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Weak Gibbs measures and the local product structure

MICHIKO YURI

Department of Business Administration, Sapporo University, Nishioka, Toyohira-ku, Sapporo 062, Japan
(e-mail: yuri@math.sci.hokudai.ac.jp)

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Abstract. We establish a version of the local product structure (*weak local product structure*) for ergodic measures $\overline{\mu}$ which are the invertible extension of ergodic weak Gibbs measures μ invariant under piecewise C^0-invertible (infinite to one) Markov maps T. As a special case, $\overline{\mu}$ possesses asymptotically 'almost' local product structure in the sense of Barreira, Pesin and Schmeling. Under piecewise conformality of T and the existence of a piecewise smooth representation of the dual map of T, the weak local product structure allows one to show that the pointwise dimension of $\overline{\mu}$ exists almost everywhere and is the sum of the pointwise dimension of μ and the pointwise dimension of the dual of μ. Our results can be applicable to a natural extension of piecewise conformal two-dimensional Markov map which is related to a complex continued fraction and admits indifferent periodic points.

1. *Introduction*

In [**4**], a thermodynamic formalism was satisfactorily established for hyperbolic systems (cf. [**19**]). Bowen's program was based on the existence of a (finite) generating Markov partition and analysis of Ruelle–Perron–Frobenius operators associated to Hölder potentials (for subshifts of finite type). Furthermore, a characterization of Gibbs measures was given via a boundedness condition on ratios of measures of cylinder sets and exponentials of partial sums of the time evolution of the Hölder potentials. Although there is another characterization of Gibbs measures given by Capocaccia [**5**] which is more closely modelled on lattice systems in statistical mechanics (cf. [**19**]), we shall only be concerned with Bowen's definition in this paper. The classical approach to the thermodynamic formalism based on either Ruelle's Perron–Frobenius Theorem [**19**] or its generalization given by Walters [**23**] relies heavily on the summability of the variation of the potentials or the bounded distortion of local Jacobians with respect to conformal measures associated to the potentials. In fact, combining the uniform hyperbolicity and

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Hölder continuity of potentials allows one to have those properties which are sufficient for the unique equilibrium states to be Gibbs measures. Furthermore, the Gibbs property allows one to establish the so-called *local product structure* [**15**] which plays an important role in showing that equilibrium states are *exact dimensional* under the existence of stable and unstable pointwise dimensions [**12**].

Recently, a version of the Gibbs property (the so-called *weak Gibbs*) was proposed in [**28**], which is more suitable to a certain class of non-hyperbolic systems. In fact, the weak Gibbs property (see the definition in §3) can be verified for Sinai–Bowen–Ruelle (SBR) measures invariant under *intermittent maps*, i.e. piecewise C^1-smooth Markov maps T defined on bounded regions $X \subset \mathbb{R}^d$ which admit indifferent periodic points and can be countable-to-one maps (see [**32**]). Although the potentials $-\log|\det DT|$ satisfy neither summable variation nor the bounded distortion, the *weak bounded variation (WBV)* property (see the definition in §3) is valid, and we can show that the SBR measures are equilibrium states for $-\log|\det DT|$ and that the uniqueness of equilibrium states typically fails (see [**30, 34**]). Furthermore, the invariant densities of the weak Gibbs measures with respect to the normalized Lebesgue measures are typically unbounded at indifferent periodic points [**25, 27, 32**] and such singularities can be connected with the polynomial behaviour of the dynamical instability [**32**, Theorem 3.1]. We refer the reader to [**30, 32, 34**] which contain a generalized thermodynamic formalism in more general non-hyperbolic situations. Also we should mention that the singularities may cause interesting phenomena from a statistical point of view (see [**31**]) which are closely related to a common phenomenon in transitions to turbulence and other complex systems (the so-called *intermittency*) (cf. [**9, 11, 13, 14, 17, 21, 29, 33**]).

In this paper, we shall consider the *invertible extension* $(\overline{T}, \overline{\mu})$ of weak Gibbs measures μ invariant under piecewise C^0-invertible Markov maps T associated to potentials ϕ of WBV and show that $\overline{\mu}$ possesses a weak local product structure which coincides with the '*asymptotically almost local product structure*' introduced by Barreira *et al* in [**3**] under certain conditions. The asymptotically almost local product structure played an important role in [**3**] in the solution of the long-standing Eckmann–Ruelle conjecture in dimension theory (cf. [**7, 8**]) on the existence of stable and unstable pointwise dimensions proved by Ledrappier and Young in [**12**], i.e. every hyperbolic measure for $C^{1+\alpha}$ -diffeomorphisms of compact Riemannian manifolds is exact dimensional. On the other hand, although our systems are far from $C^{1+\alpha}$-diffeomorphisms, the *weak local product structure* (see the definition in §3) allows one to show that $\overline{\mu}$ is exact dimensional under certain conditions.

