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Critical saddle-node cycles: Hausdorff dimension and persistence of tangencies

LORENZO J. DÍAZ† and RAUL URES‡

† *Departamento Matemática, PUC-Rio, Marquês de S. Vicente 225, 22453-900 Rio de Janeiro, RJ, Brazil*
(e-mail: lodiaz@mat.puc-rio.br)

‡ *CC30 IMERL, Facultad de Ingeniería, Universidad de la República, Uruguay*
(e-mail: ures@fing.edu.uy)

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Abstract. We consider the collapse of a saddle of a horseshoe and a sink *via* a critical saddle-node bifurcation. In this way one obtains a *saddle-node horseshoe*. We prove that there is an open set of arcs of diffeomorphisms $\{f_\mu\}_{\mu\in I}$ unfolding generically, say at μ_0, a saddle-node horseshoe with Hausdorff dimension arbitrarily close to $1/2$ so that there is an interval $(\mu_0, \mu_0+\varepsilon)$ in the parameter line such that every diffeomorphism f_μ, $\mu \in (\mu_0, \mu_0+\delta)$, has a tangency.

1. *Introduction*

We consider saddle-node bifurcations collapsing a saddle of a two-dimensional horseshoe and a sink, obtaining (at the bifurcation) a non-hyperbolic attractor Λ conjugate to the initial horseshoe (called a *saddle-node horseshoe*). We analyze the relationship between the Hausdorff dimension of the saddle-node horseshoe Λ and the phenomenon of *persistence of tangencies* following the bifurcation.

The study of saddle-node bifurcations of diffeomorphisms was initiated by Newhouse–Palis–Takens from the point of view of the structural stability, see [**NPT**]. They proved that the unfolding of a *critical saddle-node bifurcation* (see the definition below) leads to diffeomorphisms with homoclinic tangencies. Hence, phenomena such as the persistence of tangencies and the co-existence of infinitely many sinks or sources (see [**N2**]) and cascades of period doubling bifurcations (see [**YA**]) associated to homoclinic bifurcations are also displayed by arcs unfolding a critical saddle-node. Conversely, Mora has observed that critical saddle-node bifurcations are always exhibited when unfolding homoclinic tangencies, see [**PT3**, p. 150]. On the other hand, *critical saddle-node horseshoes* were first considered in [**Z**] and later studied (chronologically) in [**T**], [**DRoV**] and [**C**]. For a survey on saddle-node cycles see [**DRiV**].

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A special feature of saddle-node cycles is the interplay between global aspects of the dynamics (i.e. aspects concerning the dynamics away from the saddle node) and local aspects (i.e. the dynamics in a neighborhood of the saddle-node). In fact, the main tools for the local analysis (*adapted vector fields* and *transition maps*) were developed in [**NPT**]. Let us now explain the global aspects. For that let us recall some facts about saddle-node bifurcations.

A *saddle node* of a diffeomorphism $f: M \to M$ is a periodic point P of period k, so that the derivative Df^k of f^k at P only has an eigenvalue in the unit circle which is one. For simplicity, from now on we take the saddle node to be fixed. Here we deal with two-dimensional diffeomorphisms and assume that the saddle node P is *dissipative* (i.e. all the eigenvalues of Df at P are contained within the unit circle). Thus, the tangent space of M at P is the direct sum of two Df-invariant spaces, say $T_P M = E^c \oplus E^{ss}$, where the restrictions of Df to E^c and E^{ss} are the identity and a contraction, respectively. The stable manifold $W^s(P)$ of the saddle-node P has dimension two and the unstable manifold $W^u(P)$ is one dimensional. In $W^s(P)$ there is defined a unique f-invariant foliation $\mathcal{F}^{ss}$ with one-dimensional leaves, so that the leaf of P is the *strong stable manifold* $W^{ss}(P)$ *of* P (i.e. the unique f-invariant manifold tangent to E^{ss} at P). The foliation $\mathcal{F}^{ss}$ is the *strong stable foliation,* see [**NPT**, Theorem II.3.4] and [**HPS**].

We say that the saddle-node P is involved in a *saddle-node cycle* if $W^u(P)$ intersects the interior of $W^s(P)$.

An f-invariant set Λ is a *saddle-node horseshoe* if it is locally maximal in a neighborhood of it, conjugate to a horseshoe and contains a unique non-hyperbolic point P which is a saddle node. In this paper, all the saddle-node horseshoes we consider are attractors (the stable manifold of Λ is a neighborhood of Λ). Observe that, in this case, the unstable manifold of P intersects its strong stable manifold. In particular, the saddle node P of Λ is in a saddle-node cycle.

The first global aspect of saddle-node cycles is whether the unstable manifold of P is contained within the interior of the stable one (if so we speak of *saddle-node one cycles*). A second aspect is whether the unstable manifold intersects the strong stable manifold of the saddle-node (as in the case of the saddle-node horseshoes). Finally, a third aspect is the existence of *criticalities.* The cycle is *critical* if the unstable manifold is tangent to some leaf of the strong stable foliation. Otherwise the cycle is *non-critical.*

In this paper, we adopt the strategy suggested by Palis–Takens in [**PT3**] for studying bifurcations of diffeomorphisms: consider one-parameter families of diffeomorphisms $\{f_\mu\}_{\mu\in[-1,1]}$ unfolding a generic bifurcation (here a saddle node) and study which dynamical phenomena are more commonly displayed in terms of the Lebesgue measure of parameters following the bifurcation. According to the aspects above, the dynamics following the unfolding of a saddle-node cycle may exhibit rather different features. For instance, the bifurcation set associated to the (generic) unfolding of a critical saddle-node cycle always contains intervals, see [**DRoV**]. On the other hand, a bifurcation associated to a non-critical saddle-node horseshoe may be isolated (as in the case of the DA-maps of Smale in [**W**]), see [**CK**].

In [**T**], the features above are discussed and saddle-node cycles are presented as a mechanism for *intermittency bifurcations* (generalizing the notion of intermittency in [**PM**]).

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In [**DRoV**, Theorem A] it is proved that the unfolding of a critical saddle-node cycle leads to the existence of Hénon-like strange attractors *prevalently*: the set E of parameters μ such that f_μ exhibits a Hénon-like attractor satisfies $|E \cap [0, \mu]| > k\,\mu$ for all $\mu > 0$ small enough, where $k > 0$ is a constant independent of μ and $|A|$ denotes the Lebesgue measure of the set A. In this construction, the attractors are obtained via a renormalization mechanism and the sizes of their basins of attraction are small, thus these attractors have a local character.

In the case of one–cycles (as well as in some saddle-node horseshoes) there is an invariant neighborhood D of the unstable manifold of the saddle node. Thus, it is natural to ask about the maximal invariant set Λ_μ of f_μ in D. Still in this case, under some assumptions on the geometry of the unstable manifold of the saddle-node and on the dynamics of f_0 on it, in [**DRoV**, Theorem C] it is proved that Λ_μ is a Hénon-like strange attractor whose basin of attraction contains D prevalently at the bifurcation. This result was generalized in [**C**] for saddle-node horseshoes.

Finally, in [**DRoV**] the occurrence of hyperbolic dynamics for critical saddle-node cycles is studied.

Our result is motivated by and related to the following question. Given a diffeomorphism f, let Λ_0 be an f-invariant and transitive set which is locally maximal in a neighborhood of it (i.e. there is a compact set D containing Λ_0 in its interior such that $\Lambda_0 = \cap_{i\in Z} f^i(D)$), involved in a generic bifurcation (for instance, Λ_0 has a saddle-node, flip or Hopf bifurcation, or Λ_0 exhibits a homoclinic or heteroclinic tangency). Take an arc $\{f_\mu\}_{\mu\in[-1,1]}$ unfolding generically the bifurcation associated to Λ_0 at $\mu = 0$ (i.e. $f_0 = f$).

QUESTION. *Is there a necessary and sufficient condition involving fractal dimensions of the set Λ_0 guaranteeing the existence of an interval J of the form $(0, \mu_0)$ or $(-\mu_0, 0)$ in the parameter space, such that, for every $\mu \in J$, there is a tangency between the stable and unstable manifolds of a new hyperbolic set Λ_μ of f_μ contained in the locally maximal set of f_μ in D?*

This question was posed by Palis and Takens (see [**PT3**, Problem 3, §7]) for arcs unfolding homoclinic tangencies. Actually, in the context of homoclinic bifurcations, many papers have analyzed the relationship between the frequency of hyperbolic dynamics and bifurcations determined by the fractal dimensions (Hausdorff dimension, limit capacity and thickness) of the set involved in the bifurcation, see [**PT2, PT1, PY1, PY2, M, R**]. Here we address this problem for critical saddle-node bifurcations. Roughly speaking, we show that for critical saddle-node bifurcations the frequency of hyperbolic dynamics following the bifurcation may be independent of the Hausdorff dimension of the *saddle-node horseshoe* Λ_0.

Let us now explain all that more precisely. For arcs of diffeomorphisms unfolding a homoclinic tangency, the frequency of hyperbolic dynamics following the bifurcation is strongly related to the fractal dimension of the basic set Λ_0 associated to the tangency. Consider arcs of diffeomorphisms $\{f_\mu\}_{\mu\in I}$ unfolding generically at $\mu = 0$, a homoclinic tangency associated to a periodic point in Λ_0. On the one hand, when the Hausdorff dimension of Λ_0, $HD(\Lambda_0)$, is less than one, the bifurcation parameter value is a point of full

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(relative) density of the parameters corresponding to hyperbolic dynamics. In particular, the closure of the set of parameters corresponding to diffeomorphisms with homoclinic tangencies has (relative) density zero at the bifurcation, see [**PT2, PT1**]. On the other hand, if $HD(\Lambda_0) > 1$, then for generic arcs the bifurcation is not a point of full density of hyperbolic dynamics, see [**PY2**].

Here we consider bifurcations of horseshoes *via* saddle-node bifurcations, i.e. a sink and a saddle of a horseshoe Λ collapse, obtaining at the bifurcation a locally maximal transitive non-hyperbolic attractor Λ_0 conjugate to the initial horseshoe Λ. We construct an open set of arcs $\{f_\mu\}_{\mu\in[-1,1]}$ unfolding at $\mu = 0$ a saddle-node horseshoe Λ_0 with small Hausdorff dimension (in our context that means close to $1/2$, see the comments below and §5) exhibiting persistence of tangencies following the bifurcation. More precisely, there is a compact disk D such that for every $\mu < 0$ the limit set of f_μ in D is the disjoint union of a sink S_μ and a basic set Λ_μ, and for $\mu = 0$ the limit set of f_0 in D is a non-uniformly hyperbolic set Λ_0 (thus the basin of attraction of Λ_0 contains the neighborhood D of Λ_0) which is conjugate to Λ_μ ($\mu < 0$). We prove that there is $\mu_0 > 0$ so that for every μ in $(0, \mu_0)$ there is a hyperbolic set Σ_μ contained in the maximal invariant set of f_μ in D with a tangency.

