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101.46 Improved simple pendulum results

Introduction

The period of a simple pendulum making small oscillations (usually quoted as less than 10 degrees) is given by the standard result

$$T = 2\pi\sqrt{\frac{l}{g}}. \tag{1}$$

For larger oscillation of amplitude δ, Quadling and Ramsey [1] give the result

$$T = 4\sqrt{\frac{l}{g}}\int_0^{\frac{1}{2}\pi}\frac{d\phi}{\sqrt{1 - \sin^2\frac{1}{2}\delta\,\sin^2\phi}}. \tag{2}$$

In [2], Murphy found that by approximating this integral to the area of a single trapezium gives the much more accurate result

$$T_1 = \pi\left(1 + \sec\frac{\delta}{2}\right)\sqrt{\frac{l}{g}}. \tag{3}$$

Here we define $s = \sin\frac{1}{2}\delta$ and $I = \dfrac{2}{\pi}\displaystyle\int_0^{\pi/2}\left(1 - s^2\sin^2 x\right)^{-1/2}dx$ and find the solution to the integral as a power series in s which will provide an estimate for the error in (3) and lead to other better approximations.

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Series solution

Using the binomial expansion we find

$$\left(1 - s^2 \sin^2\phi\right)^{-1/2} = 1 + \frac{1}{2}s^2 \sin^2\phi + \frac{3}{8}s^4 \sin^4\phi + \frac{5}{16}s^6 \sin^6\phi + \frac{35}{128}s^8 \sin^8\phi + \ \ldots \ + \frac{(2n)!}{2^{2n}(n!)^2}s^{2n} \sin^{2n}\phi + \ \ldots . \tag{4}$$

Since $\int_0^{\pi/2} \sin^2\phi \, d\phi = \frac{\pi}{4}$ and if $n > 1$ then

$$\int_0^{\pi/2} \sin^{2n}\phi \, d\phi = \frac{2n - 1}{2n} \int_0^{\pi/2} \sin^{2n-2}\phi \, d\phi$$

the coefficients in (4) appear again in the result

$$\int_0^{\pi/2} \sin^{2n}\phi \, d\phi = \frac{\pi}{2}\left\{\frac{1 \times 3 \times 5 \times \ \ldots \ \times (2n - 1)}{2 \times 4 \times 6 \times \ \ldots \ \times 2n}\right\} = \left\{\frac{(2n)!}{2^{2n}(n!)^2}\right\}\frac{\pi}{2}.$$

Hence

$$I = 1 + \sum_{n=1}^{\infty}\left(\frac{(2n)!\,s^n}{2^{2n}(n!)^2}\right) = 1 + \frac{1}{4}s^2 + \frac{9}{64}s^4 + \frac{25}{256}s^6 + \frac{1225}{16384}s^8 + \frac{3969}{65536}s^{10} + \ \ldots \tag{5}$$

which is easily calculated on a spreadsheet using $t_0 = 1$, $t_{n>0} = t_{n-1}\left(\frac{2n - 1}{2n}\right)^2 s^2$.

The equivalent series from (3) is

$$I_1 = \frac{1 + (1 - s^2)^{-1/2}}{2} = 1 + \frac{1}{4}s^2 + \frac{3}{16}s^4 + \frac{5}{32}s^6 + \frac{35}{256}s^8 + \ \ldots$$

so that for small s the error is given by

$$E(I_1) = I - I_1 \approx -\frac{3}{64}s^4 - \frac{15}{256}s^6 - \frac{1015}{16384}s^8.$$

Other approximations

Another expansion which gives the first two terms of (5) correctly is

$$I_2 = \sqrt{\sec\tfrac{1}{2}\delta} = \left(1 - s^2\right)^{-1/4} = 1 + \frac{1}{4}s^2 + \frac{5}{32}s^4 + \frac{15}{128}s^6 + \frac{195}{2048}s^8 + \ \ldots .$$

This is an improvement on (3) since

$$E(I_2) \approx -\frac{1}{64}s^4 - \frac{5}{256}s^6 - \frac{335}{16384}s^8.$$

The obvious linear combination $I_3 = \frac{1}{2}(3I_2 - I_1) = \frac{1}{4}\left(6\sqrt{\sec\frac{1}{2}\delta} - \sec\frac{1}{2}\delta - 1\right)$ eliminates both 4 th order and 6 th order terms so that the error is $E(I_3) \approx \frac{5}{16384}s^8$.

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Other expressions may be found to generate the first few terms of I but they do tend to get increasingly complicated and much less memorable. For example:

$$I_4 = (\cos\tfrac{1}{2}\delta)^{-9/16}(\cos\delta)^{1/64} \quad \text{gives} \quad E(I_4) \approx \frac{1}{64}s^6 \text{ (provided } \delta << \tfrac{1}{2}\pi)$$

$$I_5 = (1 - \tfrac{7}{8}s^2)^{-2/7} \quad \text{gives} \quad E(I_5) \approx \frac{1}{256}s^6$$

$$I_6 = (\cos\tfrac{1}{2}\delta)^{-3/8}(1 - \tfrac{1}{2}s^2)^{-1/64} \quad \text{gives} \quad E(I_6) \approx -\frac{9}{16384}s^8$$

and if $h = \dfrac{21 + \sqrt{97}}{32}$, $k = \dfrac{21 - \sqrt{97}}{32}$, $\alpha = -\dfrac{1}{43}\left(7 + \dfrac{25}{\sqrt{97}}\right)$ and $\beta = -\dfrac{1}{43}\left(7 - \dfrac{25}{\sqrt{97}}\right)$

$$I_7 = (1 - hs^2)^{\alpha}(1 - ks^2)^{\beta} \quad \text{gives} \quad E(I_7) \approx \frac{9}{65536}s^{10}.$$

