. η does not need to be any particular kind of resistivity; it may be collisional,

https://doi.org/10.1017/S0022377803002423 Published online by Cambridge University Press

anomalous or turbulent, and it may depend on the point, so as to be larger in the vicinity of the sheet. The addition of an hyperresistivity transforms the induction equation into

$$\frac{\partial A}{\partial t} = -\eta_2 \nabla^4 A + \eta \nabla^2 A - \mathbf{u} \cdot \nabla A, \tag{2}$$

where η_2 is the hyperresistivity coefficient and ∇^4 the iterated Laplacian.

The fact that $(x, f(x,t))$ is a field line means that $A(x, f(x,t), t) = A_0$. Assuming that all the magnitudes are differentiable, we get

$$\begin{aligned} &\frac{\partial A}{\partial x}(x, f(x,t), t) + \frac{\partial A}{\partial y}(x, f(x,t), t)\frac{\partial f}{\partial x}(x,t) = 0, \\ &\frac{\partial A}{\partial y}(x, f(x,t), t)\frac{\partial f}{\partial t}(x,t) + \frac{\partial A}{\partial t}(x, f(x,t), t) = 0. \end{aligned} \tag{3}$$

The induction equation implies that

$$\begin{aligned} \frac{\partial A}{\partial t} &= \eta \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) + u_x B_y - u_y B_x \\ &= -\eta J + u_x B_y - u_y B_x, \end{aligned} \tag{4}$$

where J is the (scalar) current density. Thus (3) becomes

$$\begin{aligned} &B_y - B_x \frac{\partial f}{\partial x} = 0, \\ &-B_x \frac{\partial f}{\partial t} - \eta J + u_x B_y - u_y B_x = 0. \end{aligned} \tag{5}$$

2.1. The ideal case

The analysis of this equation for infinite conductivity is interesting. Since the velocity $\mathbf{u}$ is also solenoidal, there exists a flux function ψ such that $\mathbf{u} = (-\partial\psi/\partial x, \partial\psi/\partial y)$. Then (5) implies, when $B_x \neq 0$,

$$\frac{\partial f}{\partial t} = u_x \frac{\partial f}{\partial x} - u_y. \tag{6}$$

On the other hand,

$$\frac{d}{dx}(\psi(x, f(x,t), t)) = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{\partial f}{\partial x} = u_y - u_x \frac{\partial f}{\partial x} = -\frac{\partial f}{\partial t}. \tag{7}$$

Let us denote by $S(t)$ the area between the interval $[a, b]$ of the neutral sheet and the field line between the points $(a, f(a,t))$, $(b, f(b,t))$. The diagram is illustrated in Fig. 1.

We have

$$S(t) = \int_a^b f(x,t)\, dx.$$

Thus

$$\begin{aligned} S'(t) &= \int_a^b \frac{\partial f}{\partial t}(x,t)\, dx = -\int_a^b \frac{d}{dx}\psi(x, f(x,t), t)\, dx \\ &= \psi(a, f(a,t), t) - \psi(b, f(b,t), t). \end{aligned} \tag{8}$$

https://doi.org/10.1017/S0022377803002423 Published online by Cambridge University Press

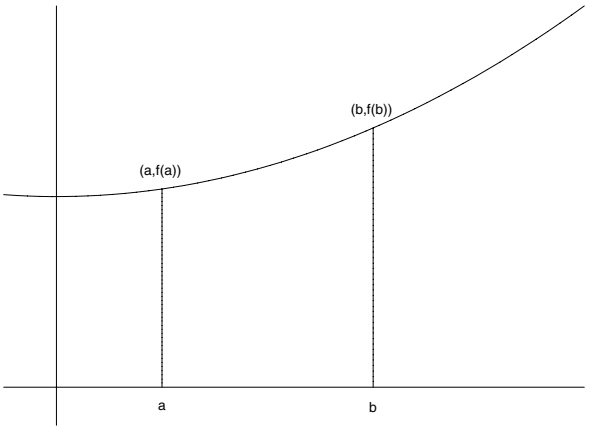


Figure 1. Diagram of the points in the field line.

This calculation was used with a different purpose in [25]. Thus the evolution of the area is determined by the streamlines cutting our field line at its extrema. Since $\nabla\psi$ is the velocity vector rotated by $\pi/2$, and ψ grows in the direction of $\nabla\psi$, this means that the area decreases if the velocity points downwards from the magnetic field line, and increases if upwards. This is obvious in the classical configurations, where the flow flattens the field line against the current sheet, but our deduction is independent of any particular geometry.

2.2. The resistive case

Let us study first the behavior of field lines whose tangent is vertical at some points.