This result means that in the saddle-node bifurcations setting, the fractal dimensions do not play the key role that they do in the case of homoclinic bifurcations for determining the prevalent dynamics following the bifurcation. In fact, in our construction the persistence of tangencies follows from the dynamics in parts of the center manifold of the saddle-node outside the set Λ_0. This is because the fractal dimension of Λ_0 is not so decisive for determining the structure of the bifurcation set here.

Let us now return to the problem posed above on the relationship between fractal dimensions of a saddle-node horseshoe, prevalence of hyperbolic dynamics and persistence of tangencies following the bifurcation.

An arc of diffeomorphisms $\{f_\mu\}_{\mu\in I}$ *generically unfolds a critical saddle-node horseshoe* at $\mu = 0$ if f_0 has a *generic critical saddle-node horseshoe* Λ_0 (i.e. a generic saddle-node horseshoe with a critical saddle node) and the arc $\{f_\mu\}_{\mu\in I}$ generically unfolds the saddle node in Λ_0. Denote by $\mathcal{S}(M)$ the set of arcs in $\mathcal{P}^\infty(M)$ (the space of arcs of $\mathcal{C}^\infty$-diffeomorphisms defined on M endowed with the usual topology) generically unfolding a critical saddle-node horseshoe.

For arcs $\{f_\mu\}_{\mu\in I}$ in $\mathcal{S}(M)$, independently of the dimension of Λ_0 (and even in the case in which Λ_0 is a point), the bifurcation $\mu = 0$ is not a point of full density of hyperbolic dynamics: a critical saddle node is a point of positive density of parameters corresponding to diffeomorphisms with Hénon-like attractors, see [**DRoV**].

On the one hand, the Hausdorff dimension of any saddle-node horseshoe Λ_0 is always lower bounded by $1/2$. To see why this is so, first observe that Λ_0 is the closure of the transverse intersections between $W^u(P)$ and $W^{ss}(P)$, where P is the saddle node in Λ_0. Take now any point x in this transverse intersection, then $f^{-n}(x)$ converges to P as $1/n$ go to zero (i.e. the intervals $[f^{-n}(x), f^{-n-1}(x)]$ of the central manifold have sizes of the order of $1/n^2$). Since the limit capacity of the set $\{0\}\cup\{1/n\}_{n\in\mathbb{N}}$ is $1/2$, this implies that the capacity of Λ_0 is at least $1/2$, for details see [**DV**, §5]. Actually, the results in [**DV**] imply that the limit capacity and the Hausdorff dimension of Λ_0 are equal, thus $HD(\Lambda_0) > 1/2$.

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In Lemma 5.1 we prove that given any $\beta > 1/2$, there is a saddle-node horseshoe with Hausdorff dimension less than β.

On the other hand, in [**DRoV**] (see Theorem B and the comments on p. 58) it is also proved that for every $\kappa \in (1/2, 2)$, there is an open subset in $\mathcal{S}(M)$ of arcs unfolding a saddle-node horseshoe Λ_0 with $HD(\Lambda_0) \geq \kappa$ such that the parameters of hyperbolic dynamics have positive density at the bifurcation. Let us remark that there are similar results for homoclinic tangencies of basic sets with Hausdorff dimension bigger than one, see [**M**].

In this paper we address the converse situation: the problem of the existence of an open set in $\mathcal{S}(M)$ unfolding a saddle-node horseshoe Λ_0 with small Hausdorff dimension (from the comments above that means that $HD(\Lambda_0)$ is close to $1/2^+$), so that there is an interval $(0, \mu_0)$ of parameters with tangencies related to a hyperbolic set Σ_μ of f_μ contained in the basin of attraction of Λ_0. More precisely, let D be an open disk where the saddle-node horseshoe is locally maximal, thus D is contained in the stable manifold of Λ_0. Define by $C(\mu)$ the set of parameters ν in $(0, \mu]$ so that there is a hyperbolic set Σ_ν of f_ν contained in the locally maximal set of f_ν in D with a tangency.

THEOREM 1.1. *For every $\varepsilon > 0$ there is a non-empty open subset $\mathcal{C}_\varepsilon(M)$ of $\mathcal{S}(M)$ of arcs of diffeomorphisms $\{f_\mu\}_{\mu \in I}$ generically unfolding at $\mu = 0$ a critical saddle-node horseshoe Λ_0 so that:*

- $\mathrm{HD}(\Lambda_0) < (1/2) + \varepsilon$*; and*
- $C(\mu_0) = (0, \mu_0]$ *for every $\mu_0 > 0$ small enough.*

We observe that our construction can be carried in higher dimensions assuming that the saddle node is dissipative.

Let us say a few words about the proof of the theorem. First let us observe that the usual way (and also the only known for surface diffeomorphisms) to get persistence of tangencies is to exhibit *thick hyperbolic sets* (i.e. a hyperbolic set Λ such that the product of its *stable and unstable thickness* is bigger than one, see Definition 3.2 and (1)) with a tangency (see [**N1**], [**N2**] and [**PT3**]).

The key idea of our proof is, following [**NPT**], to introduce *ghost dynamics* at the bifurcation. Roughly speaking, we select a fundamental domain Δ of the central manifold of the saddle node and using the transition maps of [**NPT**] (see §2.2) define a one parameter family F_σ of maps from an interval J of Δ into the whole Δ. We construct our saddle-node horseshoe in such a way that for each σ the map F_σ has a hyperbolic set Ω_σ contained in J with a big thickness (see §2.3 and §2.4). Using the results in [**DRoV**] we see that for every small $\mu > 0$ the diffeomorphism f_μ has a hyperbolic set (which is the continuation of some Ω_σ) with a big thickness. In this way we get the thick hyperbolic sets.

We point out that our construction does not work when Λ_0 is a trivial set (a periodic orbit). Thus, let us end this introduction by posing a question. First, let us recall that a *critical saddle-node one-cycle* is a critical saddle-node bifurcation associated to an isolated point P of the limit set satisfying $W^u(P) \subset W^s(P)$.

Problem. *Let $\mathcal{O}(M)$ be the set of arcs of diffeomorphisms generically unfolding a critical saddle-node one-cycle (say at $\mu = 0$). Does there exist a non-empty open subset $\mathcal{C}_\mathcal{O}(M)$ of $\mathcal{O}(M)$ of arcs $\{f_\mu\}_{\mu \in I}$ with $C(\mu_0) = (0, \mu_0]$ for all $\mu_0 > 0$ small enough?*

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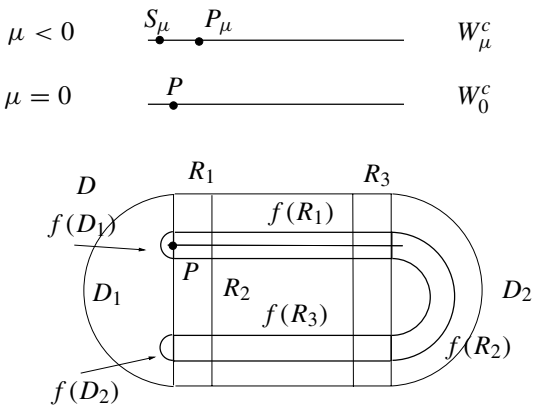


FIGURE 1. A saddle-node horseshoe.

2. *Construction and outline of the proof of the theorem*

In this section we sketch and introduce the mains ingredients of the proof of the Theorem. We begin by taking a two-dimensional disk D which is the disjoint union of a square R and two disjoint disks D_1 and D_2, as in Figure 1. In D we consider the usual Smale's horseshoe diffeomorphism $f: D \to D$ mapping D into its interior so that the limit set of f in D is a sink S, contained in the interior of the disk D_1, a linear horseshoe Λ contained in R (in fact in $R_1 \cup R_3$) and conjugate to the complete shift of two symbols. Moreover, the disks D_1 and D_2 are in the stable manifold of the sink S. In Figure 1 the dynamics of the diffeomorphism f in D are depicted. Let us observe that we can extend f to a diffeomorphism defined on any surface.

Denote by P the *accessible fixed saddle* of Λ (i.e. there are connected components of $(W^s(P) \setminus \{P\})$ and of $(W^u(P) \setminus \{P\})$ that do not intersect Λ, see Figure 1). We collapse the saddle P and the sink S by a saddle-node bifurcation. That is, we consider an arc $\{g_\mu\}_{\mu\in[-1,1]}$ with $g_{-1} = f$, such that for every $\mu < 0$ the diffeomorphisms g_μ are hyperbolic and its non-wandering set in D is the union of a sink S_μ (the continuation of S) and a horseshoe Λ_μ (the continuation of Λ_μ) conjugate to Λ and g_0 has a saddle-node horseshoe Λ_0 obtained by collapsing the continuations P_μ and S_μ of P and S. Moreover, we have that the set Λ_0 is an attractor conjugate to Λ containing the whole disk D in its stable manifold. For the details of this construction we refer the reader to [**DRoV**, §4.2].

2.1. *Bounds for the stable thickness.* We take the arc $\{g_\mu\}_{\mu\in[-1,1]}$ above with $g_\mu = f$ outside a small vicinity U of P and S. Thus

$$\bigcap_{\mathbb{Z}} g_\mu^i(D \setminus U) = \bigcap_{\mathbb{Z}} f^i(D \setminus U) = \Lambda',$$

where Λ' is a hyperbolic set for all $\mu \in [-1, 1]$. Observe that we can assume that the other fixed saddle of Λ, say Q, is not in U, so Q belongs to Λ'.

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Denote by $\tau^u = \tau^u(\Lambda')$ the *unstable thickness* of Λ', i.e. the *thickness* (see Definition 3.2) of the set $W^s(Q) \cap \Lambda'$. Clearly, there is $\ell > 0$ such that

$$\tau^u = \tau^u(\Lambda') = \tau(W^s(Q) \cap \Lambda') > \ell > 0. \tag{1}$$

2.2. *Transition maps.* A key ingredient of our construction is the *transition maps* introduced by Newhouse–Palis–Takens in [**NPT**]. The saddle node $P_0 = S_0$ of g_0 has a g_0-invariant normally hyperbolic *central manifold* W_0^c of dimension one (see [**HPS**]) and the set $(W_0^c \setminus \{P_0\})$ has two connected components, one contained in $W^s(P_0)$ and the other one in $W^u(P_0)$. Moreover, for μ close to zero, the continuation W_μ^c of W_0^c is defined. Then, if the saddle node is generic, there are coordinates such that the restriction of g_μ to W_μ^c, denoted by φ_μ, is of the form

$$\varphi_\mu(x) = x + \mu + \alpha x^2 + \beta x\mu + \gamma\mu^2 + \mathcal{O}(|\mu|^3 + |x|^3).$$

In our case, we are assuming that in our coordinates $P_0 = 0$, $\{x < 0\}$ is contained in $W^s(P_0)$ and $\alpha > 0$.