This paper is organized as follows. In §2, we collect definitions and previous results on piecewise C^0-invertible systems and their invertible extensions. In §3, we establish the weak local product structure for the invertible extension of weak Gibbs measures for potentials of WBV and compare our results with those of Barreira *et al* in [**3**]. The proofs of all results in this section are postponed until §7. In §4, we summarize the properties of weak Gibbs measures for potentials of WBV. In §5, we discuss a construction of jump transformations T^* which are uniformly expanding and possess invariant ergodic probability measures μ^* equivalent to conformal measures ν associated to ϕ. We can then establish a formula of the pointwise dimension of T-invariant ergodic measure μ equivalent to ν under piecewise conformality of T. In §6, we shall clarify sufficient

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conditions for the existence of the pointwise dimension of weak Gibbs measures for potentials of WBV and establish that the pointwise dimension of $\overline{\mu}$ is equal to the sum of the pointwise dimension of μ and the pointwise dimension of the dual of μ. In §8, we apply our results to a two-dimensional piecewise C^1-conformal Markov map with indifferent periodic points which arises from a complex continued fraction algorithm.

2. *Piecewise C^0-invertible systems and their invertible extensions*

Let $(T, X, Q = \{X_a\}_{a\in I})$ be a *piecewise C^0-invertible system*, i.e. X is a compact metric space, $T : X \to X$ is a non-invertible map which is not necessarily continuous, and $Q = \{X_a\}_{a\in I}$ is a countable disjoint partition of X such that $\bigcup_{a\in I} \operatorname{int} X_a$ is dense in X and satisfies the following properties.

(O1) For each $a \in I$ with $\operatorname{int} X_a \neq \emptyset$, $T|_{\operatorname{int} X_a} : \operatorname{int} X_a \to T(\operatorname{int} X_a)$ is a homeomorphism and $(T|_{\operatorname{int} X_a})^{-1}$ extends to a homeomorphism ψ_a on $cl(T(\operatorname{int} X_a))$.

(O2) $T\big(\bigcup_{\operatorname{int} X_a=\emptyset} X_a\big) \subset \bigcup_{\operatorname{int} X_a=\emptyset} X_a$.

(O3) $\{X_a\}_{a\in I}$ generates $\mathcal{F}$, the sigma algebra of Borel subsets of X.

Let a sequence $(a_1 \dots a_n) \in I^n$ satisfy $\operatorname{int}(X_{a_1} \cap T^{-1}X_{a_2} \cap \cdots \cap T^{-(n-1)}X_{a_n}) \neq \emptyset$. Then we define $X_{a_1\dots a_n} := X_{a_1} \cap T^{-1}X_{a_2} \cap \cdots \cap T^{-(n-1)}X_{a_n}$ which is called a cylinder of rank n and we say that the sequence $(a_1 \dots a_n)$ is *T-admissible*. By (O1), $T^n|_{\operatorname{int} X_{a_1\dots a_n}} : \operatorname{int} X_{a_1\dots a_n} \to T^n(\operatorname{int}(X_{a_1\dots a_n}))$ is a homeomorphism and $(T^n|_{\operatorname{int} X_{a_1\dots a_n}})^{-1}$ extends to a homeomorphism $\psi_{a_1} \circ \psi_{a_2} \circ \cdots \circ \psi_{a_n} = \psi_{a_1\dots a_n} : cl(T^n(\operatorname{int} X_{a_1\dots a_n})) \to cl(\operatorname{int} X_{a_1\dots a_n})$.

We impose on (T, X, Q) the next condition which gives a nice countable-state symbolic dynamics similar to sofic shifts (cf. [**24, 26–28**]).

Finite range structure (FRS). $\mathcal{U} := \{\operatorname{int}(T^n X_{a_1\dots a_n}) : \forall X_{a_1\dots a_n}, \forall n \geq 1\}$ consists of finitely many open subsets $U_1 \dots U_N$ of X.

In particular, we say that (T, X, Q) satisfies the Bernoulli property if $cl(T(\operatorname{int} X_a)) = X (\forall a \in I)$ so that $\mathcal{U} = \{\operatorname{int} X\}$ and (T, X, Q) satisfies the Markov property if $\operatorname{int}(cl(\operatorname{int} X_a) \cap cl(\operatorname{int} T X_b)) \neq \emptyset$ implies $cl(\operatorname{int} T X_b) \supset cl(\operatorname{int} X_a)$. A system (T, X, Q) satisfying the Bernoulli (Markov) property is called a piecewise C^0-invertible Bernoulli (Markov) system.

Let ν be a non-singular Borel probability measure on $\mathcal{F}$ which is non-atomic. As we will see in §5, even if $(T, X, Q = \{X_a\}_{a\in I})$ with FRS fails the bounded distortion property of local Jacobians with respect to ν, we can show that there exists a T-invariant σ-finite measure μ which is ergodic and absolutely continuous with respect to ν under a local bounded distortion (see [**2, 20, 24**]). Let us assume that μ is a probability measure. Under these situations, we can construct an invertible extension of (T, μ) as follows. Let Σ_- be the set of all semi-infinite sequences of $I^{\mathbb{N}}$, $\underline{a} = (a_{-1}, a_{-2}, \dots, a_{-n}, \dots)$ such that there exists a sequence of points in X, $\{x_{-n}\}_{n\geq 0}$, satisfying $Tx_{-n} = x_{-n+1}$ and $x_{-n} \in X_{a_{-n}}$ for all $n \geq 1$. Let $\overline{X}$ be a subset of $X \times \Sigma_-$ composed of all pairs $(x_0, \underline{a})$ and define $\overline{T} : \overline{X} \to \overline{X}$ by $\overline{T}(x_0, \underline{a}) = (Tx_0, v_{a_0}\underline{a})$ where $x_0 \in X_{a_0}$ and $v_{a_0}(a_{-1}, a_{-2}, \dots) = (a_0, a_{-1}, a_{-2}, \dots)$. It is known that the projection π_+ onto X commutes with the map T and $\overline{T}$, i.e. $\pi_+\overline{T} = T\pi_+$, and that there exists a unique ergodic invariant measure $\overline{\mu}$ on $\overline{X}$ whose image by π_+ is μ, i.e. $\overline{\mu}\pi_+^{-1} = \mu$ (see [**18**]). In particular, if (T, X, Q) satisfies