2.2.1. Field lines orthogonal to the neutral sheet. Assume that for a point x of the neutral sheet there exists a sequence of field lines f_n and points x_n tending to x such that the tangent of f_n at x_n is vertical. This means that $|(\partial f_n/\partial x)(x_n, t)| = \infty$, and since both components of the magnetic field are finite, by (5) we have $B_x(x_n, f_n(x_n, t), t) = 0$. Since the component B_y tends to zero when approaching the neutral sheet, the plasma velocity is assumed uniformly bounded and the velocity of displacement of the field lines $\partial f_n/\partial t$ is also finite, we see that (5) implies that $J(x, t) = 0$. This is indeed what happens in the ideal model of Somov and Syrovatskii [26]. There the current in the interval $[-L, L]$ is given by the formula

$$J = \frac{2(a^2 - x^2)}{(L^2 - x^2)^{1/2}},$$

so that, for $a < L$, the current changes sign at the intervals $[-L, -a]$, $[-a, a]$, $[a, L]$. At $x = \pm a$ the current vanishes and the magnetic field, given in complex notation by

$$(B_x + iB_y)(z) = \frac{a^2 - z^2}{(L^2 - z^2)^{1/2}},$$

is also vertical. Since this is an ideal model there is no smoothness and the current is given by the jump of B along the interval $[-L, L]$.

Let us consider the remaining possibility: we assume that the field line does not tend to become vertical at any time when approaching the point $(x, 0)$ of the neutral

https://doi.org/10.1017/S0022377803002423 Published online by Cambridge University Press

sheet. The main equation deduced from (5) is now

$$\frac{\partial f}{\partial t} = -\frac{\eta J}{B_x} + u_x \frac{\partial f}{\partial x} - u_y. \tag{9}$$

Our hypothesis on the non-verticality means that $\partial f/\partial x$ is bounded. Thus, near the current sheet, all the terms involving the velocity are small: therefore the motion there can be approximated by

$$\frac{\partial f}{\partial t} = -\frac{\eta J}{B_x}. \tag{10}$$

It is also reasonable to assume that we must expect the sharpest variation of the field across the sheet, and not parallel to it. This standard hypothesis implies that

$$\left|\frac{\partial B_x}{\partial y}\right| \gg \left|\frac{\partial B_y}{\partial x}\right|, \tag{11}$$

so that we are left with

$$\frac{\partial f}{\partial t} = -\frac{\eta}{B_x}\frac{\partial B_x}{\partial y} = -\eta \frac{\partial \log |B_x|}{\partial y}. \tag{12}$$

We see that the evolution of the line is as expected: since $|B_x|$ grows away from the sheet, so does $\log |B_x|$ and therefore $\partial f/\partial t < 0$, which means that the field line approaches the sheet.

2.2.2. Constant resistivity. The dependence of $|B_x|$ on the distance y to the neutral sheet is not crucial in this case. In the Petschek model, it was taken as linear in y; whereas a more refined analysis using Taylor expansions [12, 18] indicates that $|B_x|$ varies in the central region of the sheet as $|y|^3$. The functional dependence is different near the sheet edge. Let us take $|B_x| = \lambda y^\alpha$ (for $y > 0$), where λ could depend on x. We obtain

$$\frac{\partial f}{\partial t} = -\eta\alpha\frac{1}{y}, \tag{13}$$

or more accurately

$$\frac{\partial f}{\partial t}(x,t) = -\eta(x, f(x,t), t)\alpha\frac{1}{f(x,t)}. \tag{14}$$

This is an ordinary differential equation in the y coordinate. Its integral depends upon the unknown dependence of the resistivity on the point. Assuming it to be constant, it may be integrated easily to yield

$$f(x,t)^2 = f(x,0)^2 - 2\eta\alpha t. \tag{15}$$

The same formula is valid on the lower half-plane. We see that in contrast to what happens in the ideal case, field lines actually reach the neutral sheet at a finite time given by

$$t = \frac{f(0)^2}{2\eta\alpha}. \tag{16}$$

As could be expected, the time decreases with the size of the resistivity and the order of the zero of the field. As for the evolving shape of the field lines, if we start from any field line at the instant $t = 0$, then the graph at time t is given by

$$f(x,t) = (f(x,0) - 2\eta\alpha t)^{1/2},$$

https://doi.org/10.1017/S0022377803002423 Published online by Cambridge University Press

