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Lambda theories allowing terms with a finite number of fixed points

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A natural question in the λ-calculus asks what is the possible number of fixed points of a combinator (closed term). A complete answer to this question is still missing (Problem 25 of TLCA Open Problems List) and we investigate the related question about the number of fixed points of a combinator in λ-theories. We show the existence of a recursively enumerable lambda theory where the number is always one or infinite. We also show that there are λ-theories such that some terms have only two fixed points. In a first example, this is obtained by means of a non-constructive (more precisely non-r.e.) λ-theory where the range property is violated. A second, more complex example of a non-r.e. λ-theory (with a higher unsolvability degree) shows that some terms can have only two fixed points while the range property holds for every term.

1. Introduction

One of the most important features of λ-calculus is the fact that every combinator (i.e. closed term) has a fixed point. This fact is at the heart of the λ-calculus as a model of computation, in which every computable function is a *total* function. In the theory of recursive functions, as it is well known, *diagonalization* shows that some recursive functions must be *partial functions*. In λ-calculus, diagonalization shows that every combinator must have a *fixed point*.

To see this, let M be any combinator and let the term F enumerate the range of M as follows:

$$F\underline{n}\ \underline{m} = M(\mathbf{E}\underline{n}\ \underline{m}),$$

where $\mathbf{E}$ is Barendregt universal enumerator (see Barendregt (1984) Theorem 8.1.6) and we adopt the usual convention that $\underline{n}$ represents the nth Church numeral.

Then the diagonalization $\lambda x.Fxx$ must have some index n_0 and therefore

$$F\underline{n}_0\ \underline{n}_0 = M(\mathbf{E}\underline{n}_0\ \underline{n}_0) = M((\lambda x.Fxx)\ \underline{n}_0) = M(F\underline{n}_0\ \underline{n}_0).$$

Due to the importance of fixed points, the following question seems therefore very natural and basic:

Question 1. How many fixed points can a combinator have?

This question was first posed by the first author of the present paper in Intrigila and Biasone (2000). In Intrigila and Biasone (2000), the following partial result was obtained: if the combinator M has a fixed point in normal form, then M has either exactly one fixed

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point or an infinite number of fixed points. However, the general problem is still open and has been inserted in the TLCA List of Open Problems (TLCA 2014), as Problem 25.

Also in Intrigila and Biasone (2000), the question is discussed whether this problem is related to recursion theory. This is to be compared with the well-known *range property*, (see Barendregt (1984) Definition 20.2.2), that is the range of a combinator is either infinite or a singleton. According to Barendregt (see Barendregt (1993)), this property is essentially of recursion theoretic nature. A recent, noteworthy use of recursion theoretic methods in λ-calculus is in Polonsky (2012), where the range property is shown not to hold in the λ-theory $\mathcal{H}$. Such result might suggest that, for recursion theoretic reasons, a very complicated example could exist with, say, exactly two fixed points.

Therefore, it seems reasonable to analyse the following related question:

Question 2. Is there a recursively enumerable λ-theory T such that some closed term M has in T only a finite number n of fixed points, with $n \geqslant 2$?

Also this problem is a natural one, in consideration of the fact that every recursively enumerable λ-theory has the range property (see Barendregt (1984) Theorem 20.2.5). However, it does not appear to be easy, as it is required to control the behaviour of the various *fixed point combinators* (see Barendregt (1984) Definition 6.1.12). Actually one can even doubt that the existence of terms with, say, only two fixed points, is *consistent* with β-conversion.

The main contribution of the paper is to solve such consistency question in the affirmative, by showing the existence of *non-r.e.* λ-theories allowing terms with only two fixed points. In the first theory, we violate the range property for some terms. By Theorem 17.1.16 of Barendregt (1984) this implies that this theory cannot be r.e. Actually the theory turns out to be Σ_2^0 complete.

Using a second non-r.e. theory, we show that we can enforce the existence of terms with only two fixed points but with infinite range. Moreover in such theory, every term has the *range property*.

To obtain such results, we extend the syntax of λ-calculus with more *constants*.

However as both the theories are non-r.e., Question 2 (as well as Question 1) remains open.

On the other hand, in Proposition 1, we show that when we add infinitely many constants to the syntax, then the recursively enumerable theory of β-conversion has the property that every combinator has either exactly one fixed point or an infinite number of fixed points. This seems to give evidence that Question 1 has the same solution. However, as stated above, some complicated counterexample could exist.

2. Lambda theories with constants

Notation will be standard and we refer to Barendregt (1984), for terminology and results on λ-calculus. In particular:

— $\equiv$ denotes syntactical identity;
— $\underline{k}$ denotes the kth Church numeral;
— $\longrightarrow_\beta$ denotes β-reduction and $\longrightarrow_\beta^*$ its reflexive and transitive closure;

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— we will use boldface capital letters to denote the familiar combinators such as **I**, **Ω**, etc.

By $\lambda\beta$, we denote the theory of β-convertibility (see Barendregt (1984)). Following Barendregt (1984), we define:

Definition 1. A *λ-theory* T is an extension of $\lambda\beta$ such that:

— T is obtained adding to $\lambda\beta$ a set of closed equations;
— T is closed w.r.t. $\lambda\beta$ derivation rules;
— T is consistent (Barendregt (1984) page 33).

Equality in a theory T will be denoted by $=_T$.

We shall also consider λ-theories with *constants*, obtained by adding to the signature of $\lambda\beta$ a set A of new constants. We use a, a_0, a_1, b etc. as metavariables for the new constants. In the following, we assume that the notion of *λ-theory* has been extended in the obvious way in the presence of constants. In particular, we shall call $\lambda\beta A$ the theory of β-convertibility in the extended signature.

Let the set A of constants be infinite, then we have the following result:

Proposition 1. In $\lambda\beta A$, every combinator has either exactly one fixed point or an infinite number of fixed points.

Proof. Let M be any closed term. We define
$F \equiv (\lambda xyz.z(xxyz))$;
$H_a \equiv FFa$.
It is immediate that H_aM is a fixed point of M. Indeed $H_aM =_{\lambda\beta A} FFaM =_{\lambda\beta A} M(FFaM)$. Now, if for every constant a and b, not occurring in M, we have that H_aM is different from H_bM then M has an infinite number of fixed points. So, assume that for some constants a and b, not occurring in M, we have that $H_aM = H_bM$. Since the Church–Rosser Theorem obviously holds in $\lambda\beta A$, consider a common reduct N of H_aM and H_bM. As the constant a does not occur in H_bM and the constant b does not occur in H_aM, neither a nor b can occur in N. As a consequence, neither H_aM nor H_bM can occur in N. Now, let P be any fixed point of M and consider the reduction sequence $\sigma : H_aM \longrightarrow_\beta^* N$.

First observe that without loss of generality, we can assume that every reduction cycle

$$H_aM \equiv (\lambda yz.z(FFyz))aM \longrightarrow_\beta (\lambda z.z(FFaz))M \longrightarrow_\beta M(FFaM),$$

has been completed in the reduction sequence σ. Indeed, we can freely remove every reduction step belonging to a proper segment of an uncompleted cycle. Moreover, we can freely rearrange σ so that the completion of a cycle is made by consecutive reduction steps.

Now starting from the fixed point P, we imitate the reduction σ, maintaining at the same time a one-to-one correspondence between occurrences of P and occurrences of H_aM, as follows:

— at the beginning, the unique occurrence of P corresponds to the unique occurrence of H_aM;

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— if the reduction step is the first of a reduction cycle of the form above, which expands an occurrence of H_aM to $M(H_aM)$ then replace the corresponding occurrence of P with MP and ignore the remaining part of the cycle (the one-to-one correspondence between occurrences of H_aM and P is preserved by letting the occurrence of H_aM inside $M(H_aM)$ corresponds to the occurrence of P inside MP);
— if the reduction step has the form:

$$\mathcal{C}_1[H_aM,\ldots,H_aM] \longrightarrow_\beta \mathcal{C}_2[H_aM,\ldots,H_aM],$$

where $\mathcal{C}_1[H_aM,\ldots,H_aM]$ and $\mathcal{C}_2[H_aM,\ldots,H_aM]$ are two contexts with all the respective occurrences of H_aM displayed, then perform the reduction

$$\mathcal{C}_1[P,\ldots,P] \longrightarrow_\beta \mathcal{C}_2[P,\ldots,P],$$

(the one-to-one correspondence between occurrences of H_aM and P is preserved by letting residuals correspond to residuals in the obvious way);

now every step preserves β-convertibility, and moreover the final term of the defined sequence is still N. This proves that $P =_{\lambda\beta A} H_aM$. So, M has a unique fixed point. □

3. Terms with two fixed points

In this section, we define a λ-theory such that some term has, in the theory, only two fixed points. We define the theory as the convertibility relation generated by β and other special reduction rules.

Let a, b and c be constants. The theory T_c is obtained by adding to $\lambda\beta$ the following derivation rule:

Rule $\mathcal{R}_c$: let M be a closed term such that the constant b does not occur in M, then

$$aM \longrightarrow_{\mathcal{R}_c} c.$$

Remark. Observe that the theory T_c actually depends on all the three constants a, b, c. With the notation T_c, we show the dependency on the reduction rule $\mathcal{R}_c$.

Proposition 2. $\mathcal{R}_c$ is CR.

Proof. We want to show that $\mathcal{R}_c$ commutes with itself, i.e. let $\Delta_1 \equiv aM_1$ and $\Delta_2 \equiv aM_2$ be two redexes of $\mathcal{R}_c$ occurring inside a term M. If they are disjoint, then it is immediate. Otherwise, we have that, say, $\Delta_1 \equiv aM_1 \equiv a(\mathcal{C}[aM_2])$, for some context $\mathcal{C}[\ ]$. Performing the internal reduction we get $a(\mathcal{C}[c])$ and then we get c, as $\mathcal{C}[c]$ still does not contain b. It follows that $\mathcal{R}_c$ commutes with itself. □

As usual, we denote by $\mathcal{R}_c^*$ the reflexive and transitive closure of $\mathcal{R}_c$.