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the Markov property and admits a dual system $(T_-, X_-, Q_- = \{Y_a : a \in I\})$ which is a piecewise C^0-invertible system satisfying

$$(a_1, a_2, \ldots, a_n) \in I^n \text{ is } T\text{-admissible} \iff (a_n, a_{n-1}, \ldots, a_1) \in I^n \text{ is } T_-\text{-admissible},$$

then (T_-, X_-, Q_-) also satisfies the Markov property. Furthermore, by using the T-invariant probability measure μ on X, we can define a set function μ_- on the cylinders of (T_-, X_-, Q_-) by $\mu_-(Y_{a_n \ldots a_1}) = \mu(X_{a_1 \ldots a_n})$ and μ_- can be extended to a σ-additive set function which is T_--invariant (see [**20**, pp. 157–158]). We can construct naturally an invertible extension $\overline{T}$ of T on a subset $\overline{X}$ of the compact metric space $X \times X_-$ as follows. Let

$$\overline{X} := \bigcup_{a \in I} X_a \times T_- Y_a \left(= \bigcup_{a \in I} T X_a \times Y_a \right)$$

and define $\overline{T} : \overline{X} \to \overline{X}$ by

$$\overline{T}(x, y) = (T|_{X_a} x, \psi_{-,a} y) \quad \text{if } (x, y) \in X_a \times T_- Y_a,$$

where $\psi_{-,a} : T_- Y_a \to Y_a$ is the local inverse to $T_-|_{Y_a}$. Then $\overline{T}^{-1} : \overline{X} \to \overline{X}$ is given by

$$\overline{T}^{-1}(x, y) = (\psi_a x, T_-|_{Y_a} y) \quad \text{if } (x, y) \in T X_a \times Y_a.$$

Let $\pi_+ : X \times X_- \to X$ and $\pi_- : X \times X_- \to X_-$ be the natural projections. Then it is easy to see that $\pi_+ \overline{T} = T \pi_+$ and $\pi_- \overline{T}^{-1} = T_- \pi_-$.

We define

$$\overline{X_a} = \overline{X} \cap \pi_+^{-1} X_a \left(= \bigcup_{Y_b \subset T_- Y_a} (X_a \times Y_b) \right),$$

and

$$\overline{Y_a} = \overline{X} \cap \pi_-^{-1} Y_a \left(= \bigcup_{X_b \subset T X_a} (X_b \times Y_a) \right).$$

Then both $\overline{Q}_+ := \{\overline{X_a}\}_{a \in I}$ and $\overline{Q}_- := \{\overline{Y_a}\}_{a \in I}$ are countable generating partitions of $\overline{X}$. Let $Z_{(a,b)} := \overline{X} \cap (X_b \times Y_a) (= \overline{X} \cap \overline{X}_b \cap \overline{Y}_a)$. Then $\overline{Q} := \{Z_{(a,b)} : (a, b) \in I^2$ is T-admissible$\}$ is also a countable generating partition of $\overline{X}$. Since $\overline{X} (\subset X \times X_-)$ possesses the product metric naturally, all $(\overline{T}, \overline{X}, \overline{Q}_+)$, $(\overline{T}, \overline{X}, \overline{Q}_-)$ and $(\overline{T}, \overline{X}, \overline{Q})$ are piecewise C^0-invertible Markov systems which are invertible. Furthermore, if μ is ergodic then there exists a unique ergodic invariant measure $\overline{\mu}$ on $\overline{X}$ satisfying $\overline{\mu} \pi_+^{-1} = \mu$ and $\overline{\mu} \pi_-^{-1} = \mu_-$. (See [**18**].) Since $(\overline{T}^{-1}, \overline{\mu})$ is the natural extension of (T_-, μ_-), μ_- is ergodic, too.

3. *Weak Gibbs measures and weak local product structure*

Let $(T, X, Q = \{X_a\}_{a \in I})$ be a piecewise C^0-invertible system with FRS.