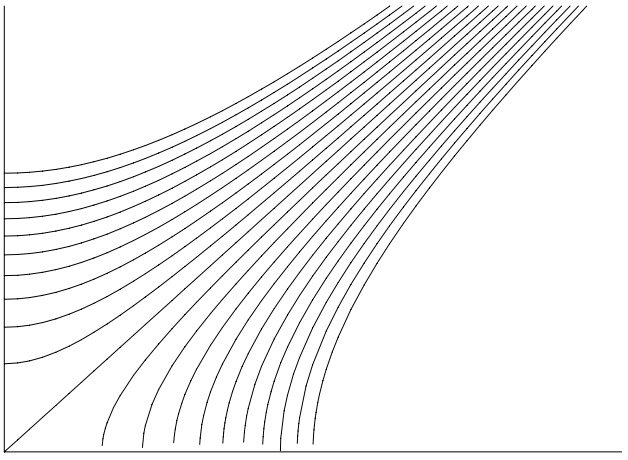


Figure 2. Evolution of a typical field line with constant resistivity.

for as long as the line does not reach the sheet. We present an illustration of the evolution of a typical field line (starting with the shape of an hyperbola) near the edge of the sheet (Fig. 2).

The line reaches the sheet, reconnects with one in the lower half-plane and translates to the right. This is a standard picture recognizable in several modelizations.

2.2.3. Anomalous resistivity. If there is an anomalous resistivity located in the vicinity of the sheet, η is a non-constant function. Let us consider a simple model: we assume that the effective resistivity is essentially proportional to the current density [27]. Thus, (14) becomes

$$\frac{\partial f}{\partial t} = -\frac{kJ^2}{B_x}. \tag{17}$$

With the same hypotheses as before, $J = \alpha\lambda y^{\alpha-1}$ and the equation to solve is

$$\frac{\partial f}{\partial t} = -k\alpha\lambda f^{\alpha-2}. \tag{18}$$

For $\alpha < 3$, the solution is

$$f(x,t) = (f(x,0)^{3-\alpha} - \alpha(3-\alpha)k\lambda t)^{1/(3-\alpha)}. \tag{19}$$

Hence the field line actually reaches the neutral sheet at a time

$$t = \frac{1}{\alpha(3-\alpha)k\lambda} f(x,0)^{3-\alpha}. \tag{20}$$

The time necessary for reconnection grows when α approaches 3. The field lines for $\alpha = 2$ near the sheet edge evolve as illustrated in Fig. 3.

Notice that while reconnection occurs, the field lines flatten near the sheet more than in the case of constant resistivity, indicating that the process is more difficult. This is even more obvious when we plot a field-line behavior for $\alpha = 2.5$ (Fig. 4).

https://doi.org/10.1017/S0022377803002423 Published online by Cambridge University Press

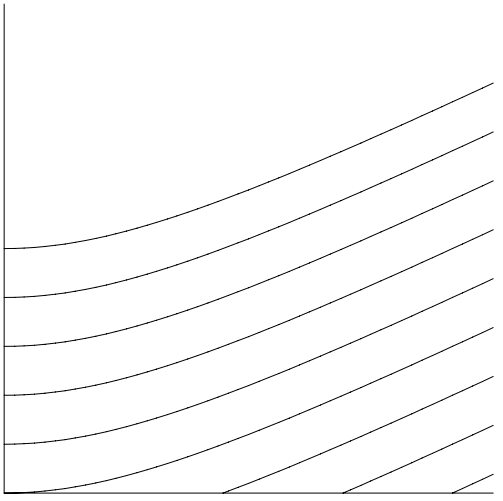


Figure 3. Evolution of a field line with anomalous resistivity and $\alpha = 2$.

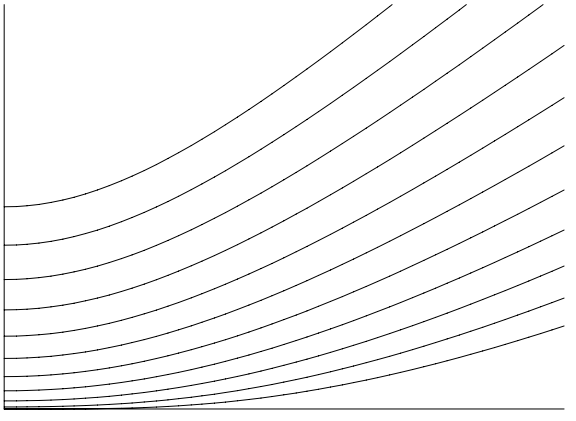


Figure 4. Evolution of a field line with anomalous resistivity and $\alpha = 2.5$.