The local dynamics (in W_μ^c) may be analyzed considering the *adapted saddle-node vector fields* of [**NPT**], i.e. a *saddle-node arc of vector fields* $X(\mu, x)$ of the form

$$X(\mu, x) = \mu + \alpha x^2 + \beta x\mu + \gamma\mu^2 + \mathcal{O}(|\mu|^3 + |x|^3), \quad \text{where } \alpha > 0.$$

Then one can assume that $\varphi_\mu(x) = X_1(\mu, x)$, where $X_s(\mu, \cdot)$ is the time-s map of the vector field $X(\mu, \cdot)$, see for instance [**DRoV**, §2.1].

In each component of $(W_0^c \setminus \{P\})$ one takes a fundamental domain of g_0, say $D^\ell = [a, g_0(a)]$ and $D^r = [b, g_0(b)]$, respectively, where $a < 0 < b$ and $X(0, x) > 0$ for all $x \in ([g_0^{-2}(a), g_0^2(b)] \setminus \{0\})$.

For every big $k \in \mathbb{N}$ and small strictly positive μ, one defines the parameters μ_k (big k) by the relation

$$X_k(\mu_k, a) = b.$$

The parameters μ_k are uniquely defined (for small μ) and form a decreasing sequence converging to 0^+ as $k \to \infty^+$. Moreover, for each k there is a C^∞ (decreasing) reparameterization $\hat{\mu}_k \colon [0, 1] \to [\mu_{k+1}, \mu_k]$ so that the sequence of *transitions* $\{T_k\}_k$ defined by

$$T_k \colon [0, 1] \times D^\ell \to W_0^c, \quad T_k(\sigma, x) = \varphi^k_{\hat{\mu}_k(\sigma)}(x) \tag{2}$$

converges in the $\mathcal{C}^\infty$ topology to the $\mathcal{C}^\infty$ *transition map* T_∞ defined by

$$T_\infty(\sigma, x) = X_{t(x)-\sigma}(0, b),$$

where $t(x)$ is given by the relation $X_{t(x)}(0, a) = x$. For the proof of these assertions see [**DRoV**, Proposition 2.1 and Theorem 2.3].

2.3. *Ghost dynamics.* Let us suppose that $R = R_1 \cup R_2 \cup R_3$, where R_1, R_2 and R_3 are three closed rectangles as in Figure 1 such that $\Lambda_\mu \subset R_1 \cup R_3$ for all $\mu \leq 0$. Moreover, we can assume that b is in the boundary of R_1. For notational simplicity let us suppose that $b = 1$ and $a = -1$. We also can assume that $f(D^r) = g_\mu(D^r)$ intersects D_2

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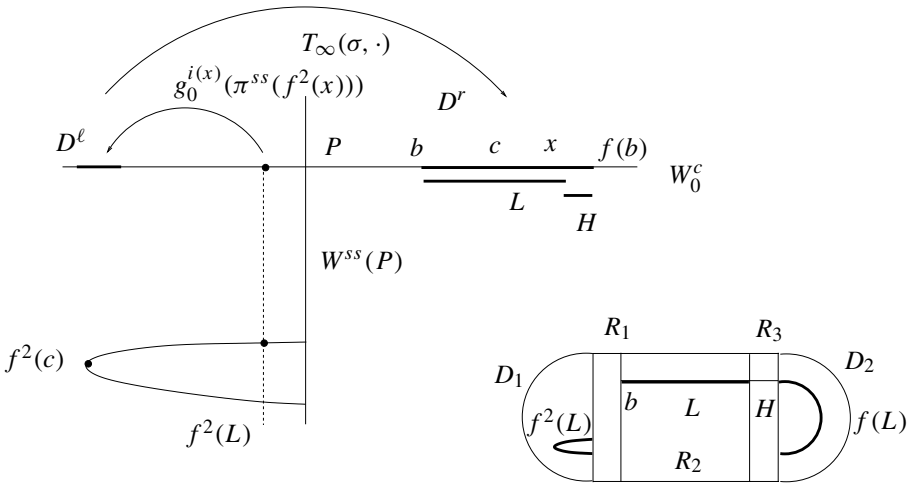


FIGURE 2. Ghost dynamics.

and that $(f(D^r) \setminus D_2) = (g_\mu(D^r) \setminus D_2)$ is an interval. Denote by H the preimage of such an interval. Moreover, the segment H can be taken to be arbitrarily small, for that it is enough to take the rectangles R_1 and R_3 containing the hyperbolic sets Λ_μ to be sufficiently narrow. For each small positive μ, consider the interval

$$L = L_\mu = \text{int}(D^r \setminus H).$$

Observe that by construction the two extremes of L are contained in the strong stable manifold of P_0, see Figure 2.

Since $f(D_2)$ is contained in D_1, which is contained in the stable manifold of P_0, we can assume that $f(D_2)$ is contained in the local stable manifold of P_0, otherwise it is enough to consider $f^i(D_2)$ for some i sufficiently large. Thus, let us assume that $f^2(L) = g_\mu^2(L)$ is contained in the local stable manifold of P_0 (recall again that the extremes of $f^2(L)$ are contained in $W^{ss}(P_0)$).

For each $\sigma \in [0,1]$ we define the *ghost return map* G_σ from L into the whole fundamental domain D^r as follows. Associated with the strong stable foliation, we define the projection π^{ss} along the leaves of the strong stable foliation, from $W^s_{\text{loc}}(P_0)$ into an interval of W_0^c of the form $[A, P_0)$, $D^\ell \subset [A, P_0)$. We can assume that $\pi^{ss}(f^2(c)) \in D^\ell$. Observe that $\pi^{ss}(f^2(L)) = [\pi^{ss}(f^2(c)), P_0)$, where c is the point of L corresponding to the tangency between $W^u(P_0)$ and the strong stable foliation of the saddle node (see condition (Q) below).

For each $x \in L$ we have $f^2(x)$ in the interior of $W^s_{\text{loc}}(P_0)$, thus the first non-positive integer $i(x)$ is defined such that

$$g_0^{i(x)}(\pi^{ss}(g_0^2(x))) = g_0^{i(x)}(\pi^{ss}(f^2(x))) \in D^\ell.$$

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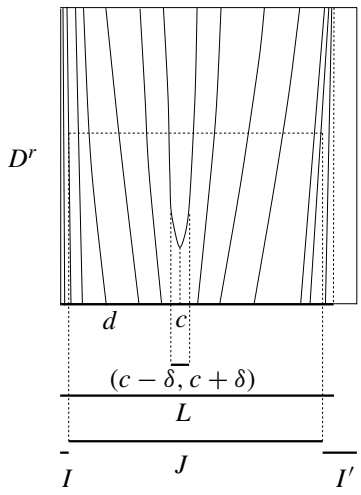


FIGURE 3. The maps G_σ and F_σ.

Observe that $i(x) \to -\infty$ as x approximates to the boundary of L. By construction,

$$T_\infty(\sigma, g_0^{i(x)}(\pi^{ss}(g_0^2(x)))) \in D^r \cup g_0^{\pm 1}(D^r)$$

for all $x \in L$. Given $y \in D^r \cup g_0^{\pm 1}(D^r)$ define $y/(\mathrm{mod}\, g_0) \in D^r$ by y if $y \in D^r$ or $g_0^{\pm 1}(y)$ if otherwise. We now let

$$G_\sigma : L \to D^r, \quad x \mapsto T_\infty(\sigma, g_0^{i(x)}(\pi^{ss}(g_0^2(x))))/(\mathrm{mod}\, g_0).$$

Observe that the maps G_σ have infinitely many branches, see Figure 3. Now for a fixed large k we consider a compact subinterval $J_k = [j_k^\ell, j_k^r]$ of L containing c in its interior such that projection of $f^2(J_k)$ by π^{ss} in $W^c(P_0)$ is such that:

- $\pi^{ss}(f^2(J_k)) \cap \mathrm{int}(D^\ell) \neq \emptyset$;
- $\pi^{ss}(f^2(J_k))$ is not contained in $\cup_{i=0}^{k-1} g_0^i(D^\ell) \neq \emptyset$, thus $\pi^{ss}(f^2(J_k)) \cap g_0^k(D^\ell) \neq \emptyset$;
- $\pi^{ss}(f^2(J_k)) \subset \bigcup_{i=0}^{k} g_0^i(D^\ell)$; and
- $\pi^{ss}(f^2(j_k^\ell)) = \pi^{ss}(f^2(j_k^r))$.

In this case we say that $\pi^{ss}(f^2(J_k))$ *intersects* $(k+1)$ *fundamental domains of* $W^c(P_0)$.

For a fixed large k (see condition (E) below) we let $J = J_k$ and denote by F_σ the restriction of G_σ to J. Finally, write $D^r = J \cup I \cup I'$, where I and I' are two intervals.

We now formulate some properties that the one parameter family of maps $\{F_\sigma\}_{\sigma \in [0,1]}$ will satisfy.

(Q) *Unique (quadratic) critical point.* We assume that the curve $f^2(L) \subset W^s_{\mathrm{loc}}(P_0)$ has a unique point of tangency with the strong stable foliation of $W^s(P_0)$ (which is a quadratic generic tangency) at $f^2(c)$, c in the interior of L. Then the point c is the (unique) critical point of F_σ for all $\sigma \in [0, 1]$ (in some cases, for finitely many parameters σ, the point c may also be a discontinuity of F_σ, see (D) below).

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(D) *Discontinuities.* Observe that a point d of J is a discontinuity of F_σ if and only if $T_\infty(\sigma, g_0^{i(x)}(\pi^{ss}(g_0^2(d))))$ belongs to the boundary of D^r. This implies that F_σ has at most $2(k+2)$ discontinuities. List $d_1(\sigma), d_2(\sigma), \ldots, d_m(\sigma)$; $d_i(\sigma) < d_{i+1}(\sigma)$, the discontinuities of F_σ. Suppose that $d_j(\sigma) \le c < d_{j+1}(\sigma)$ (observe that for at least one and at most two values of $\sigma \in [0,1]$ the critical point corresponds to a discontinuity of F_σ). Then for all $2 \le i < (j-1)$ and $j+1 \le i \le (m-2)$ the image of the interval $[d_i(\sigma), d_{i+1}(\sigma)]$ by F_σ is the whole D^r. Moreover, F_σ is monotone in $(d_i(\sigma), d_{i+1}(\sigma))$ (decreasing if $i \le j-1$ and increasing if $i \ge j+1$). So we can extend F_σ to obtain a map (which is bi-valued at the discontinuities) such that $F_\sigma([d_i(\sigma), d_{i+1}(\sigma)]) = D^r$ (if $i \ne j$). Finally, the restriction of F_σ to $[d_j(\sigma), d_{j+1}(\sigma)]$ is unimodal.