Definition. We say that ϕ is a potential of WBV if there exists a sequence of positive numbers $\{C_n\}_{n \geq 1}$ satisfying $\lim_{n \to \infty} (1/n) \log C_n = 0$ and $\forall n \geq 1, \forall X_{a_1 \ldots a_n} \in \bigvee_{i=0}^{n-1} T^{-i} Q$,

$$\frac{\sup_{x \in X_{a_1 \ldots a_n}} \exp\left(\sum_{i=0}^{n-1} \phi(T^i x) \right)}{\inf_{x \in X_{a_1 \ldots a_n}} \exp\left(\sum_{i=0}^{n-1} \phi(T^i x) \right)} \leq C_n.$$

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Remark A. If (T, X, Q) satisfies the Markov property and aperiodicity, and ϕ is a potential of weak bounded variation, then the following limit exists [**34**]: $P_{\text{top}}(T, \phi) := \lim_{n\to\infty} 1/n \log Z_n(X, \phi) \le \infty$, where

$$Z_n(X, \phi) := \sum_{(a_1 \dots a_n):\text{int}(TX_{a_n}) \supset \text{int}\, X_{a_1}} \left(\sum_{\psi_{a_1\dots a_n} x = x \in cl(\text{int}\, X_{a_1\dots a_n})} \exp\left[\sum_{h=0}^{n-1} \phi T^h(x) \right] \right).$$

Definition. A Borel probability measure ν is called a *weak Gibbs measure* for ϕ with a constant $-P$ if there exists a sequence $\{K_n\}_{n>0}$ of positive numbers with $\lim_{n\to\infty}(1/n) \log K_n = 0$ such that ν-a.e. x

$$K_n^{-1} \le \frac{\nu(X_{a_1\dots a_n}(x))}{\exp\left(\sum_{i=0}^{n-1} \phi T^i(x) - nP\right)} \le K_n,$$

where $X_{a_1\dots a_n}(x)$ denotes the cylinder containing x.

In [**32**], it was shown that typical piecewise C^1-smooth maps T with indifferent periodic points admit ergodic invariant weak Gibbs measures for $-\log|\det DT|$ which are absolutely continuous with respect to the Lebesgue measure and $-\log|\det DT|$ satisfies the WBV property. Also we can show that some examples of such maps admit the dual systems which are piecewise C^1-invertible Markov systems (see §8). In this section, first we show the following.

THEOREM 3.1. *Let $(T, X, Q = \{X_a\}_{a\in I})$ be a piecewise C^0-invertible Markov system with FRS. Let $\phi : X \to \mathbb{R}$ be a potential of WBV with $P_{\text{top}}(T, \phi) < \infty$. Suppose that μ is a weak Gibbs measure for ϕ with $-P_{\text{top}}(T, \phi)$ invariant under T. Then $\overline{\mu}$ is a weak Gibbs measure for $\pi_+^*\phi := \phi\pi_+ : \overline{X} \to \mathbb{R}$ with $-P_{\text{top}}(\overline{T}, \pi_+^*\phi)$ $(= -P_{\text{top}}(T, \phi))$.*

We denote the unique cylinder of rank $2(n + 1)$ with respect to $\overline{T}$ containing (x, y) by $\left(\bigcap_{i=-n}^{n} \overline{T}^{-i} Z_{(a_{i-1},a_i)}\right)(x, y)$ and we let $Y_{a_{-1}\dots a_{-(n+1)}}(y)$, $X_{a_0\dots a_n}(x)$ denote the unique cylinders of rank $n + 1$ with respect to T_-, T containing y, x, respectively. Then we can establish a key theorem.

THEOREM 3.2. *Suppose that all conditions in Theorem 3.1 are satisfied. Then there exists a sequence $\{K_n\}_{n>0}$ of positive numbers with $\lim_{n\to\infty}(1/n) \log K_n = 0$ such that $\overline{\mu}$-a.e. $(x, y) \in \overline{X}$*

$$\begin{aligned} K_n^{-1} &\le \frac{\overline{\mu}\left(\left(\bigcap_{i=-n}^{n} \overline{T}^{-i} Z_{(a_{i-1},a_i)}\right)(x, y)\right)}{\overline{\mu}\left(\left(\bigcap_{i=-n}^{-1} \overline{T}^{-i} Z_{(a_{i-1},a_i)}\right)(x, y)\right)\overline{\mu}\left(\left(\bigcap_{i=1}^{n} \overline{T}^{-i} Z_{(a_{i-1},a_i)}\right)(x, y)\right)} \\ &= \frac{\overline{\mu}(X_{a_0\dots a_n}(x) \times Y_{a_{-1}\dots a_{-(n+1)}}(y))}{\mu_-(Y_{a_{-1}\dots a_{-(n+1)}}(y))\mu(X_{a_0\dots a_n}(x))} \le K_n. \end{aligned}$$

In [**3**], the Eckmann–Ruelle conjecture in the dimension theory of smooth dynamical systems was proved by observing that any hyperbolic measures μ invariant under a $C^{1+\alpha}$ diffeomorphism possesses an *asymptotically 'almost' local product structure*, i.e. $\forall\delta > 0$, $\exists\Gamma \subset X$ with $\mu(\Gamma) > 1 - \delta$ and $k \ge 1$ such that $\forall x \in \Gamma, \forall r$ sufficiently small (depending on x):

$$r^\delta \mu_x^s(B^s(x, r/k))\mu_x^u(B^u(x, r/k)) \le \mu(B(x, r)) \le r^{-\delta}\mu_x^s(B^s(x, kr))\mu_x^u(B^u(x, kr)), \tag{1}$$

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where μ_x^s, μ_x^u denote the systems of conditional measures of μ defined on the leaves of stable and unstable manifolds, respectively. Barreira *et al* used (1) in [**3**] to show that μ is exact dimensional and its pointwise dimension d_μ is equal to the sum of the stable and unstable pointwise dimension, i.e. μ-a.e. $x \in X$

$$d_\mu(x) = \overline{d}_\mu(x) = \underline{d}_\mu(x) = d_\mu^s(x) + d_\mu^u(x). \tag{2}$$

For the natural extension $\overline{\mu}$ of the weak Gibbs measure μ invariant under the piecewise invertible system (T, X, Q), although $C^{1+\alpha}$-smoothness fails we come to the following weak notion of *local product structure* [**15**].