Proposition 3. $\mathcal{R}_c^*$ commutes with β-reduction.

Proof. The proof is immediate. As the $\mathcal{R}_c$-redexes are closed terms they can only be permuted, erased or duplicated by β-reductions. On the other hand, after a β-reduction is performed inside a $\mathcal{R}_c$-redex, we still have a $\mathcal{R}_c$-redex. □

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By the previous propositions, the following holds:

Proposition 4. The theory T_c is CR.

The theory $T_{b,c}$ is obtained by adding to T_c the following (non-constructive) derivation rule:

Rule $\mathcal{R}_b$: let M be a closed term such that for every N, with $M \longrightarrow^*_{\beta\mathcal{R}_c} N$, the constant b occurs in N then

$$aM \longrightarrow_{\mathcal{R}_b} b.$$

In the following, we shall need a notion of residual. Using such notion, we obtain a version of the well-known 'parallel moves' lemma (see Terese (2003)) suitable for our context.

Definition 2. Let M be a term of the form $\mathcal{C}[\Delta]$ for some context $\mathcal{C}[\ ]$ and some $\mathcal{R}_b$-redex Δ. Assume that M is reduced by the rule $\mathcal{R}_c$ to a term N. Then the *residual* Δ' *of* Δ *w.r.t. the reduction step* $M \longrightarrow_{\mathcal{R}_c} N$ is defined as follows:

— assume first that the $\mathcal{R}_c$-redex Δ_1 is a subterm of Δ. It follows that Δ_1 is a proper subterm of Δ and therefore Δ has the form $\mathcal{C}_1[\Delta_1]$ for some context $\mathcal{C}_1[\ ]$; in this case we set $\Delta' \equiv \mathcal{C}_1[c]$;
— then assume that the $\mathcal{R}_c$-redex Δ_1 is not a subterm of Δ. It follows that Δ_1 is disjoint from Δ which contains the constant b; therefore M has the form $\mathcal{C}_1[\Delta, \Delta_1]$ for some context $\mathcal{C}_1[\ ,\ ]$; in this case we set $\Delta' \equiv \Delta$.

Definition 3. Let M be a term of the form $\mathcal{C}[\Delta]$ for some context $\mathcal{C}[\ ]$ and some $\mathcal{R}_b$-redex Δ. Assume that M is reduced by the β-rule to a term N. Then the *residuals* $\Delta'_1, \ldots, \Delta'_k$ *of* Δ *w.r.t. the reduction step* $M \longrightarrow_\beta N$ are defined as follows:

— assume first that the β-redex Δ_1 is a subterm of Δ. It follows that Δ_1 is a proper subterm of Δ and therefore Δ has the form $\mathcal{C}_1[\Delta_1]$ for some context $\mathcal{C}_1[\ ]$; in this case we set $k = 1$ and $\Delta'_1 \equiv \mathcal{C}_1[\bar{\Delta}_1]$, where $\bar{\Delta}_1$ is the reduct of Δ_1;
— then assume that the β-redex Δ_1 is disjoint from Δ; therefore M has the form $\mathcal{C}_1[\Delta, \Delta_1]$ for some context $\mathcal{C}_1[\ ,\ ]$; in this case we set $k = 1$ and $\Delta'_1 \equiv \Delta$;
— finally assume that Δ is a subterm of the β-redex Δ_1. Since Δ is a closed term, it follows that the reduct $\bar{\Delta}_1$ of Δ_1 has the form $\mathcal{C}_1[\Delta, \ldots, \Delta]$ for some context $\mathcal{C}_1[\ , \ldots, \ ]$ and n occurrences of Δ, for some $n \geqslant 0$; in this case we set $k = n$ and Δ'_i is the ith occurrence of Δ in $\mathcal{C}_1[\Delta, \ldots, \Delta]$.

Definition 4. Let M be a term of the form $\mathcal{C}[\Delta]$ for some context $\mathcal{C}[\ ]$ and some $\mathcal{R}_b$-redex Δ. Let σ be a reduction sequence such that $\sigma : M \longrightarrow^*_{\beta\mathcal{R}_c} N$. Then the *residuals of* Δ *w.r.t.* σ are defined by induction on the length $|\sigma|$ of σ:

— if $|\sigma| = 1$, then it is composed of one $\mathcal{R}_c$-reduction or one β-reduction, then the residuals of Δ are those defined by Definition 2 or, respectively, Definition 3;
— let $|\sigma| = \ell + 1$, and assume that $\bar{\Delta}_1, \ldots, \bar{\Delta}_r$ are the residuals of Δ w.r.t. the reduction sequence σ' composed of the first ℓ reduction steps of σ; then the residuals of Δ are

$$\bar{\Delta}'_{1,1}, \ldots, \bar{\Delta}'_{1,s_1}, \bar{\Delta}'_{2,1}, \ldots, \bar{\Delta}'_{1,s_2}, \ldots, \bar{\Delta}'_{r,1}, \ldots, \bar{\Delta}'_{r,s_r},$$

where $\bar{\Delta}'_{j,1}, \ldots, \bar{\Delta}'_{j,s_j}$ are the residuals of $\bar{\Delta}_j$, for $j = 1, \ldots, r$.

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Remark. Observe that residuals specified by Definition 4 are unambiguously determined as the following holds:

— the one-step residuals of Definition 3 are pairwise disjoint subterms: this follows from the fact that a $\mathcal{R}_b$-redex is a closed term;
— it is easily seen that, by the same reason, pairwise disjoint residuals originate pairwise disjoint residuals.

Definition 5. Let $M \equiv \mathcal{C}[\Delta]$, with Δ a $\mathcal{R}_b$-redex. Let σ be a sequence of T_c reductions starting from $\mathcal{C}[b]$ and ending in a term N. The *candidate corresponding reduction sequence* σ' *starting from* $\mathcal{C}[\Delta]$ is defined by recursion on the length of σ as follows:

— if σ is composed of one reduction step of the form $\mathcal{C}[b] \longrightarrow_{T_c} \mathcal{C}_1[b,\ldots,b]$ for some context $\mathcal{C}_1[\ ,\ldots,\ ]$, where in $\mathcal{C}_1[b,\ldots,b]$ are displayed all the residuals of b, then σ' is composed of the pair $(\mathcal{C}[\Delta],\mathcal{C}_1[\Delta,\ldots,\Delta])$;
— assume now that the length of σ is $k+1$, and assume that the $k+1$-th reduction step of σ is composed of the reduction

$$\mathcal{C}_2[b,\ldots,b] \longrightarrow_{T_c} \mathcal{C}_3[b,\ldots,b],$$

where in $\mathcal{C}_2[b,\ldots,b]$ and $\mathcal{C}_3[b,\ldots,b]$ are displayed all the residuals of b w.r.t. k steps and, respectively, $k+1$ steps of reduction in σ, then: the $k+1$-th pair in σ' is $(\mathcal{C}_2[\Delta,\ldots,\Delta],\mathcal{C}_3[\Delta,\ldots,\Delta])$.

Proposition 5. Let Δ be a $\mathcal{R}_b$-redex occurring inside a term M. Let M be of the form $\mathcal{C}[\Delta]$, for some context $\mathcal{C}[\ ]$. Let σ be a sequence of T_c reductions starting from $\mathcal{C}[b]$ and ending in a term N. Then the candidate corresponding reduction sequence σ' starting from $\mathcal{C}[\Delta]$ is a sequence of T_c reductions.

Proof. The proposition follows by an easy induction on the length of σ using the fact that the T_c redexes never overlap with $\mathcal{R}_b$-redexes. □

By the previous proposition, we shall use the terminology *corresponding reduction sequence* instead of *candidate corresponding reduction sequence*.

Proposition 6. $\mathcal{R}_b$ is CR.

Proof. We show that $\mathcal{R}_b$ commutes with itself. Let Δ_1 and Δ_2 be two redexes of $\mathcal{R}_b$ occurring inside a term M. Without loss of generality, assume that Δ_1 is a subterm of Δ_2 and that Δ_2 has the form $a(C[\Delta_1])$. If we reduce Δ_1 then $a(\mathcal{C}[b])$ can still be reduced to b. By contradiction, assume that there exists a sequence σ of T_c reductions starting from $\mathcal{C}[b]$ and ending in a term N without occurrences of b. Let σ' be the reduction sequence which is composed by the same reduction step. We make the following claim:

Claim.
Performing from $\mathcal{C}[\Delta_1]$, the corresponding sequence σ' of reductions we can reduce $\mathcal{C}[\Delta_1]$ to N, which is impossible by assumption that $\Delta_2 = a(C[\Delta_1])$ is a R_b redex and that N contains no occurrences of b.

To prove the claim, consider to a reduction step in σ of the form

$$\mathcal{C}_1[b,\ldots,b] \longrightarrow_{T_c} \mathcal{C}_2[b,\ldots,b],$$

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where $\mathcal{C}_1[b,\ldots,b]$ and $\mathcal{C}_2[b,\ldots,b]$ are two contexts with all the respective occurrences of residuals of b are displayed, we show that the corresponding step in σ' is of the form:

$$\mathcal{C}_1[\Delta_1,\ldots,\Delta_1] \longrightarrow_{T_c} \mathcal{C}_2[\Delta_1,\ldots,\Delta_1],$$

by induction hypothesis the left hand side of the step in σ' has the form $\mathcal{C}_1[\Delta_1,\ldots,\Delta_1]$ with all the occurrences of residuals of Δ_1 are displayed.