(E) *Expansiveness of F_σ.* For every σ the map F_σ is continuously differentiable in the intervals $(d_i(\sigma), d_{i+1}(\sigma))$, $[j^r, d_1(\sigma))$ and $(d_m(\sigma), j^\ell]$, where $J = J_k = [j^r, j^\ell]$.

If k is sufficiently large we can modify f in the interior of D_2 to get a new diffeomorphism $\tilde f$ such that $\tilde f^2(J) = f^2(J)$, c is the unique critical point of $\tilde F_\sigma$ for all σ (here $\tilde F_\sigma$ is the ghost return map for $\tilde g_0$), the critical point c is a continuity point of $\tilde F_0$ and there is small $\delta > 0$, with $d_j(0) < c - \delta < c + \delta < d_{j+1}(0)$, such that the absolute value of the derivatives $\tilde F_0'(x^\pm)$ belong to $[k, 2k]$ for every $x \in J \setminus (c-\delta, c+\delta)$ (here $\tilde F'(x^+)$ and $\tilde F'(x^-)$ are the derivatives of $\tilde F$ at the right and at the left of x).

Observe that by definition,

$$\tilde F_\sigma(x) = X_\sigma(0, \tilde F_0(x))/(\mathrm{mod}\, g_0),$$

that is, the maps $\tilde F_\sigma$ are (essentially) *translations* of $\tilde F_0$ by the adapted vector field (this is not completely true, because for some points in addition to the translation by the vector field one needs to consider an extra iteration by $g_0^{\pm 1}$ in $g_0^\pm(D^r)$, but this derivative is bounded). Thus, we can assume that the expansion property above,

$$k \le |F_\sigma'(x^\pm)| \le 2k,$$

holds for all σ and $x \in J \setminus (x-\delta, x+\delta)$.

Remark 2.1. Observe that the new diffeomorphism $\tilde f$ may be far from f. However, it is obtained by a local (non-necessarily small) perturbation of f in the interior of D_2, so this perturbation does not affect the locally maximal invariant sets in R for $\mu \le 0$. So we can suppose that our initial arc (thus the transitions F_σ) satisfy the properties (Q), (D) and (E) above.

2.4. *Ghost thick horseshoes.* Having in mind properties (Q), (D) and (E), we will introduce in §3.1 the families of maps $\mathcal{L}_{\varepsilon,d}$, corresponding to an affine limit model for the families F_σ. In fact, the maps of our family F_σ are smooth perturbations of maps of this affine model category. The main consequence of our construction is that, heuristically, the diffeomorphism g_0 has a *ghost horseshoe* with a large thickness and small gaps: naïvely a (unidimensional) thick horseshoe that is not defined for g_0, but for the ghost return maps F_σ and it has a continuation for g_μ for every small μ.

The first part of the previous assertion is the following somewhat technical result (see §3.2 for the proof).

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LEMMA 2.2. *Consider any (large) $K > 0$ and (small) $\varepsilon > 0$. Then there are $\varepsilon_0 > 0$ and $k_0 > 0$ with the following property.*

Let $\{g\}_{\mu\in[-1,1]}$ be a saddle-node arc of diffeomorphisms generically unfolding a critical saddle-node horseshoe, as above, such that its associated one-parameter family of ghost return maps $\{F_\sigma\}_{\sigma\in[0,1]}$, $F_\sigma : J \to D^r$, satisfy (Q), (D) and (E) with (using the notations above):

- $\max\{l(I), l(I'), \delta\} < \varepsilon_0$*; and*
- $F'_\sigma(x^{\pm}) \in [k, 2k]$*, $k \geq k_0$, for all $x \in J \setminus (c-\delta, c+\delta)$ and $\sigma \in [0,1]$.*

Then, for every $\sigma \in [0,1]$, the map F_σ has a hyperbolic Cantor set Ω_σ contained in the interior of D^r such that:

- *the thickness of Ω_σ is bigger than K;*
- *any gap of Ω_σ in J has a length of less than ε;*
- *the distance from Ω_σ to the boundary of D^r is upper bounded by ε.*

In this lemma we will take $K > 2/\ell$, where ℓ is the unstable thickness of the set Λ' (see equation (1) above) and ε_0 will depend on the size of the unstable gaps of Λ' (for details see comments in §§2.5 and 4).

For the continuations of the horseshoes Ω_σ for g_μ, small μ, with large stable thickness we need to be much more precise. Let us first recall that the stable and unstable thicknesses of a hyperbolic set depend continuously on the diffeomorphism, see [**PT3**, Chapter 4, §3]. Consider now the sequence of maps

$$F_{k,\sigma} : J \to D^r,$$

$$F_{k,\sigma}(x) = T_k(\sigma, g^{i(x)}_{\hat{\mu}_k(\sigma)}(\pi^{ss}(g^2_{\hat{\mu}_k}(x))))/(\mathrm{mod}\, g_\mu) = g^{i(x)+k}_{\hat{\mu}_k(\sigma)}(\pi^{ss}(g^2_{\hat{\mu}_k}(x)))/(\mathrm{mod}\, g_\mu),$$

see (2) for the definition of T_k.

Since the sequence of maps T_k converges to T_∞ in the C^∞-topology (recall the result above in [**DRoV**]) and $i(x)$ is bounded (recall the definition of the subinterval J of L), we have that $F_{k,\sigma}$ converges to F_σ in the C^∞-topology. Finally, observe that $F_{k,\sigma}(x)$ and $g^{i(x)+k+2}_{\hat{\mu}_k(\sigma)}(x)/(\mathrm{mod}\, g_\mu)$ are close if k is sufficiently large. Thus, for every small μ, $\mu = \hat{\mu}_k(\sigma)$, $\sigma \in [0,1]$ and large k, one has that g_μ has a hyperbolic set Ω_μ with stable thickness $\tau^s(\Omega_\mu)$ bigger than $2/\ell$. Here Ω_μ, $\mu = \hat{\mu}_k(\sigma)$, is the continuation of the hyperbolic set Ω_σ of F_σ.

2.5. *Linked hyperbolic sets.* Finally, we prove that the sets Ω_μ and Λ' are *linked* for every $\mu > 0$ small enough (roughly speaking, each set is not contained in a gap of the other). This will be done in §4. First let us recall the Gap Lemma (see [**N2**]).

LEMMA 2.3. (Gap Lemma) *Let K_1 and $K_2 \subset \mathbb{R}$ be two Cantor sets with thickness τ_1 and τ_2 such that $\tau_1 \cdot \tau_2 > 1$. Then one of the following possibilities hold: either K_1 is contained in a* gap *of K_2 (a connected component of the complement of K_2, see Definition 3.2) or vice-versa, or $K_1 \cap K_2 \neq \emptyset$.*

Since by construction $\tau^u(\Lambda') \cdot \tau^s(\Omega_\mu) > 1$ and these sets will be linked (see §4 for details) the Gap Lemma will imply that there is a tangency between a stable leaf of Ω_μ and a unstable leaf of Λ', see §4. Moreover, since the sets Λ' and Ω_μ are homoclinically

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related (Lemma 4.1) one gets a hyperbolic set Σ_μ in the disk D with a tangency between its stable and unstable manifolds.

2.6. *Hausdorff dimension of* Λ_0. To end the proof of the theorem we are left to check that in our construction the set Λ_0 can be taken with Hausdorff dimension arbitrarily close to $1/2$. Actually, our construction becomes easier as the dimension of Λ_0 goes to $1/2$. Intuitively, to have small (close to $1/2^+$) *stable Hausdorff dimension* for Λ_0 (i.e. the Hausdorff dimension of $W^u(P)\cap\Lambda_0$, P the saddle-node of Λ_0, denoted $HD^s(\Lambda_0)$) it is enough to take the rectangles R_1 and R_3 containing the saddle-node horseshoe (or equivalently, the intervals I and I') small enough. This will be done precisely in §5.

On the other hand, to have small (close to zero) *unstable Hausdorff dimension* for Λ_0, (i.e. the Hausdorff dimension of $W^s(P)\cap\Lambda_0$, denoted $HD^u(\Lambda_0)$) it is enough to suppose that the (hyperbolic) contraction in the stable direction of Λ_0 is sufficiently large. Since $HD(\Lambda_0) = HD^s(\Lambda_0) + HD^u(\Lambda_0)$ (see the arguments in [**DV**]), $HD(\Lambda_0)$ is close to $1/2$. This will be done in §5.

Finally, let us observe that, by the continuous dependence in the C^∞-topology of all estimates involved in the proofs, our construction works for an open family of arcs.

In the next three sections we will prove the results mentioned in §§2.4, 2.5 and 2.6.

3. *Thickness of one-dimensional piecewise affine maps*

3.1. *The set* $\mathcal{L}_{\varepsilon,d}$. In this section we will introduce a class of piecewise affine maps of the interval depending on two parameters d and ε, denoted by $\mathcal{L}_{\varepsilon,d}$, such that for any fixed (large) $M > 0$ there are d_0 and $\varepsilon_0 > 0$ such that every map $f \in \mathcal{L}_{\varepsilon,d}$, $d \geq d_0$ and $\varepsilon \in (0, \varepsilon_0)$, has a hyperbolic set Λ whose thickness is bigger than M (see Proposition 3.1).

Definition 3.1. (The class $\mathcal{L}_{\varepsilon,d}$) Given $\varepsilon > 0$ and $d > 0$ we say that a function f is in $\mathcal{L}_{\varepsilon,d}$ if:

(1) $f : [\delta_1, 1-\delta_2] \to [0,1]$, where $\delta_i \in (0,\varepsilon)$, $i = 1, 2$;

(2) the map f is piecewise affine;

(3) there exists a point q (that we name the *critical point of* f) of continuity of f such that the derivatives at the right and at the left of f at x denoted by $f'(x^+)$ and $f'(x^-)$, respectively, are defined for every $x \in ([\delta_1, 1-\delta_2] \setminus \{q\})$ and satisfy

$$2d > |f'(x^\pm)| > d, \text{ for every } x \in (\delta_1, 1-\delta_2) \setminus \{q\},$$

(4) the map f has a finite number of *discontinuities*, say $\{p_i; i = 1, \ldots, n\}$, such that either $f(p_i^+) = 0$, $f(p_i^-) = 1$ if $p_i < q$ and $f(p_i^+) = 1$, $f(p_i^-) = 0$ if $p_i > q$ or $f(p_i^+) = 1$, $f(p_i^-) = 0$ if $p_i < q$, and $f(p_i^+) = 0$, $f(p_i^-) = 1$ if $p_i > q$ where $f(p^+) = \lim_{x\searrow p} f(x)$ and $f(p^-) = \lim_{x\nearrow p} f(x)$.