Definition. A Borel probability measure $\overline{\mu}$ is said to have *weak local product structure* if $\exists\{K_n\}_{n\geq 1}$ with $\lim_{n\to\infty} 1/n \log K_n = 0$ such that $\overline{\mu}$-a.e. $(x, y) \in \overline{X}$

$$K_n^{-1} \leq \frac{\overline{\mu}\big(\big(\bigcap_{i=-n}^{n} \overline{T}^{-i} Z_{(a_{i-1},a_i)}\big)(x, y)\big)}{\overline{\mu}\big(\big(\bigcap_{i=-n}^{-1} \overline{T}^{-i} Z_{(a_{i-1},a_i)}\big)(x, y)\big)\overline{\mu}\big(\big(\bigcap_{i=1}^{n} \overline{T}^{-i} Z_{(a_{i-1},a_i)}\big)(x, y)\big)} \leq K_n.$$

COROLLARY 3.1. *Suppose that all conditions in Theorem 3.1 are satisfied. Then* $\overline{\mu}$ *possesses weak local product structure.*

In the case of hyperbolic ergodic measures μ for $C^{1+\alpha}$-diffeomorphisms of a compact smooth Riemanian manifold X without boundaries, in [**12**] Young and Ledrappier constructed a special countable partition of X with finite entropy satisfying for all $0 < \epsilon < 1$, there exists $\Gamma \subset X$ with $\mu(\Gamma) > 1 - \epsilon/2$ and there exists $n_0 > 0$ such that for all $x \in \Gamma$, for all $n \geq n_0$ the diameters of cylinders of rank n containing x can be bounded from below and above by sequences decaying exponentially fast with rates determined by the Lyapunov exponents. In [**3**] Barreira *et al* essentially used such a partition to establish (2) and therefore the Eckmann–Ruelle conjecture. On the other hand, in our situation, not only $C^{1+\alpha}$-smoothness but also exponential estimates of sizes of cylinders with respect to the generating partition Q fail under the appearance of certain non-hyperbolic periodic orbits, e.g. indifferent periodic points cause polynomial decay of cylinder sizes. Even in such a non-standard situation, we will see in §6 that the weak local product structure allows one to establish an analogy of (2) for the natural extension of piecewise C^1-conformal Markov maps with indifferent periodic points. We should mention that the failure of bi-Lipschitz continuity of $T(T_-)$ does not allow us to apply Ledrappier–Young's approach in [**12**], although we have piecewise conformality. A new method that we shall use in this paper relies heavily on the existence of a derived system (the so-called *jump transformation*) which is uniformly expanding (see Lemma 6.2 in §6). As a special case we have an 'asymptotically almost local product structure' in the following sense.

COROLLARY 3.2. *Assume further that* $\exists 0 < \theta_1, \theta_2 < 1$ *and* $\exists 0 < \overline{C_i}, \underline{C_i} < \infty$ $(i = 1, 2)$ *satisfying*

$$\underline{C_1} \leq \frac{\operatorname{diam} X_{a_1 \dots a_n}(x)}{\theta_1^n} \leq \overline{C_1} \ (\mu \text{ a.e. } x \in X),$$

$$\underline{C_2} \leq \frac{\operatorname{diam} Y_{a_{-1} \dots a_{-n}}(y)}{\theta_2^n} \leq \overline{C_2} \ (\mu_- \text{ a.e. } y \in X_-).$$

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Then $\overline{\mu}$-a.e. $(x, y) \in \overline{X}$, $\forall r > 0$ sufficiently small (depending on (x, y)), $\exists k > 0$ satisfying (1) for all $\delta > 0$. This implies that $\overline{\mu}$ possesses asymptotically almost local product structure in the sense of Barreira et al.

For example, we can apply Corollary 3.2 to (countable) Markov shifts. We note that a lack of information on Lyapunov exponents of T and T_- causes a difficulty in choosing k uniformly in (x, y).

4. *Properties of weak Gibbs measures for potentials of WBV*

We recall Bowen's definition of Gibbs measures [**4**]. A Borel probability measure ν is called a Gibbs measure for ϕ with a constant $-P$ if there exists $K \geq 1$ such that $\forall x \in X$ and $\forall n \geq 1$,

$$K^{-1} \leq \frac{\nu(X_{a_1 \ldots a_n}(x))}{\exp\left(\sum_{i=0}^{n-1} \phi T^i(x) - nP\right)} \leq K,$$

where $X_{a_1 \ldots a_n}(x)$ denotes the cylinder containing x. In particular, we shall be interested in the case of $P = P_{\text{top}}(T, \phi)(< \infty)$. If ϕ is a potential of summable variation, then an $\exp[P_{\text{top}}(T, \phi) - \phi]$-conformal measure ν (i.e. the local Jacobian $(d\nu T/d\nu)|_{X_a}$ of ν is $\exp[P_{\text{top}}(T, \phi) - \phi]$ $\forall X_a \in Q$) satisfies the Gibbs property. Such a situation typically happens for uniformly expanding maps T and Hölder potentials ϕ. When neither the uniformly expanding property of T nor the summability of variations of ϕ hold, although the conformal measure ν fails the Gibbs property, the weak Gibbs property of ν still holds if ϕ is a potential of weak bounded variation [**32, 34**].