Assume first that the reduction step reduces a β-redex, this case is immediate; then assume that reduction step reduces a $\mathcal{R}_c$-redex of the form aP. Now, since Δ_1 is a $\mathcal{R}_b$-redex, $\Delta_1 \equiv aQ$ for some closed term Q with a occurring in Q. As every occurrence of Δ_1 in $\mathcal{C}_1[\Delta_1,\ldots,\Delta_1]$ corresponds to an occurrence of b in $\mathcal{C}_1[b,\ldots,b]$ it is impossible that aP is a subterm of some aQ. On the other hand, as b does not occur in P, aP is disjoint from every occurrence of b in $\mathcal{C}_1[b,\ldots,b]$ and therefore is disjoint from every occurrence of Δ_1 in $\mathcal{C}_1[\Delta_1,\ldots,\Delta_1]$. The claim follows. □

Proposition 7. $\mathcal{R}_b^*$ commutes with $\mathcal{R}_c^*$ and β-reduction.

Proof. We show that $\mathcal{R}_b$ commutes with $\mathcal{R}_c$. Let Δ_1 and Δ_2 be a $\mathcal{R}_b$-redex and, respectively, a $\mathcal{R}_c$-redex occurring inside a term M. Assume that Δ_2 is a subterm of Δ_1. Therefore, Δ_1 has the form $a(\mathcal{C}[\Delta_2])$. If we reduce Δ_2 then $\mathcal{C}[c]$ can still be reduced to b. Indeed, by an already used argument, if there exists a sequence of T_c reductions starting from $\mathcal{C}[c]$ and ending in a term N without occurrences of b, then performing from $\mathcal{C}[\Delta_1]$ first the reduction $\Delta_1 \longrightarrow c$ and then the corresponding sequence of reductions we can reduce $\mathcal{C}[\Delta_1]$ to N which is impossible.

Now assume that Δ_1 is a subterm of Δ_2. This is however impossible since Δ_1 contains b. The case of two disjoint redexes is immediate.

The proof that $\mathcal{R}_b^*$ commutes with β-reduction is similar to the proof that $\mathcal{R}_c^*$ commutes with β-reduction. □

By the previous propositions, the following holds:

Proposition 8. The reduction system composed by β-reduction and rules $\mathcal{R}_b$ and $\mathcal{R}_c$ is CR.

Recall that the theory $T_{b,c}$ has been defined as the convertibility theory generated by rules β-reduction, $\mathcal{R}_b$ and $\mathcal{R}_c$. As a corollary of Proposition 8, we obtain:

Proposition 9. In the theory $T_{b,c}$:

1. constants b and c are different;
2. constant a has the finite range $\{b,c\}$;
3. constant a has only two fixed points:
   the constant b and the constant c.

Proof. The proof is immediate. By the CR property constants b and c are different. Constant a has a finite range $\{b,c\}$ as either rule $\mathcal{R}_b$ or rule $\mathcal{R}_c$ applies to a redex of the form aM, with M closed. Observe that, on the contrary, both b and c have infinite range. □

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It follows, from the previous proposition, that the theory $T_{b,c}$ is the claimed example of non-r.e. lambda theory that violates the range property and in which some term has exactly two fixed points.

3.1. *The degree of unsolvability of theory* $T_{b,c}$

Now, we shall consider the degree of unsolvability of the equality relation in the theory $T_{b,c}$.

We need a preliminary result.

Proposition 10 (Postponement of constant rules). Assume that σ is a sequence of reductions in $T_{b,c}$ which reduces a term M to a term N. Then, there exist terms P and Q such that

$$M \longrightarrow_{\beta}^{*} P \longrightarrow_{\mathcal{R}_c}^{*} Q \longrightarrow_{\mathcal{R}_b}^{*} N.$$

Proof. By the previous results on reciprocal commutativity of the rules. □

Upper bound. Now, to provide an *upper bound* to the degree of unsolvability of the equality relation in the theory $T_{b,c}$, we argue as follows.

Let M and N be two terms, by the results already obtained, $M =_{T_{b,c}} N$ is equivalent to the following:

There are two contexts $\mathcal{C}_1$ and $\mathcal{C}_2$, two sequences of terms $T_1, \ldots, T_k$ and $V_1, \ldots, V_r$ and two reductions sequences σ_1 and σ_2 such that:

— σ_1 and σ_2 are composed of β and $\mathcal{R}_c$ reductions;
— $\sigma_1 : M \longrightarrow^{*} \mathcal{C}_1[aT_1, \ldots, aT_k]$;
— $\sigma_1 : N \longrightarrow^{*} \mathcal{C}_2[aV_1, \ldots, aV_r]$;
— $\mathcal{C}_1[b, \ldots, b] \equiv \mathcal{C}_2[b, \ldots, b]$;

and such that a certain Π_1^0-condition holds.

This condition is

for every reduction sequence σ, which is composed of β and $\mathcal{R}_c$ reductions, σ has not the form $\sigma : U \longrightarrow^{*} L$ for some $U \in \{T_1, \ldots, T_k, V_1, \ldots, V_r\}$ and some term L such that b does not occur in L.

Observe that given a reduction sequence σ, it is a recursive relation to test if σ has the required properties, so that the condition is actually Π_1^0.

It follows that the equality relation in the theory $T_{b,c}$ is expressible as a Σ_2^0-relation.

Lower bound. Now, to provide a *lower bound* to the degree of unsolvability of the equality relation in the theory $T_{b,c}$, we argue as follows.

In the following, we use the notation $\longrightarrow_{\mathcal{R}_{b,c}}$ to denote the reduction notion generated by the two rules $\mathcal{R}_b$ and $\mathcal{R}_c$, without β-reduction.

We also need the notion of *applicative component* of a term. This notion is defined as follows:

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Every term M can be written uniquely in one of the following forms (where n may be 0):

1. $CX_1 \dots X_n$ for a constant C,
2. $xX_1 \dots X_n$ for a variable x,
3. $(\lambda x.X)X_1 \dots X_n$,

then the applicative components of M are, respectively,

1. $C, X_1, \dots, X_n$,
2. $x, X_1, \dots, X_n$,
3. $(\lambda x.X), X_1, \dots, X_n$.

Recall that a term M has *order* 0 iff has no β-reduct of the form $\lambda x.N$, for some term N (see Barendregt (1984), Definition 17.3.2). We need also the following result, where following Rogers (1967), we denote by $\varphi_{\mathbf{e}}$ the recursive function with code number $\mathbf{e}$:
Fact. There exists a closed term P such that:

— no constant occurs in P;
— $P\underline{\mathbf{e}}\,\underline{n}$ has order 0 if $\varphi_{\mathbf{e}}(n)$ diverges;
— $P\underline{\mathbf{e}}\,\underline{n} = \lambda xy.y$ if $\varphi_{\mathbf{e}}(n)$ converges.

Let
$\mathbf{s} \equiv \lambda xyz.y(xyz)$ (the successor function for Church numerals)
$\mathbf{\Theta} \equiv (\lambda xy.y(xxy))(\lambda xy.y(xxy))$ (Turing fixed point combinator)
$A \equiv \lambda fgxyz.fx(a(P\underline{\mathbf{e}}xb))[f(\mathbf{s}x)y(g(\mathbf{s}x))z]$
$B \equiv \lambda fgx.f(\mathbf{s}x)(a(P\underline{\mathbf{e}}(\mathbf{s}x)b))(g(\mathbf{s}x))(gx)$
$G \equiv \mathbf{\Theta}(\lambda u.B(\mathbf{\Theta}(\lambda v.Auv))u)$
$F \equiv \mathbf{\Theta}(\lambda u.AuG)$
$H \equiv \lambda x.F\underline{0}(ax)(G\underline{0})$.

Now, we consider the class of all recursive functions which are defined on input 0. On such class, we have the following result, whose proof makes use of a technique introduced by the second author of the present paper in Statman (1999).

Proposition 11. $Hc =_{T_{b,c}} Hb$, if and only if $\varphi_{\mathbf{e}}$ is not total.

Proof. Assume first that for some $k > 0$, $\varphi_{\mathbf{e}}(k)$ diverges. Then, $a(P\underline{\mathbf{e}}\,\underline{k}) = b$. To show how the computation works, assume $k = 1$. Then, we have

$$Hb = F\underline{0}(ab)(G\underline{0}) = F\underline{0}b(G\underline{0}) = F\underline{0}(a(P\underline{\mathbf{e}}\,\underline{0}b))[F\underline{1}b(G\underline{1})(G\underline{0})],$$

now observe that $a(P\underline{\mathbf{e}}\,\underline{1}b) = b$, as $\varphi_{\mathbf{e}}(1)$ diverges. Therefore

$$F\underline{1}b(G\underline{1})(G\underline{0}) = F\underline{1}(a(P\underline{\mathbf{e}}\,\underline{1}b))(G\underline{1})(G\underline{0}),$$

on the other hand

$$F\underline{1}(a(P\underline{\mathbf{e}}\,\underline{1}b))(G\underline{1})(G\underline{0}) = G\underline{0},$$

and so

$$F\underline{0}(a(P\underline{\mathbf{e}}\,\underline{0}b))[F\underline{1}b(G\underline{1})(G\underline{0})] = F\underline{0}(a(P\underline{\mathbf{e}}\,\underline{0}b))(G\underline{0}) = F\underline{0}c(G\underline{0}) = Hc.$$

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In the general case, iteratively using the definition of F up to k, we compute:

$$Hb = F\underline{0}(a(P\underline{\mathbf{e}}\,\underline{0}b))(F\underline{1}(a(P\underline{\mathbf{e}}\,\underline{1}b))(\ldots F(\underline{k-1})(a(P\underline{\mathbf{e}}\,(\underline{k-1})b))[F\underline{k}b(G\underline{k})(G(\underline{k-1}))](G(\underline{k-2}))\ldots)(G\underline{1})(G\underline{0}))$$

now, as in the case $k = 1$, $a(P\underline{\mathbf{e}}\,\underline{k}b) = b$ as $\varphi_{\mathbf{e}}(k)$ diverges. Therefore

$$F\underline{k}b(G\underline{k})(G(\underline{k-1})) = F\underline{k}(a(P\underline{\mathbf{e}}\,\underline{k}b))(G\underline{k})(G(\underline{k-1})) = G(\underline{k-1}).$$

Moreover

$$F(\underline{k-1})(a(P\underline{\mathbf{e}}\,(\underline{k-1})b))(G(\underline{k-1}))(G(\underline{k-2})) = G(\underline{k-2}),$$

and so on, until we eventually get $G\underline{0}$. Therefore

$$Hb = F\underline{0}(a(P\underline{\mathbf{e}}\,\underline{0}b))(F\underline{1}(a(P\underline{\mathbf{e}}\,\underline{1}b))(\ldots[F(\underline{k-1})(a(P\underline{\mathbf{e}}\,(\underline{k-1})b))(G(\underline{k-1}))(G(\underline{k-2}))]\ldots)(G\underline{1})(G\underline{0})) = \cdots\cdots = F\underline{0}(a(P\underline{\mathbf{e}}\,\underline{0}b))(G\underline{0}) = Hc.$$

Now, assume that $Hb = Hc$.