We now recall the definition of *thickness* of a Cantor set.

Definition 3.2. (Gaps, bridges and thickness of a Cantor Set) Let Λ be a Cantor set contained in the interval $I = [a, b]$ such that $a, b \in \Lambda$. A *gap of* Λ is a connected component of the complement of Λ in $[a, b]$.

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Let U be a gap of Λ. A closed interval B contained in I is a *bridge for the gap* U if $B \cap \overline{U}$ consists of exactly one point, B does not intersect any gap of Λ of length greater than the length of U and B is the maximal closed subinterval of I with these properties. Observe that there are exactly two bridges for each gap of Λ. The *thickness of* Λ is defined as

$$\tau(\Lambda) = \inf\left\{\frac{l(B)}{l(U)}; \quad \text{where } U \text{ is a gap for } \Lambda \text{ and } B \text{ is a bridge for } U\right\},$$

where $l(J)$ denotes the length of the interval J.

PROPOSITION 3.1. *For all $M > 0$ there exists $d_0 \in \mathbb{N}$ such that for every $d > d_0$ and $0 < \varepsilon < 1/d$, every function $f \in \mathcal{L}_{\varepsilon,d}$ has an f-invariant hyperbolic set $\Lambda = \Lambda(f)$ such that:*

- *the thickness of Λ is strictly larger than M, $\tau(\Lambda) > M$;*
- *any gap of Λ has length less than $5/d$; and*
- *the distance from Λ to the boundary of $[0, 1]$ is upper bounded by $5/d$, i.e.*

$$\max\{\text{dist}(0, \Lambda), \text{dist}(1, \Lambda)\} < 5/d.$$

Proof. Consider any function f in $\mathcal{L}_{\varepsilon,d}$ defined in the interval $[\delta_1, 1 - \delta_2]$, where $\delta_1, \delta_2 < \varepsilon < 1/d$. For simplicity let us assume that f is increasing in (δ_1, q) and decreasing in $(q, 1 - \delta_2)$. Let $\{p_i; i = 1, \ldots, n\}$ be the points of discontinuity of f, where $p_i < p_{i+1}$. Write $I_i = [p_i, p_{i+1}], i = 1, \ldots, n - 1$, $I_0 = [\delta_1, p_1]$ and $I_n = [p_n, 1 - \delta_2]$. Suppose that the critical point q of f belongs to the interval I_j.

We shall construct the hyperbolic set Λ with large thickness in the interval $[p_1, p_{n-1}]$. To obtain this set we shall construct a Markov partition of it. Let us suppose $j \neq 1, n - 2$, otherwise we work with the interval $[p_2, p_{n-2}]$ and the calculations below work in an easier way. So let us consider here the more difficult case $j \neq 1, n - 2$.

By hypothesis, the absolute value of the derivative of f in any interval I_i belongs to $[d, 2d]$. Thus, observing that $f(I_i) = [0, 1]$ for every $i = 1, \ldots, n - 1$, $i \neq j$, and that $f(I_j) = [0, f(q)]$, one has that the lengths of the intervals $I_i, i = 1, \ldots, n - 1$, satisfy the following:

$$l(I_i) \in [1/2d, 1/d] \quad \text{if } i \neq j \quad \text{and} \quad l(I_j) \in [f(q)/d, (2f(q))/d]. \tag{3}$$

Moreover, the choices of ε and d and the definition of $\mathcal{L}_{\varepsilon,d}$ give

$$\max\{l([0, p_1]), l([p_{n-1}, 1])\} < \frac{2}{d}. \tag{4}$$

Using the monotonicity of f in each interval I_i $(i \neq j)$ and that $f(I_i) = [0, 1]$ for all $i = 1, \ldots, n - 1$, $i \neq j$, we get a unique fixed point t_i of f in the interior of each I_i. We now focus our attention on the interval I_1 and its fixed point $t = t_1$. Observe that for each $i \notin \{j, 0, n\}$ there are points r_i and s_i of I_i with $f(r_i) = t$ and $f(s_i) = r_{n-2}$. Since the restriction of f to any I_i, $i \notin \{j, 0, n\}$, is injective and its image is $[0, 1]$ these points are unique (see Figure 4).

To construct the Cantor set Λ we will distinguish two cases, according to the position of the critical value $f(q)$.

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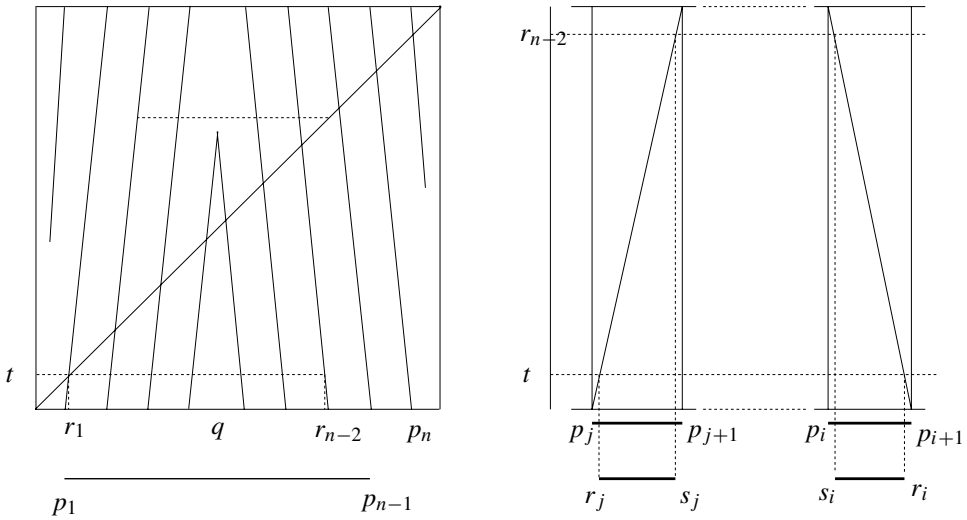


FIGURE 4. The maps $\mathcal{L}_{\varepsilon,d}$ and their Markov partition.

First case: $f(q) < \sqrt{d}/d$. Let $\mathcal{J} = \{1, 2, \ldots, j-1, j+1, \ldots, n-2\}$. For each $i \in \mathcal{J}$, let $K_i = [r_i, s_i]$ if $r_i < s_i$ $(i < j)$ or $K_i = [s_i, r_i]$ if otherwise $(i > j)$. The set Λ of the proposition will be the maximal invariant set of f in the union of the intervals $\{K_i\}_{i\in\mathcal{J}}$. The hyperbolicity of this set is obvious.

LEMMA 3.2. *The family of intervals* $\{K_i\}_{i\in\mathcal{J}}$ *is a Markov partition of the set* Λ*, that is:*

(a) *the map* f *is defined in a neighborhood of each* K_i*;*
(b) *the set* Λ *is contained in the union of the* K_i*;*
(c) *the image of each* K_i *by* f *is the convex hull of a finite number of intervals of the Markov partition;*
(d) *for each* i *and* m *sufficiently big* $f^m(\Lambda \cap K_i) = \Lambda$ *(topologically mixing).*

Proof. Conditions (a) and (b) are both obvious. To see that (c) holds just observe that the interval $[r_1, r_{n-2}]$ is the convex hull of the sets K_i and that, by definition, $f(K_i)$ is the interval $[t, r_{n-2}] = [r_1, r_{n-2}]$. This assertion and the expansiveness of f imply that in (d) we can take $m = 1$ (independently of the interval K_i of the Markov partition). The proof of the lemma is now complete. □

Consider the family of intervals $\mathcal{G}$ defined by

$$\mathcal{G} = \{(s_i, r_{i+1})\}_{i=1}^{j-2} \cup \{(r_i, s_{i+1})\}_{i=j+2}^{n-3} \cup (s_{j-1}, r_{j+1}).$$

By definition of Λ, this family is a *generating family of gaps* for Λ (i.e. any gap of Λ is of the form $f^{-k}(J)$ for some interval J of $\mathcal{G}$ and $k \geq 0$). We claim that the family

$$\mathcal{B} = \{[r_i, s_i]\}_{i=1}^{j-1} \cup \{[s_i, r_i]\}_{i=j+1}^{n-1}$$

are the sets of bridges for $\mathcal{G}$. To see this we need to estimate the lengths of the gaps in $\mathcal{G}$.

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Let us observe that since the map f is affine in $\{K_i\}_{i\in\mathcal{J}}$ the thickness of Λ is given by

$$\inf\left\{\frac{l(B)}{l(U)} \quad \text{where } U \text{ is a gap in } \mathcal{G} \text{ and } B \text{ is a bridge of } U \text{ (i.e. } B\in\mathcal{B})\right\}. \tag{5}$$

We first obtain an estimate for the lengths of the bridges of $\mathcal{B}$. For notational simplicity, from now on let $[a,b]$ be the closed interval whose extremes are a and b (i.e. if $b<a$ one has $[a,b]=[b,a]$). Observe that for each $i\neq j$ one has $f[r_i,s_i]=[r_1,r_{n-2}]\supset[p_2,p_{n-2}]$ and that by (3) and (4) the length of $[p_2,p_{n-2}]$ is larger than or equal to $(1-6/d)$. Since the modulus of the derivative of f in each I_i $(i\neq j)$ is upper bounded by $2d$ it follows that

$$l([r_i,s_i])\geq\frac{1-(6/d)}{2d} \quad \text{for every } i\notin\{0,j,n\}. \tag{6}$$

We now estimate the lengths of the gaps in $\mathcal{G}$. The length of the main gap (s_{j-1},r_{j+1}) is

$$l([s_{j-1},r_{j+1}])=l(I_j)+l([s_{j-1},p_j])+l([p_{j+1},r_{j+1}]).$$

Observing that $f(p_j^-)=1$ and $f(s_{j-1})=r_{n-2}>p_{n-2}>1-3/d$, arguing as above one has that $l([s_{j-1},p_j])<3/d^2$. Similarly, since $f(p_{j+1}^+)=0$ and $f(r_{j+1})=t<p_2<3/d$, it follows $l([p_{j+1},r_{j+1}])<3/d^2$. These two estimates, the upper bound for the size of I_j in (3), and $f(q)<\sqrt{d}/d$ give

$$l([s_{j-1},r_{j+1}])<\frac{6}{d^2}+2\frac{f(q)}{d^2}<\frac{6}{d^2}+2\frac{\sqrt{d}}{d^2}.$$

Arguing analogously one has that

$$\max\{\{l([s_i,p_{i+1}]),l([p_i,r_i])\}_{i<j},\{l([p_i,s_{i+1}]),l([r_i,p_{i+1}])\}_{i>j}\}<\frac{3}{d^2}.$$

Thus the length of any gap in $\mathcal{G}$ different from (s_{j-1},r_{j+1}) (i.e. the intervals (s_i,r_{i+1}) if $i<j$ and (r_i,s_{i+1}) if $i>j$) is bounded from above by $6/d^2<(6+2\sqrt{d})/d^2$. That is,

$$l(U)<\frac{6+2\sqrt{d}}{d^2} \quad \text{for all } U\in\mathcal{G}. \tag{7}$$

These estimates and the fact that f is piecewise affine imply that (if d is big enough) $\mathcal{B}$ is the family of bridges for the family $\mathcal{G}$ of gaps. Finally, from (5), (6) and (7) we have

$$\tau(\Lambda)>\frac{d(1-(6/d))}{2(6+2\sqrt{d})}>\sqrt{d}\frac{1-(6/d)}{4+(12/\sqrt{d})}\xrightarrow[d\to+\infty]{}+\infty,$$

ending the proof of the first assertion of the proposition in the first case.