LEMMA 4.1. [**32**] *If $\phi : X \to \mathbb{R}$ satisfies $\text{Var}_n(\phi) \to 0$, then ϕ is a potential of WBV. In fact we can take that $C_n = \exp\left[\sum_{i=0}^{n-1} \text{Var}_i(\phi)\right]$.*

LEMMA 4.2. [**32**] *If ν is an $\exp[P_{\text{top}}(T, \phi) - \phi]$-conformal measure for a potential ϕ of WBV with $\{C_n\}_{n\geq 1}$, then ν is a weak Gibbs measure for ϕ with $-P_{\text{top}}(T, \phi)$ and the weak Gibbs sequence $\{K_n\}_{n\geq 1}$ is given by $\{C_n\}_{n\geq 1}$.*

Definition. Let ν be an $\exp[P_{\text{top}}(T, \phi) - \phi]$-conformal measure. We say that a periodic point x_0 with period q is an *indifferent periodic point with respect to ν* if $q^{-1} \sum_{i=0}^{q-1} \phi T^i(x_0) = P_{\text{top}}(T, \phi)$.

In particular, for piecewise C^1-smooth maps T and for the Lebesgue measure ν, the definition coincides with the usual one (i.e. $|\det DT^q(x_0)| = 1$). Summability of variations of potentials ϕ typically fails under the existence of indifferent periodic points.

LEMMA 4.3. *Let ν be an $\exp[P_{\text{top}}(T, \phi) - \phi]$-conformal measure and suppose that there exists an indifferent periodic point x_0 with respect to ν. Then $C_n \to \infty$ as $n \to \infty$. (Cf.* [**33**].*)*

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Proof. So as to simplify our argument, we assume that x_0 is a fixed point. Then we see that

$$\begin{aligned}
&\sup_{x,y\in X_{a_1\dots a_n}(x_0)} \frac{\exp\left[\sum_{i=0}^{n-1}(P_{\mathrm{top}}(T,\phi)-\phi)T^i(x)\right]}{\exp\left[\sum_{i=0}^{n-1}(P_{\mathrm{top}}(T,\phi)-\phi)T^i(y)\right]}\\
&\quad\geq \frac{\sup_{x\in X_{a_1\dots a_n}(x_0)}\exp\left[\sum_{i=0}^{n-1}(P_{\mathrm{top}}(T,\phi)-\phi)T^i(x)\right]}{\exp\left[\sum_{i=0}^{n-1}(P_{\mathrm{top}}(T,\phi)-\phi)T^i(x_0)\right]}\\
&\quad= \frac{\sup_{x\in X_{a_1\dots a_n}(x_0)}\exp\left[\sum_{i=0}^{n-1}(P_{\mathrm{top}}(T,\phi)-\phi)T^i(x)\right]}{\exp[n(P_{\mathrm{top}}(T,\phi)-\phi(x_0))]}\\
&\quad\geq \frac{\int_{X_{a_1\dots a_n}(x_0)}\exp\left[\sum_{i=0}^{n-1}(P_{\mathrm{top}}(T,\phi)-\phi)T^i(z)\right]d\nu(z)}{\nu(X_{a_1\dots a_n}(x_0))\exp[n(P_{\mathrm{top}}(T,\phi)-\phi(x_0))]}\\
&\quad= \frac{\nu(T^nX_{a_1\dots a_n}(x_0))}{\nu(X_{a_1\dots a_n}(x_0))}\exp[-n(P_{\mathrm{top}}(T,\phi)-\phi(x_0))].
\end{aligned}$$

Then the required fact immediately follows. □

PROPOSITION 4.1. [**30**, Theorem 2.2] *Let ν be a weak Gibbs measure for ϕ with $-P_{\mathrm{top}}(T,\phi)$. If there exists a T-invariant ergodic probability measure μ equivalent to ν with $H_\mu(Q)<\infty$ and $\phi\in L^1(\mu)$, then $P_{\mathrm{top}}(T,\phi)=h_\mu(T)+\int_X\phi\,d\mu$.*

Remark B. If ν is a weak Gibbs measure for ϕ with $-P_{\mathrm{top}}(T,\phi)$, then cylinder measures decay exponentially fast at any fixed point x with $P_{\mathrm{top}}(T,\phi)>\phi(x)$. On the other hand, if T admits an indifferent fixed point x_0, then the weak Gibbs sequence $\{K_n^{-1}\}_{n\geq1}$ gives lower bounds of cylinder measures at x_0, i.e. $\nu(X_{a_1\dots a_n}(x_0))\geq K_n^{-1}$, which may imply subexponential decay of cylinder measures at x_0. The slowness may be related to the slow decay of the correlation functions with respect to T-invariant ergodic measures μ equivalent to ν. (Cf. [**9, 11, 13, 17, 29, 33**].)