Note here that $Fxyz$ has order 0. Suppose that $F\underline{0}c(G\underline{0}) = F\underline{0}b(G\underline{0})$ and $\varphi_{\mathbf{e}}$ is total. Then, by the Church–Rosser property and Proposition 10, there exist a Church numeral $\underline{l}$ and terms L, M, N such that

$$F\underline{l}(ac)(G\underline{l}) \longrightarrow_{\beta}^{*} L \longrightarrow_{\mathcal{R}_{b,c}}^{*} N$$

$$F\underline{l}(ab)(G\underline{l}) \longrightarrow_{\beta}^{*} M \longrightarrow_{\mathcal{R}_{b,c}}^{*} N,$$

where the beta reductions are standard.

Now, a head β-reduction sequence beginning with a term of the form $F\underline{k}UV$ cycles through segments which are eight terms long; namely,

$$F\underline{k}UV \equiv \mathbf{\Theta}(\lambda u.AuG)\underline{k}UV \longrightarrow_{\beta} (\lambda x.x(\mathbf{\Theta}x))(\lambda u.AuG)\underline{k}UV \longrightarrow_{\beta}$$

$$\longrightarrow_{\beta} (\lambda u.AuG)F\underline{k}UV \longrightarrow_{\beta} AFG\underline{k}UV \longrightarrow_{\beta}$$

$$\longrightarrow_{\beta} (\lambda gxyx.Fx(a(P\underline{\mathbf{e}}\,xb))[F(\mathbf{s}x)y(g(\mathbf{s}x))z])G\underline{k}UV \longrightarrow_{\beta}$$

$$\longrightarrow_{\beta} (\lambda xyz.Fx(a(P\underline{\mathbf{e}}\,xb))[F(\mathbf{s}x)y(G(\mathbf{s}x))z])\underline{k}UV \longrightarrow_{\beta}$$

$$\longrightarrow_{\beta} (\lambda yz.F\underline{k}(a(P\underline{\mathbf{e}}\,\underline{k}b))[F(\mathbf{s}\underline{k})y(G(\mathbf{s}\underline{k}))z])UV \longrightarrow_{\beta}$$

$$\longrightarrow_{\beta} (\lambda z.F\underline{k}(a(P\underline{\mathbf{e}}\,\underline{k}b))[F(\mathbf{s}\underline{k})U(G(\mathbf{s}\underline{k}))z])V \longrightarrow_{\beta}$$

$$\longrightarrow_{\beta} F\underline{k}(a(P\underline{\mathbf{e}}\,\underline{k}b))[F(\mathbf{s}\underline{k})U(G(\mathbf{s}\underline{k}))V]$$

where $a(P\underline{\mathbf{e}}\,\underline{k}b)$ becomes the new U and $F(\mathbf{s}\underline{k})U(G(\mathbf{s}\underline{k}))V$ the new V.

We first observe that, in the terms $F\underline{l}(ac)(G\underline{l})$ and $F\underline{l}(ab)(G\underline{l})$, we start with a term U which begins with the constant a and, moreover, this property is preserved at the end of the 8 terms long cycle displayed above; therefore, we can assume that U begins with the constant a.

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Then, we observe that the head β-reductions in the sequences:

$$\sigma_1 : F\underline{l}(ac)(G\underline{l}) \longrightarrow_\beta^* L$$
$$\sigma_2 : F\underline{l}(ab)(G\underline{l}) \longrightarrow_\beta^* M,$$

cannot terminate at different spots in the 8 term cycle. Indeed, observe that, in the sequences σ_1 and σ_2, in the above mentioned eight term cycle U is the only applicative component that begins with the constant a. Therefore, the positions of components which begin with the constant a are resp. 5, 4, 4, 5, 4, 3, 2, and finally not at all. On the other hand, the two different positions with constant a at the fifth applicative component and the three positions with constant a at the fourth applicative component cannot match up by internal β reductions since

$$\lambda xy.y(xxy) \neq_\beta A$$

$$\lambda x.x(\mathbf{\Theta}x) \neq_\beta \lambda u.AuG$$

$$\lambda x.x(\mathbf{\Theta}x) \neq_\beta \lambda gxyx.Fx(a(P\underline{\mathbf{e}}\,xb))[F(\mathbf{s}x)y(g(\mathbf{s}x))z]$$

$$\lambda u.AuG \neq_\beta \lambda gxyx.Fx(a(P\underline{\mathbf{e}}\,xb))[F(\mathbf{s}x)y(g(\mathbf{s}x))z].$$

All the displayed inequalities are easily seen to hold, as there is not a common β reduct.

We have proved that the head reductions in the sequences σ_1 and σ_2 terminate at the same spot in the eight term cycle.

Moreover, if σ_1 fails to complete the first eight term cycle outside of of the eighth case ('not at all'), then $c = b$ contradicting the Church–Rosser property. In the eighth case, we have shorter standard reductions from $F(\mathbf{s}\underline{l})(ab)(G(\mathbf{s}\underline{l}))$ and $F(\mathbf{s}\underline{l})(ac)(G(\mathbf{s}\underline{l}))$. Arguing by induction on the sum of the lengths of σ_1 and σ_2, it is impossible that $F\underline{l}(ab)(G\underline{l})$ and $F\underline{l}(ac)(G\underline{l})$ have a common reduct. This contradiction ends the proof. □

Summing up the previous results, we obtain.

Proposition 12. The theory $T_{b,c}$ is Σ_2^0-complete.

Proof. By Proposition 11, using Theorem VIII Chapter 13 of Rogers (1967). □

4. Terms with two fixed points and the range property

In the theory $T_{b,c}$, the constant a has exactly two fixed points and, at the same time, such fixed points are the entire range of the constant a. So, a violates the *range property* (see Barendregt (1984) Definition 20.2.2), which states that a combinator has either an infinite range or a range which is a singleton. Since the range property holds in the *Closed Term Model* of β-convertibility (see Barendregt (1984) Definition 4.1.17), a natural question is whether it is possible to have terms with a finite number (greater than one) of fixed points and preserving the range property for all terms. In the following, we show that this assumption is consistent with β-convertibility; however, we obtain this again by means of a *non-r.e.* theory. We do not know whether this can hold in r.e. λ-theories.

In this section, let A be the following set of constants $A = \{a, c\} \cup \{b_1, b_2, \ldots\} \cup \{b_\infty\}$. Observe that we have *positive* integers (as well as the infinity) as indexes of constants.

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We extend the notion of Böhm tree (see Barendregt (1984), 10.1.3) as follows. (*Equality* of Böhm trees is denoted by $=$).

Definition 6. Let M be a term of $\lambda\beta A$ then the *Böhm tree* of M, notation $\mathcal{B}(M)$, is defined as follows:

1. if M is solvable and has as principal hnf $\lambda x_1 \dots x_n.\xi M_1 \dots M_m$ then
   $\mathcal{B}(M) =$
   $$\begin{array}{ccc} \lambda x_1 \dots x_n. & \xi & \\ & / & \backslash \\ \mathcal{B}(M_1) & \dots & \mathcal{B}(M_m), \end{array}$$
   where ξ is either a variable or a constant, and $m, n \geqslant 0$.
2. if M is unsolvable then $\mathcal{B}(M) = \bot$

Therefore, in particular, if, say, $M \equiv a$ then $\mathcal{B}(M) = a$, i.e. the tree with only one node, labelled by a.

The following lemma will be useful.

Lemma 1. Let N be a term of $\lambda\beta A$ of the form $\mathcal{C}[aM]$ for some closed term M and some context $\mathcal{C}[\ ]$. Let z be a fresh variable not occurring in N. Then the Böhm tree of N is obtained from the Böhm tree of $\mathcal{C}[z]$, by replacing every node labelled by z with the Böhm tree of aM.

Proof. First observe that, since M is closed, and the constant a behaves like a free variable (since we do not use any rule for constants), in every reduct of N the residuals of aM have all the same Böhm tree. Now, arguing by contradiction, assume that $\mathcal{B}(N)$ differs at some path θ from the tree specified by the statement of the lemma that we denote by $\mathcal{B}(\mathcal{C}[z := aM])$.

First assume that $\mathcal{B}(N)$ is undefined at path θ, whilst path θ is in the domain of $\mathcal{B}(\mathcal{C}[z := aM])$. Therefore, there exists an ancestor θ_0 of θ such that $\mathcal{B}(\mathcal{C}[z])$ has node labelled by z at path θ_0. Let σ be a sequence of β-reductions which reduces $\mathcal{C}[z]$ to a term $P[zR_1 \dots R_t]$ – for some terms $R_1, \dots, R_t$, and some $t \geqslant 0$ – which has the variable z occurring at path θ_0 of its head normal form. In $P[zR_1 \dots R_t]$, we have displayed such occurrence of z by the context notation. If we perform from N the corresponding sequence of reductions σ', we get the term $P[aMR_1 \dots R_t]$, where the term aM occurs at the path θ_0.