Second case: $f(q)\geq\sqrt{d}/d$. Observe that if d is sufficiently large then $\sqrt{d}/d>6/d>p_3$. Thus, there is $k\geq1$ such that $f(q)\in I_{k+2}$.

Consider the family of closed intervals

$$\mathcal{B}=\{[r_i,s_i];\,i\neq j\}\cup\{(f|_{I_j})^{-1}([r_1,r_k])\} \quad \text{if } k>j$$

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or

$$\mathcal{B} = \{[r_i, s_i]; i \neq j\} \cup \{(f|_{I_j})^{-1}([r_1, r_{k-1}])\} \quad \text{if } k \leq j.$$

The set Λ will be the maximal (forward) f-invariant set in the union of the intervals of $\mathcal{B}$. We will do the calculations in the first case, the proof in the second one follows in the same way. Observe that the hyperbolicity of Λ is obvious.

As in the first case, the intervals of $\mathcal{B}$ define a Markov partition of Λ and the complement of $\mathcal{B}$ in $[r_1, r_{n-2}]$ is a family $\mathcal{G}$ of generating gaps of Λ. We will omit the proof of these two assertions. We will see that $\mathcal{B}$ is the family of bridges for $\mathcal{G}$ and estimate the thickness of Λ.

We now obtain estimates for the lengths of the gaps in $\mathcal{G}$ and bridges in $\mathcal{B}$. As in the first case, since f is piecewise affine, to get a lower bound for the thickness of Λ it is enough to consider these intervals, recall (5).

Observe that $(f|_{I_j})^{-1}([r_1, r_k])$ is the union of two closed intervals $[a, b]$ and $[c, d]$, where $p_j < a < b < q < c < d < p_{j+1}$. Thus the generating gaps of Λ are the intervals

$$\mathcal{G} = \{(s_i, r_{i+1})\}_{i=1}^{j-2}, \{(r_i, s_{i+1})\}_{i=j+1}^{n-2}, (s_{j-1}, a), (b, c), (d, s_{j+1}).$$

The estimates for the lengths of the gaps $\{(s_i, r_{i+1})\}_{i=1}^{j-2}$ and $\{(r_i, s_{i+1})\}_{i=j+1}^{n-2}$ and the bridges $\{[r_i, s_i]\}_{i=1}^{j-1}$ and $\{[s_i, r_i]\}_{i=j+1}^{n-1}$ are exactly the same as in the first case (i.e. the gaps have length bounded from above by $6/d^2$ and the bridges have length larger than $(1 - 6/d)/2d$). So we are left to estimate the lengths of the gaps (s_{j-1}, a), (b, c) and (d, s_{j+1}) and of the bridges $[a, b]$ and $[c, d]$.

CLAIM 3.3. $\max\{l(s_{j-1}, a), l(b, c), l(d, s_{j+1})\} \leq 12/d^2$.

Proof. By construction $f(a) = f(d) = r_1$ and $f(b) = f(c) = r_k$. Observe that the distance between $f(q)$ and r_k is upper bounded by $6/d$ (the segment $[r_k, f(q)]$ is contained in $I_k \cup I_{k+1} \cup I_{k+2}$ and, by (3), the lengths of these intervals are upper bounded by $2/d$). Since $f([b, q]) = f([q, c]) = [r_k, f(q)]$ and the modulus of the derivative of f in $[b, q]$ and $[q, c]$ is in between d and $2d$, it follows

$$l([b, c]) = l[b, q]) + l([q, c]) \leq 2\frac{6/d}{d} = \frac{12}{d^2}.$$

We now estimate the length of (s_{j-1}, a) (the estimate for $l(d, s_{j+1})$ is the same, so we will omit it). Write $l([s_{j-1}, a]) = l([s_{j-1}, p_j]) + l([p_j, a])$. Observe that $f(s_{j-1}) = r_{n-2} \geq p_{n-3} \geq 1 - 3/d$ and $f(p_j^+) = 1$. Thus, $l(f([s_{j-1}, p_j])) \leq 3/d$ and then, as above, $l([s_{j-1}, p_j]) \leq 3/d^2$. On the other hand, since $f(p_j^+) = 0$ and $f(a) = r_1 < p_2 \leq 3/d$, it follows that $l([p_j, a]) < 3/d^2$. This completes the proof of the claim. □

CLAIM 3.4. $\min\{l([a, b]), l([c, d])\} \geq (\sqrt{d} - 7)/(2d^2)$.

Proof. Let us estimate, for instance, the length of $[a, b]$. Using that $r_k \in I_k$, $f(q) \in I_{k+2}$, $j < k$ and that $l(I_i) \leq 1/d$ for all $i > j$ (see (3)) it follows

$$r_k \geq f(q) - 3/d \geq (\sqrt{d}/d) - (3/d).$$

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On the other hand, since $r_1 \leq 3/d$, we have

$$f([a, b]) = [r_1, r_k] \supset [3/d, \sqrt{d}/d - 3/d].$$

Now, since f is monotonous in $[a, b]$ and its derivative has modulus less than $2\,d$, it follows

$$l([a, b]) > \frac{\sqrt{d} - 7}{2d^2},$$

ending the proof of the claim. □

By the claims above and the previous estimates, if d is large enough, then the length of any bridge of $\mathcal{B}$ is larger than $((\sqrt{d} - 7)/2\,d^2)$ and the length of any gap of $\mathcal{G}$ is at most $(12/d^2)$. Thus, the quotient between the lengths of any gap of $\mathcal{G}$ and its bridges is greater than

$$\frac{(\sqrt{d} - 7)/(2d^2)}{12/d^2}.$$

Thus, as in the first case recall (5), we have that

$$\tau(\Lambda) > \frac{\sqrt{d} - 7}{24} \xrightarrow[d \to +\infty]{} +\infty.$$

This ends the calculation of the thickness of Λ in the second case and of the first part of the proposition.

To finish the proof of the proposition we need to estimate the size of the gaps of Λ and the distance from Λ to the boundary of $[0, 1]$. Observe that it is enough to see the assertion for the gaps in $\mathcal{G}$. However, in the two cases, the size of these gaps of Λ are bounded by $12(1 + 2\sqrt{d})/d^2$, which is less than $5/d$ if d is large enough.

To calculate the distance from Λ to $\{0, 1\}$ just observe that the points r_1 and r_{n-2} belong to Λ. Now since $r_1 < p_2 \leq 3/d$ and $1 - 3/d \leq p_{n-3} < r_{n-2}$ we get that the distance between Λ and $\{0, 1\}$ is at most $3/d$. The proof of the proposition is now complete. □

3.2. *C^2-perturbations of maps of $\mathcal{L}_{\varepsilon,d}$.* We now introduce the set of maps $\mathcal{L}^{\xi}_{\varepsilon,d}$; these maps set can be thought of as *regularizations* of the maps of $\mathcal{L}_{\varepsilon,d}$. The set $\mathcal{L}^{\xi}_{\varepsilon,d}$ of *regular ξ-perturbations* of maps of $\mathcal{L}_{\varepsilon,d}$ is defined as follows.

Definition 3.3. Consider a map $f\colon I = [\delta_1, 1 - \delta_2] \to [0, 1]$ in $\mathcal{L}_{\varepsilon,d}$ with critical point q and set of discontinuities $\mathcal{D}_f$.

We say that the map $F\colon I' = [\delta_1', 1 - \delta_2'] \to [0, 1]$ is a *regular ξ-perturbation of f* if there are a point q', a finite subset $\mathcal{D}'$ and a C^2-diffeomorphism $h\colon I' \to I$ which is ξ-close to the identity such that:

(1) the map F is C^2 in $(I' \setminus \mathcal{D}')$, $F'(x^{\pm})$ is defined for all $x \in I'$ and q' is a quadratic critical point of F;
(2) $h(\mathcal{D}') = \mathcal{D}_f$ and $h(q') = q$;
(3) F and $f \circ h$ are ξ-C^2-close in $I' \setminus ((q' - \xi, q' + \xi) \cup \mathcal{D}_f))$;
(4) if p and p' are two consecutive discontinuities of F such that $q' \notin [p, p']$, then $F([p, p']) = [0, 1]$.

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We have the following result whose proof follows immediately from the proof of Proposition 3.1.

PROPOSITION 3.5. *Given $M > 0$ consider d_0 and $\varepsilon_0 > 0$ such that every map $f \in \mathcal{L}_{\varepsilon,d}$, with $d \geq d_0$ and $\varepsilon \in (0, \varepsilon_0)$, has a hyperbolic Cantor set Λ in $[0, 1]$ with thickness larger than $2M$, gaps of length upper bounded by $5/d$ and distance to $\{0, 1\}$ less than $5/d$.*

Then there is $\xi > 0$ such that every map $F \in \mathcal{L}^{\xi}_{\varepsilon,d}$ has a hyperbolic Cantor set Σ with thickness larger than M, gaps of length upper bounded by $6/d$, and distance to $\{0, 1\}$ less than $6/d$.