Remark C. Let μ be a T-invariant weak Gibbs measure for ϕ with $-P$ and the weak Gibbs sequence be $\{K_n\}_{n\geq1}$. Then

$$\frac{1}{n}\log\frac{\mu(X_{a_1\dots a_n}(x))}{\exp\left(\sum_{i=0}^{n-1}\phi T^i(x)-nP\right)}$$

converges to zero uniformly on a set of full measures with respect to μ; we therefore have uniform convergence of

$$\frac{1}{n}\log\mu(X_{a_1\dots a_n}(x))-\frac{1}{n}\sum_{i=0}^{n-1}\phi T^i(x)$$

to $-P$ on a set of full measures and the rate of convergence is $1/n\log K_n$.

In [**32**] and [**34**], under certain conditions the weak Gibbs property was established for T-invariant probability measures μ equivalent to the conformal measures ν associated to potentials ϕ of WBV. In particular, for piecewise C^1-smooth maps with indifferent periodic points, the equilibrium states μ for $-\log|\det DT|$, which are equivalent to the Lebesgue

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measures, are weak Gibbs measures with polynomial sequences $\{K_n\}_{n\geq 1}$. This implies that the convergence of

$$-\frac{1}{n}\log \mu(X_{a_1\dots a_n}(x)) + \frac{1}{n}\sum_{i=0}^{n-1}\phi T^i(x)$$

to the difference between the measure theoretical entropy and the measure theoretical Lyapunov exponent

$$h_\mu(T) - \int_X \log|\det DT|\, d\mu \ (= P_{\text{top}}(T, -\log|\det DT|) = 0)$$

is uniform on a set of full measures and the rates of convergence are given by $O(n^{-1}\log n)$. For example, the well-known Manneville–Pomeau maps defined by $T_\beta x = x + x^{1+\beta}$ $(x \in [0,1], 0 < \beta < 1)$ possess $K_n = (n+1)^{1+1/\beta}$ and the rate of decay of correlation functions is given by $n^{-(1/\beta-1)}$. In [**10**], we can find a relation between the rate of convergence in the pointwise ergodic theorem and the mixing rates. On the other hand, many numerical experimental results suggest slow convergence to Lyapunov exponents under the existence of non-hyperbolic periodic orbits. In general, the following questions arise naturally for weak Gibbs measures μ.

Question 1. When do each of the sequences $\{-n^{-1}\log\mu(X_{a_1\dots a_n}(x))\}$ and $\left\{n^{-1}\sum_{i=0}^{n-1}\phi T^i(x)\right\}$ converge uniformly on sets of full measure?

Question 2. Can the uniform convergence rates be described in terms of the sequence $\{1/n\log K_n\}_{n\geq 1}$?

(Cf. [**10**].)

5. *Jump transformations*

Let $(T, X, Q = \{X_a\}_{a\in I})$ be a piecewise C^0-invertible Markov system with FRS. Let ν be a non-atomic Borel probability measure on X which is non-singular with respect to T $(\nu T^{-1} \sim \nu)$ satisfying the *local invertibility*, i.e. $\forall X_a \in Q, \nu(X_a) > 0$ and $(d\nu T/d\nu)|_{X_a} > 0$ (ν-a.e.). For a constant $C \geq 1$, we define

$$\mathcal{R}(C, T, \nu) = \bigcup_{n=1}^{\infty}\left\{X_{a_1\dots a_n} \in \bigvee_{i=0}^{n-1} T^{-i}Q : \sup_{x,y\in X_{a_1\dots a_n}} \frac{(d\nu T^n/d\nu)|_{X_{a_1\dots a_n}}(x)}{(d\nu T^n/d\nu)|_{X_{a_1\dots a_n}}(y)} \leq C\right\}.$$

Suppose that $\exists C_+ \geq 1$ such that $\mathcal{R}(C_+, T, \nu) \neq \emptyset$ and satisfies the *strong playback property* (i.e. $X_{a_1\dots a_n b_1\dots b_m} \in \mathcal{R}(C_+, T, \nu)$ whenever $X_{b_1\dots b_m} \in \mathcal{R}(C_+, T, \nu)$). Let $R(x)$ be the stopping time with respect to $\mathcal{R}(C_+, T, \nu)$ defined by $R(x) := \inf\{n \in \mathbb{N} : X_{a_1\dots a_n}(x) \in \mathcal{R}(C_+, T, \nu)\}$. We call $T^* : \{x \in X : R(x) < \infty\} \to X$ defined by $T^*x = T^{R(x)}x$ the *jump transformation* over $\mathcal{R}(C_+, T, \nu)$. We define $D_n := \{x \in X : R(x) > n\}$ for $n \geq 0$ and $B_n = D_{n-1}\backslash D_n$ for $n \geq 1$. Let $X^* := X\backslash\bigcup_{m=0}^{\infty} T^{*-m}\left(\bigcap_{n\geq 0} D_n\right)$ and $I^* := \bigcup_{n\geq 1}\{(a_1\dots a_n) \in I^n : X_{a_1\dots a_n} \subset B_n\}$. Then $(T^*, X^*, Q^* = \{X_\alpha\}_{\alpha\in I^*})$ is a piecewise C^0-invertible Markov system satisfying the uniformly bounded distortion property with respect to $\nu|_{X^*}$ (i.e. $\mathcal{R}(C_+, T^*, \nu|_{X^*})$ is the set of all cylinders with respect to (T^*, X^*, Q^*) with all ranks). Let $\mathcal{U}^* := \{\text{int}(T^{*n}X_{\alpha_1\dots\alpha_n}) : \forall X_{\alpha_1\dots\alpha_n}, \forall n > 0\} = \{U_1^*, \dots, U_M^*\}$. We impose the next condition on (T^*, X^*, Q^*).