Now consider the suffix path θ_1 from θ_0 to θ. Let $\bar{\sigma}$ be a sequence of β-reductions which reduces aM to a term Q such that the path θ_1 belongs to the head normal form of Q. If we perform in $P[aMR_1 \dots R_t]$ the corresponding sequence of reductions $\bar{\sigma}'$, inside the occurrence of term aM lying at path θ_0, we get a term of the form $P[QR_1 \dots R_t]$ such that the path $\theta = \theta_0\theta_1$ belongs to its head normal form. Thus we have reached a contradiction and therefore every path θ which is in the domain of $\mathcal{B}(\mathcal{C}[z := aM])$ is in the domain of $\mathcal{B}(N)$.

The other cases are similar and left to the reader. □

If a constant occurs in the Böhm tree $\mathcal{B}(M)$ of a term M then it occurs on a unique path θ from the root of the tree. We say that the subtree of $\mathcal{B}(M)$ determined by θ is

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closed if there exists a reduct M' of M whose head normal form is defined for the path θ and such that the subterm of M' corresponding to θ is a closed term.

Assume the constant b_n, for some n, with $n = 1, 2, \ldots$, occurs at θ_1 and let θ_2 be any occurrence of the constant a such that the path θ_2 is an initial segment of the path θ_1, then we say that the occurrence of a is *in functional position* w.r.t. the occurrence of b_n if

— the suffix path from θ_2 to θ_1 starts going from a to its leftmost child;
— the subtree of $\mathcal{B}(M)$ determined by θ_2 is closed.

More generally the *functional degree* of the occurrence θ_2 of the constant a w.r.t. the occurrence θ_1 of the constant b_n, that we denote by $d(\theta_2, \theta_1)$, is the number of times the constant a occurs in functional position in the suffix path from θ_2 to θ_1.

Example. In $\mathbf{a}(\lambda x.\mathbf{a}(\mathbf{a}b_3))$ and in $\mathbf{a}(\lambda x.a(x(\mathbf{a}b_3)))$, we have displayed in bold text the occurrences of constant a in functional position. In $\mathbf{a}(\lambda x.\mathbf{a}(\mathbf{a}b_3))$, the functional degree of the head occurrence of a w.r.t. the unique occurrence of b_3 is 3, whilst in $\mathbf{a}(\lambda x.a(x(\mathbf{a}b_3)))$ is 2, since the second occurrence of a along the path is not in a closed term (in every reduct, since the term is in normal form).

The theory $T_{A,c}$ is obtained by adding to $\lambda\beta A$ the following (non-constructive) derivation rule:

Rule $\mathcal{R}_{A,c}$:
let M be a closed term such that $\mathcal{B}(M)$ is finite and does not contain any constant b_n, for $n = 1, 2, \ldots, \infty$, then

$$aM \longrightarrow_{\mathcal{R}_{A,c}} c.$$

Remark. Observe that, in contrast with the previous section, here we are considering the Böhm tree $\mathcal{B}(M)$ of the term M. So, e.g. $a(\mathbf{\Omega}b_2) \longrightarrow_{\mathcal{R}_{A,c}} c$ since $\mathcal{B}(\mathbf{\Omega}b_2) = \perp$ and no constant b_n occurs in $\perp$.

In the following, we shall use a notion of *corresponding reduction sequence* which is just a simple adaptation of the notion introduced in Definition 5. To avoid tedious repetitions, we leave the formal details to the reader.

Proposition 13. $\mathcal{R}_{A,c}$ is CR.

Proof. Let $\Delta_1 \equiv aM_1$ and $\Delta_2 \equiv aM_2$ be two redexes of $\mathcal{R}_{A,c}$ occurring inside a term M. If they are disjoint then it is immediate. Otherwise, we have that, say, $\Delta_1 \equiv aM_1 \equiv a(\mathcal{C}[aM_2])$, for some context $\mathcal{C}[\ ]$. Performing the internal reduction we get $a(\mathcal{C}[c])$. Now, we have that $\mathcal{B}(aM_1)$ is a finite tree and does not contain any occurrence of a constant b_n, for $n = 1, 2, \ldots, \infty$. Assume by contradiction that $\mathcal{B}(a(\mathcal{C}[c]))$ contains at the path θ an occurrence of a constant b_n for some n. Then, there is a sequence σ of β-reductions such that σ reduces $a(\mathcal{C}[c])$ to a term M' which is in head normal form and at the path θ of the head normal form there is an occurrence of b_n. We consider the corresponding reduction sequence σ' starting from aM_1, following the residuals of aM_2. First of all aM_2, being a $\mathcal{R}_{A,c}$-redex, is a closed term. Therefore, we have that in any residual of aM_2 the constant b_n cannot be inserted by β-reduction. Moreover, no residual of aM_2 can contribute to the generation of the path θ, as otherwise the constant b_n would occur in $\mathcal{B}(aM_2)$. It follows that the path θ can be generated also by σ' which is impossible.

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A similar arguments proves that in $\mathcal{B}(a(\mathcal{C}[c]))$ an infinite path θ cannot exist. Therefore, we can reduce $a(\mathcal{C}[c])$ to c. It follows that $\mathcal{R}_{A,c}$ commutes with itself. □

Let $\mathcal{R}^*_{A,c}$ the reflexive and transitive closure of $\mathcal{R}_{A,c}$.

Proposition 14. $\mathcal{R}^*_{A,c}$ commutes with β-reduction.

Proof. The proof is immediate. As the $\mathcal{R}_{A,c}$-redexes are closed terms they can only be permuted, erased or duplicated by β-reductions. On the other hand, after a β-reduction is performed inside a $\mathcal{R}_{A,c}$-redex, we still have a $\mathcal{R}_{A,c}$-redex. □

By the previous propositions, the following holds:

Proposition 15. The reduction system composed by β-reduction and rule $\mathcal{R}_{A,c}$ is CR.

Now, let M be a closed term such that $\mathcal{B}(M)$ is finite and some constant b_n, for some positive integer n, occurs in $\mathcal{B}(M)$. Therefore, it occurs also in the Böhm tree $\mathcal{B}(aM)$ of the term aM. Let us denote by $Path(\mathcal{B}(aM), n)$ the set of all paths θ in $\mathcal{B}(aM)$ such that $\mathcal{B}(aM)(\theta) = b_n$, that is the label occurring at path θ in $\mathcal{B}(aM)$ is b_n. Finally, observe that in $\mathcal{B}(aM)$, the head occurrence of a occurs at the empty path ϵ. We denote by $\mathcal{M}_n$ the following set of numbers:

$$\mathcal{M}_n = \{2^{d(\epsilon,\theta)} * n |\ \theta \in Path(\mathcal{B}(aM), n)\}.$$

Moreover, we define

$$m_n = LCM(\mathcal{M}_n),$$

where LCM denotes the *least common multiple*. Observe that, since $\mathcal{B}(M)$ is finite, $\mathcal{M}_n$ is a finite set.

Now, let $\mathcal{I}(aM)$ be the set of all indexes n, for $n = 1, 2, \ldots$, such that b_n occurs in $\mathcal{B}(aM)$. Let m be defined by

$$m = LCM(\{m_n \mid n \in \mathcal{I}(aM)\}),$$

then we call m the *maximum constant index of* aM.

The theory T_{A,c,b^*} is obtained by adding to $T_{A,c}$ the following (non-constructive) derivation rule: Rule $\mathcal{R}_{A,b^*}$

$$aM \longrightarrow_{\mathcal{R}_{A,b^*}} b_m.$$

where M is a closed term such that:

1. $\mathcal{B}(M)$ is finite;
2. the constant b_∞ does not occur in $\mathcal{B}(M)$;
3. some constant b_n, for some n in the interval $1, 2, \ldots$, occurs in $\mathcal{B}(M)$;
4. m is the maximum constant index of aM.

Example. In $\mathbf{a}(\lambda x.\mathbf{a}(\mathbf{a}b_3))$, we have displayed in bold text the occurrences of constant a in functional position. The maximum constant index of the whole term is 24 since the functional degree of the head occurrence of the constant a is 3, the index of the constant b_3 is 3 and so $2^3 * 3 = 24$. Therefore, $a(\lambda x.a(ab_3)) \longrightarrow_{\mathcal{R}_{A,b^*}} b_{24}$. On the other hand, in $a(\lambda x.a(ab_3))$ there are other two $\mathcal{R}_{A,b^*}$-redexes corresponding to the other two occurrences of the constant a. The maximum constant index of $a(ab_3)$ is 12, therefore

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$a(ab_3) \longrightarrow_{\mathcal{R}_{A,b^*}} b_{12}$. Performing first such internal reduction in the original term, we have: $a(\lambda x.a(ab_3)) \longrightarrow_{\mathcal{R}_{A,b^*}} a(\lambda x.b_{12})$. Now, the maximum constant index of $a(\lambda x.b_{12})$ is 24, therefore $a(\lambda x.b_{12}) \longrightarrow_{\mathcal{R}_{A,b^*}} b_{24}$.

Remark. As another exemplification of rule $\mathcal{R}_{A,b^*}$, observe that $ab_n \longrightarrow_{\mathcal{R}_{A,b^*}} b_{2n}$.

Proposition 16. $\mathcal{R}_{A,b^*}$ is CR.

Proof. We show that $\mathcal{R}_{A,b^*}$ commutes with itself. Let $\Delta_1 \equiv aM_1$ and $\Delta_2 \equiv aM_2$ be two redexes of $\mathcal{R}_{A,b^*}$ occurring inside a term M. If they are disjoint, then it is immediate. Otherwise, we have that, say, $\Delta_1 \equiv aM_1 \equiv a(\mathcal{C}[aM_2])$, for some context $\mathcal{C}[\ ]$. Performing the internal reduction we get $a(\mathcal{C}[b_r])$, where r is the maximum constant index of aM_2. Moreover, assume that reducing aM_1 we get the constant b_k, where k is the maximum constant index of aM_1.