Returning to the saddle-node horseshoe and its associated ghost dynamics, observe that after a change of coordinates we can assume that $D^r = [0, 1]$ and $J = [\delta_1, 1 - \delta_2]$, where δ_1 and δ_2 are small (this corresponds to taking the intervals I and I' in §2.3 small). We now perturb the initial f in D_2 (thus without perturbing the dynamics of the initial saddle-node horseshoe) in such a way that:

Remark 3.6. The family of ghost return maps in §2.3, $F_\sigma : J \to D^r$, is such that F_σ is a regular ξ-perturbation of some $f_\sigma \in \mathcal{L}_{\varepsilon,d}$ for each $\sigma \in [0, 1]$, where ε is small and d is large (for that, it is enough to take the expansiveness constant k in condition (E) large enough). In view of Proposition 3.5, for a fixed $M > 1/\ell$, ℓ as in (1), we can assume (taking large d and small ξ) that, for every σ, the map F_σ has a hyperbolic set Σ_σ of thickness larger than M, gaps of length less than $6/d$ and distance to $\{0, 1\}$ less than $6/d$.

4. *Linked Hyperbolic sets*

In this section, to get persistence of tangencies we will use standard arguments in homoclinic bifurcation theory (see for instance [**PT3**, Chapter 6]) in order to adequately *link* the sets Ω_μ and Λ' (see §2.5). For that we consider an extension of the unstable foliation of Λ' to a neighborhood of it. In fact, to simplify the exposition and without loss of generality, after a change of coordinates we can suppose that, for every $x \in \Lambda'$, the connected component of $W^u(x) \cap R$, $R = R_1 \cup R_2 \cup R_3$, containing x is horizontal and the strong stable leaves are vertical. Consider the horizontal foliation $\mathcal{H}$ in R and take $\mathcal{F}^u = g_\mu(\mathcal{H})$ as the mentioned extension. Under the assumptions above, we can suppose that $\mathcal{F}^u \cap R$ is formed by horizontal segments and $\mathcal{F}^u \cap D_2$ by parabolic-like curves.

Let γ be the horizontal segment joining $f^2(c)$, c is the critical point in condition (Q) in §2.2, to the boundary of R and consider the Cantor set $K(\Lambda')$ obtained projecting the set Λ' into γ along the leaves of $\mathcal{F}^u$,

$$K(\Lambda') = \{y = F^u(x) \cap \gamma, x \in \Lambda'\},$$

where $F^u(x)$ is the leaf of $\mathcal{F}^u$ containing x. These sets are depicted in Figure 5. Let us observe that (see [**PT1**] or [**N2**]) the thickness of $K(\Lambda')$ is equal to the unstable thickness of Λ'.

We now perturb f in D_2 in such a way that conditions (Q), (D) and (E) in §2.2 and Remark 3.6 remain valid and f also satisfies the following.

(i) Consider the Cantor set $K^s(\Lambda') = \pi^{ss}(K(\Lambda')) \subset W^c_0 \cap D_1$. The perturbation of the initial f is such that $K^s(\Lambda')$ is contained in D^ℓ. As above (see [**PT1**] or [**N2**]) the thickness of this set is ℓ, the unstable thickness of Λ', see equation (1).

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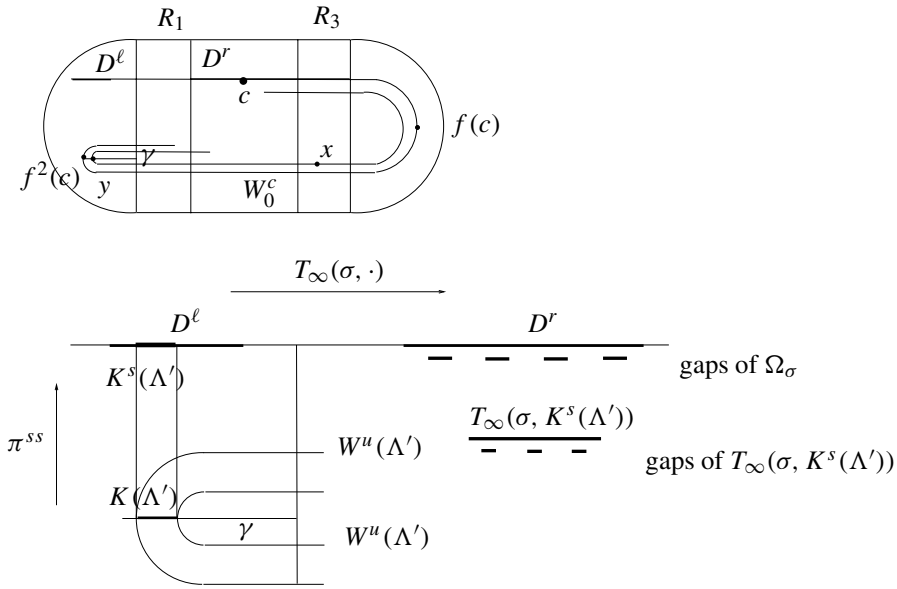


FIGURE 5. Linked Cantor sets.

(ii) The diameter of $T_\infty(\sigma, (K^s(\Lambda')))$ is greater than the length of the gaps of Ω_σ and the lengths of the gaps of $T_\infty(\sigma, (K^s(\Lambda')))$ are less than the diameter of Ω_σ. Thus the Cantor sets Ω_σ and $T_\infty(\sigma, K^s(\Lambda'))$ are linked (recall §2.5). By the definition of T_∞, $T_\infty(\sigma, x) = X_{t(x)-\sigma}(0, b)$ and since the sequence T_k converges to T_∞ in the C^∞-topology (see the comments in §2.2), it is immediate that the Cantor sets $T_k(\sigma, K^s(\Lambda'))$ and Ω_σ are linked for every k large enough.

(iii) Again, by the definition of T_∞, we can suppose that $\tau(T_\infty(\sigma, K^s(\Lambda')))$ is also greater than ℓ. Since the sequence of maps T_k converge to T_∞ in the C^∞-topology and the thickness depends continuously on the C^2-topology (see for instance [**PT3**, Theorem 3 and Remark 1, Chapter 4.3]) we have that $\tau(T_k(\sigma, K^s(\Lambda'))) > \ell$ for all k sufficiently large.

Recalling Remark 3.6, by (iii) one has,

$$\tau(\Omega_\sigma)\tau(T_k(\sigma, K^s(\Lambda'))) \geq M\ell > \frac{1}{\ell}\ell = 1$$

for every $\sigma \in [0, 1]$ and k sufficiently large. By (ii) these Cantor sets are linked, the Gap Lemma (Lemma 2.3) now implies that

$$T_k(\sigma, \pi^{ss}(K(\Lambda'))) \cap \Omega_\sigma \neq \emptyset$$

for all $\sigma \in [0, 1]$. Observe that

$$T_k(\sigma, \pi^{ss}(K^s(\Lambda'))) = g_\mu^k(K^s(\Lambda')), \quad \mu = \hat{\mu}_k(\sigma),$$

where $\hat{\mu}_k$ are the reparameterizations in §2.2.

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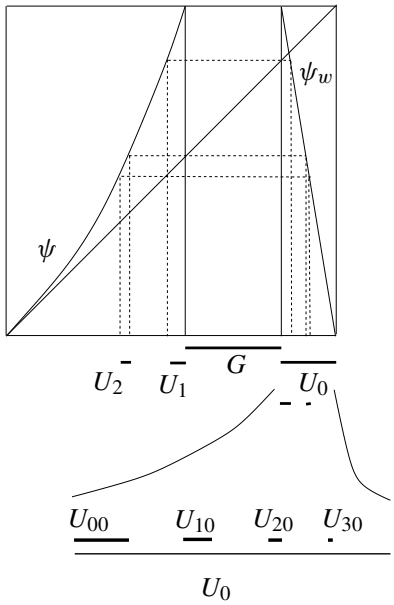


FIGURE 6. Linked Cantor sets.

Now it is not difficult to see that this implies that there are tangencies between unstable manifolds of Λ' and stable manifolds of Ω_μ, $\mu = \hat{\mu}_k(\sigma)$ for all large k and $\sigma \in [0, 1]$ (for details see [**PT3**, Chapter 6]).

Finally, we will prove the following lemma.

LEMMA 4.1. *The sets Ω_μ and Λ' are homoclinically related for every $\mu > 0$ small enough.*

This lemma implies that any hyperbolic set containing Λ' and Ω_μ has a tangency between its invariant manifolds.

Proof. To see that $W^s(\Omega_\mu)$ transversely intersects $W^u(\Lambda')$ just observe that the local stable manifold of any point of Ω_μ is almost vertical and the local unstable manifolds of the points of Λ' are almost horizontal. Now it is not difficult to see that $W^s(\Omega_\mu)$ transversely cuts the unstable manifolds of the points of Λ'.

To check that $W^s(\Lambda')$ transversely meets $W^u(\Omega_\mu)$ recall that any map $F \in \mathcal{L}^{\xi}_{\varepsilon,d}$ sends any interval I_i, with end-points corresponding to two consecutive discontinuities and not containing the critical point, onto $[0, 1]$. This fact easily implies that the unstable manifold of the continuation of the fixed point of F contained in I_i transversely intersects the stable manifolds of the points of Λ'. □

5. *Hausdorff dimension of* Λ_0

First let us observe that the arguments and results in [**DV**] imply that $HD(\Lambda_\mu) = HD^s(\Lambda_\mu) + HD^u(\Lambda_\mu)$ and that $HD^s(\Lambda_\mu)$ and $HD^s(\Lambda_\mu)$ depend continuously on μ for $\mu \leq 0$.

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It is easy to see that we can take $HD^u(\Lambda_0)$ to be as small as we want, for that it is enough to have a contraction in the stable direction sufficiently strong. Then, the only thing we should prove is that, for any $\beta > 1/2$, we can construct Λ_0 such that $HD^s(\Lambda_0) \leq \beta$. We shall only give the main steps of our construction omitting most of the proofs, which are contained in [**DV**]. In fact, in our case the proofs are simpler.

First, following [**PT3**] for instance, we can think of the set $\Lambda_0^u = \Lambda_0 \cap W^u_{\text{loc}}(P_0)$ as a dynamically defined Cantor set whose Markov partition consists of two intervals, in one interval the expanding map defining the Cantor set is uniformly hyperbolic (in fact linear) and in the second one this map is expanding, except at a saddle-node point, see Figure 6.

More precisely, take a one parameter family of C^∞-functions $(\varphi_w)_{w\in(0,1)}$, $\varphi_w : [0, v] \cup [1 - w, 1] \to [0, 1]$, such that:

- $\psi = \varphi_w|_{[0,v]}$ and $\psi([0, v]) = [0, 1]$, $\psi(0) = 0$, $\psi(v) = 1$;
- $\psi_w = \varphi_w|_{[1-w,1]}$ and ψ_w is an affine map such that $\varphi_w([1 - w, 1]) = [0, 1]$, $\psi_w(1 - w) = 1$ and $\psi_w(1) = 0$;
- $|\psi'(x)| \geq 1$, for every $x \in [0, v]$, $\psi'(x) = 1$ if and only if $x = 0$, and $\psi''(0) > 0$.