δ^0	I	$E(I_1)$	$E(I_2)$	$E(I_3)$	$E(I_4)$	$E(I_5)$	$E(I_6)$	$E(I_7)$
5	1.00047617248599	-1.70E-07	-5.67E-08	3.11E-15	1.08E-10	2.70E-11	-7.77E-15	0.00E+00
10	1.00190718814322	-2.73E-06	-9.10E-07	1.03E-12	7.00E-09	1.74E-09	-1.86E-12	2.44E-15
15	1.00430057917346	-1.39E-05	-4.63E-06	2.67E-11	8.11E-08	2.00E-08	-4.81E-11	2.06E-13
20	1.00766902579154	-4.43E-05	-1.48E-05	2.70E-10	4.68E-07	1.13E-07	-4.86E-10	3.73E-12
25	1.01203055030504	-1.09E-04	-3.64E-05	1.64E-09	1.84E-06	4.40E-07	-2.95E-09	3.55E-11
30	1.01740879759595	-2.29E-04	-7.64E-05	7.18E-09	5.73E-06	1.34E-06	-1.29E-08	2.25E-10
35	1.02383341137053	-4.31E-04	-1.44E-04	2.52E-08	1.52E-05	3.45E-06	-4.53E-08	1.08E-09
40	1.03134051913004	-7.48E-04	-2.49E-04	7.54E-08	3.59E-05	7.89E-06	-1.36E-07	4.26E-09
45	1.03997334319680	-1.22E-03	-4.07E-04	2.00E-07	7.81E-05	1.65E-05	-3.58E-07	1.44E-08
50	1.04978296062303	-1.91E-03	-6.35E-04	4.80E-07	1.60E-04	3.21E-05	-8.62E-07	4.31E-08
55	1.06082924184800	-2.86E-03	-9.53E-04	1.07E-06	3.11E-04	5.90E-05	-1.92E-06	1.17E-07
60	1.07318200714936	-4.17E-03	-1.39E-03	2.24E-06	5.88E-04	1.04E-04	-4.01E-06	2.96E-07
65	1.08692245210315	-5.92E-03	-1.97E-03	4.47E-06	1.09E-03	1.75E-04	-7.97E-06	7.00E-07
70	1.10214490963927	-8.24E-03	-2.74E-03	8.53E-06	1.99E-03	2.88E-04	-1.52E-05	1.57E-06
75	1.11895903863095	-1.13E-02	-3.75E-03	1.57E-05	3.69E-03	4.60E-04	-2.78E-05	3.35E-06
80	1.13749255992392	-1.52E-02	-5.05E-03	2.81E-05	7.11E-03	7.19E-04	-4.94E-05	6.90E-06
85	1.15789470427982	-2.03E-02	-6.73E-03	4.89E-05	1.53E-02	1.10E-03	-8.54E-05	1.38E-05
90	1.18034059901609	-2.68E-02	-8.87E-03	8.33E-05		1.67E-03	-1.44E-04	2.67E-05
95	1.20503691081477	-3.51E-02	-1.16E-02	1.40E-04		2.49E-03	-2.39E-04	5.04E-05
100	1.23222919673752	-4.56E-02	-1.51E-02	2.30E-04		3.67E-03	-3.90E-04	9.35E-05
105	1.26221161942343	-5.91E-02	-1.95E-02	3.76E-04		5.36E-03	-6.26E-04	1.71E-04
110	1.29533999887655	-7.64E-02	-2.51E-02	6.07E-04		7.78E-03	-9.94E-04	3.07E-04
115	1.33204967730809	-9.85E-02	-3.22E-02	9.75E-04		1.12E-02	-1.56E-03	5.45E-04
120	1.37288050061835	-1.27E-01	-4.13E-02	1.56E-03		1.61E-02	-2.43E-03	9.62E-04

TABLE 1

Table 1 shows the sum of the series for I, calculated correct to 14 decimal places using a Macro in an Excel spreadsheet, and the errors for the different formulae.

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Conclusions

There are two useful improved formulae for the period of the simple pendulum (those using I_2 and I_3) which give

$$T_2 = 2\pi\sqrt{\frac{l}{g\cos\frac{1}{2}\delta}} \qquad \text{and} \qquad T_3 = \frac{\pi}{2}\sqrt{\frac{l}{g}}\left(6\sqrt{\sec\tfrac{1}{2}\delta} - \sec\tfrac{1}{2}\delta - 1\right).$$

Despite T_3 giving much greater accuracy, for its simplicity and error less than 1% for acute δ, T_2 has to be the 'best buy'.

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101.47 Notes on Lagrange's interpolation

Let $a_0, \ldots, a_N$ and $b_0, \ldots, b_N$ be sequences of integers. We know that there exists a unique polynomial $f(x) \in \mathbb{Q}[x]$ of degree not larger than N such that $f(a_j) = b_j$ for $0 \leqslant j \leqslant N$. In fact, such a polynomial can be constructed using Lagrange's interpolation (to be described later), and may be called the Lagrange's polynomial of the table

a_0	a_1	…	…	a_N
b_0	b_1	…	…	b_N

TABLE 1

The coefficients of f are not necessarily integers.

A common procedure in designing a test problem in a first course of numerical analysis is to start with a polynomial $f(x)$ in $\mathbb{Z}[x]$ of degree N, and a sequence $a_0, \ldots, a_N$ of integers, and to ask students to find the Lagrange polynomial of the table

a_0	a_1	…	…	a_N
$f(a_0)$	$f(a_1)$	…	…	$f(a_N)$

TABLE 2

Of course, the answer, being nothing but the polynomial f that we started with, is sure to have integer coefficients. If we delete a column or more of the table above, then it is conceivable that the coefficients of the

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