Claim.

$$a(\mathcal{C}[b_r]) \longrightarrow_{\mathcal{R}_{A,b^*}} b_k.$$

To prove the claim, assume $a(\mathcal{C}[b_r]) \longrightarrow_{\mathcal{R}_{A,b^*}} b_{k'}$; we have to prove $b_k = b_{k'}$. Consider any constant b_s occurring in $\mathcal{B}(aM_1)$. Consider moreover the computation of the index m_s in $\mathcal{B}(aM_1)$, that we denote by $m_s^{\mathcal{B}(aM_1)}$. We recall that $m_s^{\mathcal{B}(aM_1)} = LCM(\mathcal{M}_s)$, and on the other hand $\mathcal{M}_s = \{2^{d(\epsilon,\theta)} * s |\ \theta \in Path(\mathcal{B}(aM_1), s)\}$. Therefore, there exists a path θ to an occurrence of a constant b_s in $\mathcal{B}(aM_1)$ that determines the index $m_s^{\mathcal{B}(aM_1)}$.

Now, by Lemma 1, $\mathcal{B}(aM_1)$ is obtained from the Böhm tree of $a(\mathcal{C}[z])$, by replacing every node labelled by z with the Böhm tree of aM_2.

Assume first that the path θ ends in a node which belongs to a copy of $\mathcal{B}(aM_2)$ substituted for z in $\mathcal{B}(a(\mathcal{C}[z]))$. Let moreover θ_1 be the suffix path of θ, which starts from the head occurrence of constant a in aM_2. It is easily seen that, in $\mathcal{B}(aM_2)$, the occurrence of the constant b_s at path θ_1 determines the index m_s of constant b_s in aM_2. To see this, let θ_0 be such that $\theta = \theta_0\theta_1$ and let t_0 be the number of occurrences of a in functional position which are in the prefix path θ_0 with respect to the occurrence of b_s at θ. Then for every other occurrence of the constant b_s occurring in $\mathcal{B}(aM_2)$ at some path θ', the contribution of the prefix path θ_0 is always t_0, which means that the maximization of the value $2^{d(\epsilon,\theta')} * s$ must be done inside $\mathcal{B}(aM_2)$. Therefore the occurrence of the constant b_s at path θ_1 determines the index $m_s^{\mathcal{B}(aM_2)}$ of constant b_s in aM_2.

Now in $\mathcal{B}(aM_1)$, $m_s^{\mathcal{B}(aM_1)} = 2^{t_0+t_1} * s$, where t_1 is the number of occurrences of a in functional position which are in the suffix path θ_1 with respect to the occurrence of b_s at θ. Now, since r is the maximum constant index of aM_2, we have that $m_s^{\mathcal{B}(aM_2)} = 2^{t_1} * s$ divides r. On the other hand, k' is the maximum constant index of $a(\mathcal{C}[b_r])$ and it follows that the computation of $m_r^{\mathcal{B}(a(\mathcal{C}[b_r]))}$ returns a value not lesser than $2^{t_0} * r$, and moreover $2^{t_0} * r$ is a factor of $m_r^{\mathcal{B}(a(\mathcal{C}[b_r]))}$. Therefore, $2^{t_0} * r$ divides k' and this implies that $m_s^{\mathcal{B}(aM_1)}$ divides k'.

Assume now that the path θ ends in a node which does not belong to a copy of $\mathcal{B}(aM_2)$ substituted for z in $\mathcal{B}(a(\mathcal{C}[z]))$. Then, the occurrence of constant b_s at path θ is still in $\mathcal{B}(a(\mathcal{C}[b_r]))$. Moreover, such an occurrence determines $m_s^{\mathcal{B}(a(\mathcal{C}[b_r]))}$ and therefore also in this case $m_s^{\mathcal{B}(aM_1)}$ divides k'.

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Summing up, we have proved that for every constant b_s occurring in $\mathcal{B}(aM_1)$, $m_s^{\mathcal{B}(aM_1)}$ divides k'. It follows that k divides k'.

Now to prove that, conversely, k' divides k, it suffices to prove that $m_r^{\mathcal{B}(a(\mathcal{C}[b_r]))}$ divides k. (Indeed for the other constants, it is obvious). Also in this case, there exists a path θ' to an occurrence of the constant b_r in $\mathcal{B}(a(\mathcal{C}[b_r]))$ that determines the index $m_r^{\mathcal{B}(a(\mathcal{C}[b_r]))}$. If this path is also in $\mathcal{B}(a(\mathcal{C}[z]))$, then it is obvious. Otherwise assume that such occurrence is the one substituted for z in $\mathcal{B}(a(\mathcal{C}[z]))$. By our previous assumptions such occurrence is determined by the path θ_0 and in θ_0 there are t_0 occurrences of a in functional position. Therefore, $m_r^{\mathcal{B}(a(\mathcal{C}[b_r]))} = 2^{t_0} * r$.

Now recall that r is the maximum constant index of aM_2, and consider any constant b_s occurring in $\mathcal{B}(aM_2)$. Since aM_2 is a subterm of aM_1, and considering the occurrences of a in functional position in the path θ_0, we have that $m_s^{\mathcal{B}(aM_1)}$ is not lesser than $m_s^{\mathcal{B}(aM_2)} * 2^{t_0}$ and moreover $m_s^{\mathcal{B}(aM_2)} * 2^{t_0}$ is a factor of $m_s^{\mathcal{B}(aM_1)}$. Since $m_s^{\mathcal{B}(aM_1)}$ divides k, it follows that $m_s^{\mathcal{B}(aM_2)} * 2^{t_0}$ divides k. Now r is the LCM of the values $m_s^{\mathcal{B}(aM_2)}$, as b_s varies on the constants occurring in $\mathcal{B}(aM_2)$. Therefore, using the following elementary property of the function LCM:

if $p, q, r_1, \ldots, r_u$ are natural numbers and for every i, with $1 \leqslant i \leqslant u$, pr_i divides q then $pLCM(r_1, \ldots, r_u)$ divides q

it follows that $r * 2^{t_0}$ divides k.

We have thus proved the claim. From the claim, the Proposition is immediate.

□

Proposition 17. $\mathcal{R}^*_{A,b^*}$ commutes with $\mathcal{R}^*_{A,c}$ and β-reduction.

Proof. First, we prove that $\mathcal{R}_{A,b^*}$ commutes with $\mathcal{R}_{A,c}$.

Let $\Delta_1 \equiv aM_1$ and $\Delta_2 \equiv aM_2$ be a $\mathcal{R}_{A,c}$-redex and, respectively, a $\mathcal{R}_{A,b^*}$-redex occurring inside a term M.

If they are disjoint, then it is immediate.

Otherwise, consider first the case where $\Delta_1 \equiv aM_1 \equiv a(\mathcal{C}[aM_2])$, for some context $\mathcal{C}[\ ]$. Performing the internal reduction we get $a(\mathcal{C}[b_n])$, for some n. Now, we have that $\mathcal{B}(aM_1)$ does not contain any occurrence of a constant b_n, for $n = 1, 2 \ldots, \infty$. Moreover by Lemma 1 $\mathcal{B}(aM_1)$ is obtained from $\mathcal{B}(a(\mathcal{C}[z]))$, where z is a free variable not occurring in aM_1, by replacing every occurrence of z with the tree $\mathcal{B}(aM_2)$. Therefore, there is no occurrence of z in $\mathcal{B}(a(\mathcal{C}[z]))$ and therefore

$$\mathcal{B}(a(\mathcal{C}[b_n])) = \mathcal{B}(a(\mathcal{C}[z])) = \mathcal{B}(a(\mathcal{C}[aM_2])).$$

It follows that $a(\mathcal{C}[b_n]) \longrightarrow_{\mathcal{R}_{A,c}} c$.

Then, consider the case where $\Delta_2 \equiv aM_2 \equiv a(\mathcal{C}[aM_1])$, for some context $\mathcal{C}[\ ]$.

Reducing aM_2, we get the constant b_k, where k is the maximum constant index of aM_2. On the other hand, performing the internal reduction we get $a(\mathcal{C}[c])$.

Claim.

$$a(\mathcal{C}[c]) \longrightarrow_{\mathcal{R}_{A,b^*}} b_k.$$

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By Lemma 1, $\mathcal{B}(aM_2)$ is obtained from $\mathcal{B}(a(\mathcal{C}[z]))$, where z is a free variable not occurring in aM_2, by replacing every occurrence of z with the tree $\mathcal{B}(aM_1)$. On the other hand, in $\mathcal{B}(aM_1)$ there is not any occurrence of a constant b_n, for $n = 1, 2 \ldots, \infty$. It follows that, if we replace, in $\mathcal{B}(a(\mathcal{C}[aM_1]))$, every occurrence of aM_1 with the constant c, the computation of the maximum constant index is clearly not affected, as we replace a tree without occurrences of constants b_n, $n = 1, 2 \ldots, \infty$, with a tree that still does not contain such constants. The claim follows.

It remains to prove that $\mathcal{R}^*_{A,b^*}$ commutes with β-reduction.

The proof is immediate. As the $\mathcal{R}_{A,b^*}$-redexes are closed terms they can only be permuted, erased or duplicated by β-reductions. On the other hand, after a β-reduction is performed inside a $\mathcal{R}_{A,b^*}$-redex aM, the resulting term has the same Böhm tree as aM. The computation of the maximum constant index is therefore not affected. This completes the proof. □

The theory T_{A,c,b^*,b_∞} is obtained by adding to T_{A,c,b^*} the following (non-constructive) reduction rule:

Rule $\mathcal{R}_{A,b_\infty}$:

$$aM \longrightarrow_{\mathcal{R}_{A,b_\infty}} b_\infty.$$

where M is a closed term such that $\mathcal{B}(M)$ satisfies one of the following conditions:

1. $\mathcal{B}(M)$ is infinite, or
2. the constant b_∞ occurs in $\mathcal{B}(M)$.

Proposition 18. $\mathcal{R}_{A,b_\infty}$ is CR.