For each $w \in (0, 1)$ let K_w be the dynamically defined Cantor set associated to the map φ_w.

In our construction to get small Hausdorff dimension for Λ_0^u we increase the size of the rectangle R_2 (corresponding to the central gap of the horseshoe). In the one-dimensional model this corresponds to an increase in the size of the central gap of the Cantor set K_w, i.e. to take w close to 0. We will prove the following lemma.

LEMMA 5.1. *Under the notation above,* $\text{HD}(K_w) \to 1/2$ *as* $w \to 0^+$.

In view of the lemma, in our case it is enough to take $\Lambda_0^u = K_w$ for some w small enough (i.e. the gap of the initial horseshoe is proportionally large).

Before proving the lemma let us recall the definition of Hausdorff dimension. Let K be a set contained in $\mathbb{R}$ and $\mathcal{U} = \{U_i\}_{i\in I}$ a finite covering of it by open intervals of $\mathbb{R}$. The *diameter of the covering* $\mathcal{U}$ is the maximum of $l(U_i)$, $i \in I$. Denote by $\mathcal{U}_\varepsilon(K)$ the set of coverings of K of diameter less than ε. The *Hausdorff* α*-measure of* K is

$$m_\alpha(K) = \lim_{\varepsilon\to 0} \left(\inf_{\mathcal{U}\in\mathcal{U}_\varepsilon(K)} H_\alpha(\mathcal{U}) \right), \quad \text{where} \quad H_\alpha(\mathcal{U}) = \sum_{i\in I} l(U_i)^\alpha.$$

Then there is a unique number $HD(K)$, the *Hausdorff dimension of* K, such that $m_\alpha(K) = \infty$ if $\alpha < HD(K)$ and $m_\alpha(K) = 0$ if $\alpha > HD(K)$.

Proof. Consider the set A_w defined as the union of $\{0\}$ and the pre-orbit of one by φ_w. Since this set is countable, $HD(K_w) = HD(K_w \setminus A_w)$. To prove the lemma given $\beta > 1/2$ we will get w sufficiently small such that the set $(K_w \setminus A_w)$ has a covering $\mathcal{U}$ of arbitrarily small diameter such that $H_\beta(\mathcal{U}) < \infty$. Thus, $m_\beta(K_w) < \infty$ and then $HD(K_w) < \beta$, proving the lemma.

To construct the announced covering let $G(w) = G_0(w) = (v, 1 - w)$, $U(w) = U_0(w) = (1-w, 1)$, $G_i(w) = \psi^{-i}(G_0(w))$ and $U_i(w) = \psi^{-i}(U_0(w))$. Given a (k)-tuple

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of non-negative integers $(i_0, i_1, \dots, i_{k-1})$ inductively define

$$U_{i_0,\dots,i_k}(w) = \psi_w^{-1}(U_{i_0,\dots,i_{k-1}}), \quad \text{if } i_k = 0 \text{ and}$$
$$U_{i_0,\dots,i_k}(w) = \psi^{-i_k}(U_{i_0,\dots,i_{k-1}}), \quad \text{if } i_k > 0.$$

Analogously, we define the gaps $G_{i_0,\dots,i_k}(w)$. For notational simplicity, from now on we will omit the dependence on w of these sets. For each k consider the covering of $(K_w \setminus A_w)$ defined by

$$\mathcal{U}_k = \{U; U = U_{i_0,\dots,i_k}, i_0, i_1, \dots, i_k \geq 0\}.$$

Let $a_j = \psi^{-j}(1)$ and $z_j = a_{j-1} - a_j$, $j \geq 1$, (these numbers are independent of w) and

$$n(i_0, \dots, i_k) = \#\{j; i_j = 0\} + \sum_{j=1}^{k} i_j.$$

The number $n(i_0, \dots, i_k)$ is the number of iterations of φ_w that we need to get U from $U_{i_0,i_1,\dots,i_k}$. In [**DV**] the following two lemmas are proved.

LEMMA 5.2. [**DV**, Lemma 3.2] *The Hausdorff measure $m_\alpha(\mathcal{U}_0)$ satisfies*

$$m_\alpha(\mathcal{U}_0) = \sum_0^\infty l(U_i)^\alpha < \sum_0^\infty z_j^\alpha = M(\alpha) < \infty$$

for every $\alpha > 1/2$.

LEMMA 5.3. (Bounded distortion) [**DV**, Lemma 3.3] *There exists $c > 0$, independent of $(i_0, \dots, i_k)$ and w, such that*

$$\sum_{t=1}^{n(i_0,\dots,i_k)} l(\varphi_w^t(U_{i_0,\dots,i_k} \cup G_{i_0,\dots,i_k})) < c.$$

We shall show that given any $\beta > 1/2$ there exists w_0 such that

$$m_\beta(\mathcal{U}_k) \leq m_\beta(\mathcal{U}_{k-1}), \quad \text{for every } w \text{ with } 0 < w < w_0. \tag{8}$$

Since, by construction, $\operatorname{diam}(\mathcal{U}_k) \to 0$ as $k \to \infty$ this implies $HD(K_w) \leq \beta$.

Using the bounded distortion of the derivative of φ_w, as corollaries of Lemma 5.3 one immediately gets the following.

COROLLARY 5.4. [**DV**, Corollary 3.4] *There exists $c > 0$, independent of $(i_0, \dots, i_k)$ and w, such that*

$$\frac{l(U_{i_0,\dots,i_k})}{l(G_{i_0,\dots,i_k})} \leq c \frac{l(U)}{l(G)} = a.$$

COROLLARY 5.5. [**DV**, Corollary 3.5] *There exists $b > 0$, independent of $(i_0, \dots, i_k)$ and w, such that*

$$\frac{l(U_{i_0,\dots,i_k} \cup G_{i_0,\dots,i_k})}{l(U_{i_1,\dots,i_k})} \leq b z_{i_0}.$$

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To prove (8) it is sufficient to take w small enough in order to have

$$\Pi = \frac{\sum_{i_0=0}^{\infty} l(U_{i_0,\dots,i_k})^\beta}{l(U_{i_1,\dots,i_k})^\beta} \le 1.$$

By Corollaries 5.4 and 5.5

$$\Pi \le \frac{1}{(1+a^{-1})^\beta} \frac{\sum_{i_0=0}^{\infty} (l(U_{i_0,\dots,i_k}) + l(G_{i_0,\dots,i_k}))^\beta}{l(U_{i_1,\dots,i_k})^\beta} \le \frac{\sum_{i_0=0}^{\infty} b^\beta z_{i_0}^\beta}{(1+a^{-1})^\beta} \le \frac{b^\beta M(\beta)}{(1+a^{-1})^\beta},$$

where the last inequality follows from Lemma 5.2. Now observe that if a is small enough (that is, w is close to 1^-) then

$$\frac{b^\beta M(\beta)}{(1+a^{-1})^\beta} < 1,$$

ending the proof of the lemma. □

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References

[CK] Y. Cao and S. Kiriki. An isolated saddle-node bifurcation occurring inside a horseshoe. *Dynam. & Stability Sys.* **15** (2000), 11–22.

[C] M. J. Costa. Saddle-node horseshoes giving rise to global Hénon-like attractors. *Thesis IMPA & An. Acad. Bras. Ci* **70** (1998), 393–400.

[DRiV] L. J. Díaz, I. Rios and M. Viana. The intermittency route to chaotic dynamics. *Global Anal. Dynam. Sys. (Festschrift dedicated to F. Takens for his 60th birthday)*. Eds. H. Broer, B. Krauskopf and G. Vegter. IOP, pp. 309–327.

[DRoV] L. J. Díaz, J. Rocha and M. Viana. Strange attractors in saddle-node cycles: prevalence and globality. *Invent. Math.* **125** (1996), 37–74.

[DV] L. J. Díaz and M. Viana. Discontinuity of Hausdorff dimension. *Ergod. Th. & Dynam. Sys.* **9** (1989), 403–425.

[HPS] M. Hirsch, C. Pugh and M. Shub. *Invariant Manifolds (Lecture Notes in Mathematics, 583)*. Springer, Berlin, 1997.

[M] C. G. T. A. Moreira. Stable intersections of Cantor sets and homoclinic bifurcations. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **13**(6) (1996), 741–781.

[N1] S. Newhouse. Diffeomorphisms with infinitely many sinks. *Topology* **13** (1974), 9–18.

[N2] S. Newhouse. The abundance of wild hyperbolic sets and nonsmooth stable set for diffeomorphisms. *Publ. Math. I.H.E.S.* **50** (1979), 101–151.

[NPT] S. Newhouse, J. Palis and F. Takens. Bifurcations and stability of families of diffeomorphisms. *Publ. Math. I.H.E.S.* **57** (1983), 7–71.

[PT1] J. Palis and F. Takens. Cycles and measure of bifurcation sets for two-dimensional diffeomorphisms. *Invent. Math.* **82** (1985), 397–422.

[PT2] J. Palis and F. Takens. Hyperbolicity and the creation of homoclinic orbits. *Ann. Math.* **125** (1987), 337–374.

[PT3] J. Palis and F. Takens. *Hyperbolicity and Sensitive Chaotic Dynamics at Homoclinic Bifurcations*. Cambridge University Press, 1993.

[PY1] J. Palis and J. C. Yoccoz. Homoclinic tangencies for hyperbolic sets of large Hausdorff dimension. *Acta Math.* **172**(1) (1994), 91–136.

[PM] Y. Pomeau and P. Manneville. Intermittent transitions to turbulence in dissipative dynamical systems. *Commun. Math. Phys.* **74** (1980), 189–197.

https://doi.org/10.1017/S0143385702000500 Published online by Cambridge University Press

[PY2] J. Palis and J. C. Yoccoz. On the arithmetic sum of regular Cantor sets. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **14**(4) (1997), 439–456.

[R] I. Rios. Unfolding homoclinic tangencies inside horseshoes: hyperbolicity, fractal dimensions and persistence of tangencies. *Nonlinearity* **14** (2001), 431–462.

[T] F. Takens. *Intermittency: Global Aspects (Lecture Notes in Mathematics, 1331)*. (1988), 213–239.

[W] R. Williams. The DA maps of Smale and structural stability. *Global Analysis, Proc. Symp. Pure Math.* **14** (1970), 329–334.

[YA] J. Yorke and K. T. Alligood. Cascades of period doubling bifurcations: a prerequisite for horseshoes. *Bull. Am. Math. Soc.* **9** (1983), 319–322.

[Z] C. Zeeman. *Bifurcations, Catastrophe and Turbulence (Proc. Cent. Conf., New Directions in Applied Mathematics, Case Western Reserve University, Cleveland)*. Eds. P. Hilton and G. S. Young. Springer, 1981, pp. 109–153.

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