It appears that field lines are reluctant to reconnect. The limit case occurs for $\alpha = 3$:

$$f(x,t) = f(x,0)e^{-k\alpha\lambda t}. \tag{21}$$

This implies that while the field lines tend rapidly to the neutral sheet when they are at some distance from it, they slow down and never make contact at a finite time. For $\alpha > 3$,

$$f(x,t) = \left(\frac{1}{f(x,0)^{\alpha-3}} + \alpha(\alpha-3)k\lambda t\right)^{1/(\alpha-3)}, \tag{22}$$

the approach is slower and there is no contact either. Thus, for one of these anomalous resistivities to describe with some realism magnetic reconnection, either $\alpha < 3$ or (17) does not describe accurately the resistivity up to the neutral line.

https://doi.org/10.1017/S0022377803002423 Published online by Cambridge University Press

2.3. Hyperresistivity

The addition of an hyperresistivity may be treated in the same way. In this case (5) becomes

$$B_y - B_x \frac{\partial f}{\partial x} = 0,$$
$$-B_x \frac{\partial f}{\partial t} + \eta_2 \nabla^2 J - \eta J + u_x B_y - u_y B_x = 0, \tag{23}$$

so that, for non-vertical fields,

$$\frac{\partial f}{\partial t} = \frac{\eta_2 \nabla^2 J}{B_x} - \frac{\eta J}{B_x} + u_x \frac{\partial f}{\partial x} - u_y. \tag{24}$$

The expression of $\nabla^2 J$ is

$$\nabla^2 J = \frac{\partial^3 B_x}{\partial x^2 \partial y} + \frac{\partial^3 B_x}{\partial y^3} - \frac{\partial^3 B_y}{\partial x^3} - \frac{\partial^3 B_y}{\partial x \partial y^2}.$$

Assuming as before that the resistive terms are dominant in the immediate vicinity of the sheet, and that the magnitudes vary much more rapidly across the neutral line than along it, we can approximate (24) by

$$\frac{\partial f}{\partial t} = \frac{\eta_2}{B_x} \frac{\partial^3 B_x}{\partial y^3} - \frac{\eta}{B_x} \frac{\partial B_x}{\partial y}. \tag{25}$$

Taking as before a dependence of B_x on y of the form $B_x(x, y) = \lambda y^\alpha$, the equation describing the approximate motion of the field line near the sheet is

$$\frac{\partial f}{\partial t} = \eta_2 \frac{\alpha(\alpha-1)(\alpha-2)}{f^3} - \eta \frac{\alpha}{f}. \tag{26}$$

If the resistivity η grows more than y^{-2} when approaching the sheet, its action masks completely the effect of the hyperresistivity. This rapid growth is anyway highly unlikely, so we assume that η_2 is bounded below and $y^2\eta \to 0$ when $y \to 0$. Then the first term on the right-hand side of (26) is dominant when $\alpha(\alpha-1)(\alpha-2) \neq 0$. For $\alpha(\alpha-1)(\alpha-2) > 0$, the term is positive near the neutral sheet, so that this actually repels the field line and there is no reconnection. For integer powers $\alpha = 1$ or $\alpha = 2$ the term vanishes. Only for fractional α, $\alpha < 2$ does the hyperresistivity accelerate the velocity of field lines and eventually the reconnection process. This fact casts some doubt on the presence of hyperresistivity in these phenomena.

3. Conclusions

The evolution of a magnetic field line near a current sheet in two dimensions is found to be rather independent of any specific model in the magnetohydrodynamics approximation. The distance of the line to the sheet is shown to satisfy a differential equation derived from the induction equation whose behavior depends only on the relative sizes of field and velocity. With some reasonable assumptions on them and constant resistivity it may be proved that, as expected, every field line reaches the sheet in a finite time and reconnection occurs. The shape of the evolving field line is found to coincide with certain classical models. The presence of an anomalous resistivity modifies these results: for some simple models of resistivity (e.g. $\alpha > 3$ in (18)), field lines may never reach the sheet, so these models are

https://doi.org/10.1017/S0022377803002423 Published online by Cambridge University Press

inadequate for studies of magnetic reconnection. The addition of an hyperresistive term only helps the process when the field size does not decrease too rapidly near the sheet; otherwise it actually tends to expel field lines from it, thus causing the hyperresistivity to inhibit magnetic reconnection.

References

[1] Sweet, P. A. 1958 The production of high-energy particles in solar flares. *Nuovo Cimento* **8** Ser. X 188–196.

[2] Parker, E. N. 1957 Sweet's mechanism for merging magnetic fields in conducting fluids. *J. Geophys. Res.* **62**, 509–520.

[3] Parker, E. N. 1963 The solar flare phenomenon and theory of reconnection and annihilation of magnetic fields. *Astrophys. J. Supp.* **8**, 177–211.