Proof. We show that $\mathcal{R}_{A,b_\infty}$ commutes with itself. Let $\Delta_1 \equiv aM_1$ and $\Delta_2 \equiv aM_2$ be two redexes of $\mathcal{R}_{A,b_\infty}$ occurring inside a term M. If they are disjoint, then it is immediate. Otherwise, we have that, say, $\Delta_1 \equiv aM_1 \equiv a(\mathcal{C}[aM_2])$, for some context $\mathcal{C}[\ ]$. Performing the internal reduction, we get $a(\mathcal{C}[b_\infty])$. On the other hand, reducing aM_1 we get the constant b_∞.

Claim.

$$a(\mathcal{C}[b_\infty]) \longrightarrow_{\mathcal{R}_{A,b_\infty}} b_\infty.$$

To prove the claim, observe that since aM_1 is a $\mathcal{R}_{A,b_\infty}$ redex, either $\mathcal{B}(aM_1)$ is infinite or the constant b_∞ occurs in $\mathcal{B}(aM_1)$.

Assume first that $\mathcal{B}(aM_1)$ is infinite.

Now, by Lemma 1, $\mathcal{B}(aM_1)$ is obtained from the Böhm tree of $a(\mathcal{C}[z])$, by replacing every node labelled by z with the Böhm tree of aM_2.

Led by z with the Böhm tree of aM_2.

If every infinite path in $\mathcal{B}(aM_1)$ ends in an occurrence of $\mathcal{B}(aM_2)$, this means that z occurs in $\mathcal{B}(a(\mathcal{C}[z]))$. Therefore, b_∞ occurs in $\mathcal{B}(a(\mathcal{C}[b_\infty]))$ and the claim follows. On the other hand, if some infinite path does not end in an occurrence of $\mathcal{B}(aM_2)$, this implies that $\mathcal{B}(a(\mathcal{C}[b_\infty]))$ is infinite, and also in this case the claim follows.

Assume now that b_∞ occurs in $\mathcal{B}(aM_1)$.

As before, $\mathcal{B}(aM_1)$ is obtained from the Böhm tree of $a(\mathcal{C}[z])$, by replacing every node labelled by z with the Böhm tree of aM_2.

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If every occurrence of b_∞ is in an occurrence of $\mathcal{B}(aM_2)$, this means that z occurs in $\mathcal{B}(a(\mathcal{C}[z]))$. Therefore, b_∞ occurs in $\mathcal{B}(a(\mathcal{C}[b_\infty]))$ and the claim follows. On the other hand, if some occurrence of b_∞ is not in an occurrence of $\mathcal{B}(aM_2)$, this implies that b_∞ occurs in $\mathcal{B}(a(\mathcal{C}[b_\infty]))$, and also in this case the claim follows. □

Proposition 19. $\mathcal{R}^*_{A,b_\infty}$ commutes with $\mathcal{R}^*_{A,c}$, $\mathcal{R}^*_{A,b^*}$ and β-reduction.

Proof. First, we prove that $\mathcal{R}_{A,b_\infty}$ commutes with $\mathcal{R}_{A,c}$.
Let $\Delta_1 \equiv aM_1$ and $\Delta_2 \equiv aM_2$ be a $\mathcal{R}_{A,c}$-redex and, respectively, a $\mathcal{R}_{A,b_\infty}$-redex occurring inside a term M.
If they are disjoint, then it is immediate.

Otherwise, consider first the case where $\Delta_1 \equiv aM_1 \equiv a(\mathcal{C}[aM_2])$, for some context $\mathcal{C}[\ ]$. Performing the internal reduction, we get $a(\mathcal{C}[b_\infty])$. Now, we have that $\mathcal{B}(aM_1)$ is finite and does not contain any occurrence of the constant b_∞. Moreover by Lemma 1, $\mathcal{B}(aM_1)$ is obtained from $\mathcal{B}(a(\mathcal{C}[z]))$, where z is a free variable not occurring in aM_1, by replacing every occurrence of z with the tree $\mathcal{B}(aM_2)$. Since $\mathcal{B}(aM_2)$ is either infinite or contains some occurrence of the constant b_∞, it follows that there is no occurrence of z in $\mathcal{B}(a(\mathcal{C}[z]))$ and therefore

$$\mathcal{B}(a(\mathcal{C}[b_\infty])) = \mathcal{B}(a(\mathcal{C}[z])) = \mathcal{B}(a(\mathcal{C}[aM_2])).$$

It follows that $a(\mathcal{C}[b_\infty]) \longrightarrow_{\mathcal{R}_{A,c}} c$.

Then, consider the case where $\Delta_2 \equiv aM_2 \equiv a(\mathcal{C}[aM_1])$, for some context $\mathcal{C}[\ ]$.

Reducing aM_2, we get the constant b_∞. On the other hand, performing the internal reduction we get $a(\mathcal{C}[c])$.

Claim.

$$a(\mathcal{C}[c]) \longrightarrow_{\mathcal{R}_{A,b_\infty}} b_\infty.$$

Now $\mathcal{B}(M)$ is infinite, or the constant b_∞ occurs in $\mathcal{B}(M)$.
On the other hand, by Lemma 1 $\mathcal{B}(aM_2)$ is obtained from $\mathcal{B}(a(\mathcal{C}[z]))$, where z is a free variable not occurring in aM_2, by replacing every occurrence of z with the tree $\mathcal{B}(aM_1)$. Moreover, $\mathcal{B}(aM_1)$ is finite and does not contain any occurrence of the constant b_∞. It follows that either $\mathcal{B}(a(\mathcal{C}[z]))$ is infinite, or the constant b_∞ occurs in $\mathcal{B}(a(\mathcal{C}[z]))$. Therefore if we replace, in $\mathcal{B}(a(\mathcal{C}[z]))$, every occurrence of z with the constant c then the resulting tree either is infinite or contains an occurrence of b_∞. The claim follows.

The proof that $\mathcal{R}_{A,b_\infty}$ commutes with $\mathcal{R}_{A,b^*}$ is very similar to the previous one and left to reader.

It remains to prove that $\mathcal{R}^*_{A,b_\infty}$ commutes with β-reduction.

The proof is immediate. As the $\mathcal{R}_{A,b_\infty}$-redexes are closed terms they can only be permuted, erased or duplicated by β-reductions. On the other hand, after a β-reduction is performed inside a $\mathcal{R}_{A,b_\infty}$-redex aM, the resulting term has the same Böhm tree as aM. Therefore, it remains a $\mathcal{R}_{A,b_\infty}$-redex. This completes the proof. □

By the previous results, we have obtained.

Proposition 20. The reduction system composed by β-reduction and rules $\mathcal{R}_{A,c}$, $\mathcal{R}_{A,b^*}$ and $\mathcal{R}_{A,b_\infty}$ is CR.

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As a corollary, we obtain

Proposition 21. In the theory T_{A,c,b^*,b_∞}:

1. the constants c, $b_1, b_2, \ldots$ and b_∞ are all pairwise different;
2. the constant a has infinite range;
3. the constant a has only two fixed points:
   the constant c and the constant b_∞.

4.1. *The range property*

Now, we want to prove that in the theory T_{A,c,b^*,b_∞} the range property holds for every term.

We need some preliminary results. For shortness, in the following we refer to the rules $\mathcal{R}_{A,c}$, $\mathcal{R}_{A,b^*}$ and $\mathcal{R}^*_{A,b_\infty}$ as the *constant rules*.

Proposition 22 (Postponement of constant rules). Assume that σ is a sequence of reductions in $T_{b,c}$ which reduces a term M to a term N. Then, there exists a term P such that

$$M \longrightarrow^*_\beta P \longrightarrow^*_{\mathcal{R}} N,$$

where by $\longrightarrow^*_{\mathcal{R}}$ we denote an arbitrary sequence of constant rules.

Proof. By the previous results on reciprocal commutativity of the constant rules w.r.t. the β-rule. □

The following proposition states that some kinds of indices cannot be generated by the application of the rules.

Proposition 23. Let M be a term and let p be a prime number, with $p > 2$, such that for every constant b_n occurring in M, p does not divide n. Then, for every reduct N of M and every constant b_m occurring in N, p does not divide m.

Proof. It is immediately seen that no new primes $p > 2$ can be generated by the application of the rules. □

Now we want to prove, roughly speaking, that if a term M can eliminate one or several constants with some kinds of indices then it must be *generic* on every constant. We start with a special case.

Proposition 24. Let M be a term of the form $\lambda x.aN$, where the only free variable possibly occurring in N is x. Let p be a prime number, with $p > 2$, such that for every constant b_n occurring in M, p does not divide n. Assume moreover that $Mb_p =_{T_{A,c,b^*,b_\infty}} Q$ where for every constant b_m occurring in Q, p does not divide m. Then for every ξ, with ξ varying on $c, b_1, \ldots, b_n, \ldots, b_\infty$, $M\xi =_{T_{A,c,b^*,b_\infty}} Q$.

Proof. First, we can assume that the variable x occurs in $\mathcal{B}(aN)$ for otherwise the result is obvious. Moreover, if $\mathcal{B}(aN)$ is infinite, then for every constant ξ, $\mathcal{B}(aN[x := \xi])$ is still infinite and therefore $M\xi =_{T_{A,c,b^*,b_\infty}} b_\infty$. For the same reason, b_∞ cannot occur in $\mathcal{B}(aN)$.

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However in the remaining case, $Mb_p =_{T_{A,c,b^*,b_\infty}} b_m$ for some m such that p divides m. By the Church–Rosser property, this contradicts the Proposition 23. The proposition follows. □

Actually, we need stronger forms of the previous proposition. First, we consider the case of several constants.

Proposition 25. Let M be a term of the form $\lambda x_1 \dots x_n.aN$, where the only free variables possibly occurring in N are in the list $x_1, \dots, x_n$. Let p be a prime number, with $p > 2$, such that for every constant b_m occurring in M, p does not divide m. Let $r_1, \dots, r_n$ be multiples of the prime p. If $Mb_{r_1} \dots b_{r_n} =_{T_{A,c,b^*,b_\infty}} Q$ where for every constant b_m occurring in Q, p does not divide m, then for every $\xi_1, \dots, \xi_n$, with each ξ_j, with $1 \leqslant j \leqslant n$, varying on $c, b_1, \dots, b_t, \dots, b_\infty$,

$$M\xi_1 \dots \xi_n =_{T_{A,c,b^*,b_\infty}} Q.$$

Proof. Essentially the same argument of the proof of the previous proposition. □

Finally, we remove the restriction on the form of the term.

Proposition 26. Let M be a term and let p be a prime number, with $p > 2$, such that for every constant b_m occurring in M, p does not divide m. Let $r_1, \dots, r_n$ be multiples of the prime p. If $Mb_{r_1} \dots b_{r_n} =_{T_{A,c,b^*,b_\infty}} Q$ where for every constant b_m occurring in Q, p does not divide m, then for every $\xi_1, \dots, \xi_n$, with each ξ_j, with $1 \leqslant j \leqslant n$, varying on $c, b_1, \dots, b_t, \dots, b_\infty$,
$M\xi_1 \dots \xi_n =_{T_{A,c,b^*,b_\infty}} Q$.

Proof. Suppose that $Mb_{r_1} \cdots b_{r_n} =_{T_{A,c,b^*,b_\infty}} Q$. Then, by Church–Rosser and postponement of constant rules, there exist L, N such that

$$Mb_{r_1} \cdots b_{r_n} \longrightarrow_\beta^* L \longrightarrow_{\mathcal{R}}^* N \ {}^*\!\longleftarrow Q.$$

We may write $L = L[b_{r_1}, \dots b_{r_n}]$. Since the constants behave as variables w.r.t. β-reduction $Mx_1 \dots x_n \longrightarrow_\beta L[x_1, \dots, x_n]$, for fresh variables $x_1, \dots, x_n$.

Observe that in any reduct of Q, p does not divide the index of any b_m. This follows from Proposition 23. Therefore, no finite multiple of p can occur as an index in N. This is possible only if in $L[x_1, \dots, x_n]$ every occurrence of variables $x_1, \dots, x_n$ lies in a subterm of $L[x_1, \dots, x_n]$ of the form $a(X[x_1, \dots, x_n])$, where the only free variables in $X[x_1, \dots, x_n]$ are in the list $x_1, \dots, x_n$ and for every such subterm $a(X[x_1, \dots, x_n])$, $a(X[b_{r_1}, \dots, b_{r_n}])$ reduces to a constant $\bar{\xi}$ which is not of the form b_m, with m multiple of p.

Now observe that to the subterm $a(X[x_1, \dots, x_n])$, we can apply Proposition 25 so that for every $\xi_1, \dots, \xi_n$, with each ξ_j, with $1 \leqslant j \leqslant n$, varying on $c, b_1, \dots, b_t, \dots, b_\infty$,

$$a(X[\xi_1, \dots, \xi_n]) =_{T_{A,c,b^*,b_\infty}} \bar{\xi}.$$

Now, let $\xi_1, \dots, \xi_n$ be arbitrary constants. As shown before

$$M\xi_1 \cdots \xi_n \longrightarrow_\beta^* L[\xi_1, \dots, \xi_n].$$

Moreover, every occurrence of $\xi_1, \dots, \xi_n$ lies in a subterm $a(X[\xi_1, \dots, \xi_n])$ with the properties stated above.

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It follows that each such subterm always returns the same value, so that $M\xi_1 \dots \xi_n$ always reduces to the same term. The proposition follows. □

Proposition 27. The range property holds in T_{A,c,b^*,b_∞}.

Proof. By contradiction, suppose for example that M has a range of size 2 say $\{P, Q\}$. At the end of proof, we show that if we start with a *finite* range of arbitrary size, greater than 1, the argument is the same.

Now let p be a large prime which does not divide the index of any b_m occurring in M, P or Q. Now, suppose that $Mb_p =_{T_{A,c,b^*,b_\infty}} P$. Then, there exist L, N such that

$$Mb_p \longrightarrow_\beta^* L \longrightarrow_{\mathcal{R}}^* N \ {}^*\!\longleftarrow P,$$

by Church–Rosser and postponement of constant rules. We may write $L = L[b_p]$ and $Mx \longrightarrow_\beta L[x]$.

Observe that in any reduct of P, p does not divide the index of any b_m. This follows from Proposition 23. Therefore, no finite multiple of p can occur as an index in N and this is possible only if in $L[x]$ every occurrence of x lies in a subterm of $L[x]$ of the form aX, where x is the only free variable in X.

Let $aX_1, \dots, aX_k$ be the subterms of $L[x]$ such that each aX_i is a maximal subterm of the form aX with the one free variable x. Therefore, we can represent $L[x]$ as a context of the form:

$$L[x] \equiv \mathcal{C}[aX_1, \dots, aX_k]. \tag{1}$$

Now, we consider separately each aX_i, with $i = 1, \dots, k$. We have different cases:

Case I. In $\mathcal{B}(aX_i)$, there is no occurrence of the variable x. In this case, $\lambda x.aX_i$ returns the same value Q_i for every input, so that we can freely replace aX_i with Q_i in the context $\mathcal{C}$ of (1).

Case II. If $\mathcal{B}(aX)$ has an infinite path which does not pass through some occurrence of x then for any closed M, the term $(aX_i)[x := M]$, with M replaced for x in aX_i, has an infinite Böhm tree and $\lambda x.aX_i$ is constantly b_∞. In this case we can freely replace aX_i with b_∞ in the context $\mathcal{C}$ of (1).

Case III. If in $\mathcal{B}(aX_i)$, there is a path to some occurrence of b_∞ which does not pass through some occurrence of x then for any closed M, the term $[M/x](aX_i)$, with M replaced for x in aX_i, is constantly b_∞. In this case, we can freely replace aX_i with b_∞ in the context $\mathcal{C}$ of (1).

Case IV. In this last case, the variable x occurs in $\mathcal{B}(aX_i)$. Moreover, every infinite path passes through some occurrence of x. Therefore, the set of x free paths is a finitely branching tree which by Koenig's lemma is finite. No maximal path in this tree can be extended so each next vertex in the Böhm tree must be x. There are only finitely many such occurrences of x and thus these occurrences of x have a bounded number of arguments t_i (for further references, we call t_i the *number of arguments of* aX_i). Moreover, every occurrence of b_∞ in $\mathcal{B}(aX_i)$ occurs in a subtree determined by one of such occurrences of x.

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It is easy to see that the cases considered are exhaustive. Moreover, up to the substitutions considered above in Case I, II and III, we can freely assume that in the context $\mathcal{C}$ of (1), every aX_i belongs to Case IV and the set of such aX_i is not empty. Let t be the maximum of the numbers t_i of arguments of aX_i, for $i = 1,\ldots,k$.

Consider the application of M to a term U of the form $\lambda x_1 \ldots x_t.b_p$. It follows that

$$MU =_{T_{A,c,b^*,b_\infty}} \mathcal{C}[b_{m_1},\ldots,b_{m_k}],$$

where every index m_i, with $i = 1,\ldots,k$, divides p. Now, either $MU =_{T_{A,c,b^*,b_\infty}} P$ or $MU =_{T_{A,c,b^*,b_\infty}} Q$. In both cases, we can apply Proposition 26 to conclude that there exists a term V such that:

$$\mathcal{C}[\xi_1 \cdots \xi_k] =_{T_{A,c,b^*,b_\infty}} V,$$

for every $\xi_1,\ldots,\xi_k$, with each ξ_j, with $1 \leqslant j \leqslant k$, varying on $c, b_1,\ldots,b_s,\ldots,b_\infty$.

On the other hand, for every closed term Z, $MZ =_{T_{A,c,b^*,b_\infty}} \mathcal{C}[\xi_1 \cdots \xi_k]$ for some constants $\xi_1,\ldots,\xi_k$, and we have reached a contradiction.

Observe that we have used the assumption that the range is composed by *two* terms, P and Q, only to be able to choose a large prime p which does not divide the index of any b_m occurring in M, P or Q. It is clear that a similar choice is always possible, if we start with a *finite* range of arbitrary size. Therefore the proposition follows. □

We state the following result concerning the degree of unsolvability of the theory T_{A,c,b^*,b_∞}. The proof, which follows the pattern of the proof of Proposition 12, is omitted for brevity.

Proposition 28. The theory T_{A,c,b^*,b_∞} is Σ_3^0-complete.

5. Conclusions and further research

The previous examples of λ-theories are of increasing unsolvability degrees. We do not know whether this is just a possibility or it is necessary to reach such degrees to obtain theories with the prescribed properties.

We think that, to understand this, some related questions are worth to be investigated. First of all, we do not know if in the closed term model of *Böhm Trees* (see Barendregt (1984) 18.3.10), every term has either one or an infinite number of fixed points. That is if, up to Böhm Trees equality, a term has always either one or an infinite number of fixed points. A positive answer would be, in our opinion, a strong hint that the same holds in the Term Model of β-convertibility.

Another natural question concerns the possibility of adapting Polonsky counterexample (see Polonsky (2012)) to have – in the λ-theory $\mathcal{H}$ – a term M with a range composed by exactly two terms each one a fixed point of M. In this case, one obtains the result given by Proposition 9 in a much more natural λ-theory. Interestingly enough, the unsolvabilty degree would be still a Σ_2^0-complete theory (see Barendregt (1984) 16.1.11).

We end by considering the question on how to obtain the results of the present paper without the use of constants. We conjecture that it is possible to obtain such results for pure λ-theories, by means of techniques similar to those used in Intrigila and Statman

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(2005). Informally speaking, the idea is to use terms with very little functional behaviour such as $\mathbf{\Omega}$ or $\mathbf{\Omega\Omega}$, to mimic the use of constants. The big difference is that now such terms can be generated by β-reductions; however making use of suitable modifications of Böhm Trees, as in Intrigila and Statman (2005), one can probably cope with such problem.

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