[4] Syrovatskii, S. I. 1971 Formation of current sheets in a plasma with a frozen-in magnetic field. *Sov. Phys. JETP* **33**, 933–940.

[5] Syrovatskii, S. I. 1981 Pinch sheets and reconnection in astrophysics. *Annual Rev. Astron. Astrophys.* **19**, 163–229.

[6] Syrovatskii, S. I. 1982 Model for flare loops, fast motions, and opening magnetic field in the corona. *Solar Phys.* **76**, 3–20.

[7] Petschek, H. E. 1964 Magnetic field annihilation. In: *Symp. Physics of Solar Flares* (ed. W. N. Hess). Washington, DC: NASA, pp. 425–439.

[8] Sonnerup, B. U. Ö. 1970 Magnetic reconnection in a highly conducting incompressible fluid. *J. Plasma Phys.* **4**, 161–174.

[9] Yeh, T. and Axford, W. I. 1970 On the reconnection of magnetic field lines in conducting fluids. *J. Plasma Phys.* **4**, 207–229.

[10] Vasyliunas, V. M. 1975 Theoretical models of magnetic field line merging. *Rev. Geophys. Space Phys.* **13**, 303–336.

[11] Biskamp, D. 1986 Magnetic reconnection via current sheets. *Phys. Fluids* **29**, 1520–1531.

[12] Biskamp, D. 1993 *Nonlinear Magnetohydrodynamics*. Cambridge: Cambridge University Press.

[13] Priest, E. R. and Forbes, T. G. 1986 New models for fast steady-state magnetic reconnection. *J. Geophys. Res.* **91**, 5579–5588.

[14] Forbes, T. G. and Priest, E. R. 1987 A comparison of analytical and numerical models for steadily driven reconnection. *Rev. Geophys.* **25**, 1583–1607.

[15] Jardine, M. and Priest, E. R. 1988 Weakly nonlinear theory of fast steady-state magnetic reconnection. *J. Plasma Phys.* **40**, 143–161.

[16] Jardine, M. and Priest, E. R. 1990 Energetics of compressible fast steady-state magnetic reconnection. *J. Plasma Phys.* **43**, 141–150.

[17] Priest, E. R. and Forbes, T. G. 1992 Does fast magnetic reconnection exists? *J. Geophys. Res.* **97**, 16 757–16 772.

[18] Priest, E. R. and Forbes, T. G. 2000 *Magnetic Reconnection: MHD Theory and Applications*. Cambridge: Cambridge University Press.

[19] Drake, J. F., Kleva, R. G. and Mandt, M. E. 1994 Structure of thin current layers: implications for magnetic reconnection. *Phys. Rev. Lett.* **73**, 1251–1254.

[20] Drake, J. F., Biskamp, D. and Zeiler, A. 1997 Breakup of the electron current layer during 3D collisionless magnetic reconnection. *Geophys. Res. Lett.* **24**, 2921–2924.

[21] Ji, H., Yamada, M., Hsu, S., Kulsrud, R., Carter, T. and Zaharia, S. 1999 Magnetic reconnection with Sweet–Parker characteristics in two-dimensional laboratory plasmas. *Phys. Plasmas* **6**(5), 1743–1750.

[22] Treumann, R. A. 2001 Origin of resistivity in reconnection. *Earth Planet. Space* **53**(6), 453–462.

[23] Carter, T. A., Yamada, M., Ji, H., Kulsrud, R. M. and Trintchouk, F. 2002 Experimental study of lower-hybrid drift turbulence in a reconnecting current sheet. *Phys. Plasmas* **9**(8), 3272–3288.

https://doi.org/10.1017/S0022377803002423 Published online by Cambridge University Press

[24] Numata, R. and Yoshida, Z. 2002 Chaos-induced resistivity in collisionless magntic reconnection. *Phys. Rev. Lett.* **88**(4), 045003 1–4.

[25] Cordoba, D. and Fefferman, C. 2002 Growth of solutions of the quasi-geostrophic and 2D Euler equations. *J. Amer. Math. Soc.* **15**(3), 665–670.

[26] Somov, B. V. and Syrovatskii, S. I. 1976 Hydrodynamic plasma flow in a strong magnetic field. *Proc. Lebedev Phys. Inst.* **74**, 13–72.

[27] Sato, T. and Hayashi, T. 1979 Externally driven magnetic reconnection and a powerful magnetic energy converter. *Phys. Fluids* **22**, 1189–1202.

https://doi.org/10.1017/S0022377803002423 Published online by Cambridge University Press