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Transitivity. $\operatorname{int} X^* = \bigcup_{k=1}^{M} U_k^*$ and $\forall l \in \{1, 2, \ldots, M\}, \exists 0 < s_l < \infty$ such that for all $k \in \{1, 2, \ldots, M\}$, U_k^* contains an interior of a cylinder $X^{(k,l)}(s_l)$ of rank s_l with respect to T^* such that $T^{*s_l}(\operatorname{int} X^{(k,l)}(s_l)) = U_l^*$.

Then applying [**24**] (cf. [**1, 2**]) gives a T^*-invariant finite measure μ^* equivalent to ν which is ergodic and exact. Suppose that $\nu\big(\bigcap_{n\geq 0} D_n\big) = 0$. Then Schweiger's formula

$$\mu(E) = \sum_{n=0}^{\infty} \mu^*(D_n \cap T^{-n}E) \quad (\forall E \in \mathcal{F}) \tag{3}$$

gives a T-invariant σ-finite ergodic measure μ which is equivalent to ν and exact. If $\sum_{n=0}^{\infty} \nu(D_n) < \infty$ then μ is finite. In particular, we shall be interested in the case when the reference measure ν is an $\exp[P_{\text{top}}(T, \phi) - \phi]$-conformal measure for a potential ϕ of WBV and T^* is uniformly expanding. Then the next condition imposed on ϕ guarantees the bounded distortion property for $d\nu T^*/d\nu$ under $B_{n+1} = D_n \cap T^{-n}B_1 (\forall n \geq 1)$ (cf. [**6**, **30–32**, **34**]).

Local bounded distortion (LBD). There exists $\theta > 0$ such that $\forall X_{a_1 \ldots a_n} \subset B_n \; \exists 0 < L_\phi(a_1 \ldots a_n) < \infty$ satisfying $\forall x, y \in T^n X_{a_1 \ldots a_n}$

$$|\phi(\psi_{a_1 \ldots a_n}(x)) - \phi(\psi_{a_1 \ldots a_n}(y))| \leq L_\phi(a_1 \ldots a_n) d(x, y)^\theta$$

and

$$\sup_{n>0} \sup_{X_{a_1 \ldots a_n} \subset B_n} \sum_{j=0}^{n-1} L_\phi(a_{j+1} \ldots a_n) < \infty.$$

In fact, the LBD property implies equi-Hölder continuity of $\{\phi^* \psi_\alpha^*\}_{\alpha \in I^*}$, where $\phi^*(x) = \sum_{i=0}^{R(x)-1} \phi T^i(x)$ and $\psi_\alpha{}^* = (T^*|_{X_\alpha})^{-1}$. Since T^* is uniformly expanding the standard argument allows one to establish the bounded distortion property for $d\nu T^*/d\nu = \exp[R P_{\text{top}}(T, \phi) - \phi^*]$. We recall the construction of uniformly expanding jump transformations in [**30**] (cf. [**32**]). Let $\{\psi_a\}_{a \in I}$ be a family of Lipschitz continuous maps with the Lipschitz constant $L_a \leq 1$. Given $L < 1$, we define $\mathcal{B}_1 := \{X_b : L_b \leq L\}$, $\mathcal{D}_1 := \{X_d : L_d > L\}$, and inductively define

$$\mathcal{B}_n := \{X_{b_1 \ldots b_n} : X_{b_1 \ldots b_{n-1}} \in \mathcal{D}_{n-1}, L_{b_n} \leq L\},$$
$$\mathcal{D}_n := \{X_{d_1 \ldots d_n} : X_{d_1 \ldots d_i} \in \mathcal{D}_i (i = 1 \ldots n-1), L_{d_n} > L\}.$$

Put $B_n := \bigcup_{X_{b_1 \ldots b_n} \in \mathcal{B}_n} X_{b_1 \ldots b_n}$ and $D_n := \bigcup_{X_{d_1 \ldots d_n} \in \mathcal{D}_n} X_{d_1 \ldots d_n}$. (We put $D_0 = X$.) Then $\mathcal{R} := \{X_{a_1 \ldots a_n} : X_{a_n} \in \mathcal{B}_1\}$ coincides with $\mathcal{R}(C_+, T, \nu)$ for a constant $C_+ > 1$. Furthermore, the strong playback property and $B_{n+1} = D_n \cap T^{-n}B_1$ are satisfied. As we will see in §8, for typical piecewise C^1-invertible maps with indifferent periodic points, $\mathcal{D}_n$ consists of cylinders of rank n touching indifferent periodic points and $\mathcal{B}_n$ consists of cylinders of rank n away from those periodic points.

We assume that μ is normalized and satisfies $H_\mu(Q) < \infty$. Furthermore, if $\phi \in L^1(\mu)$, then we can establish Proposition 4.1. The next result gives sufficient conditions for $H_\mu(Q) < \infty$ and $\phi \in L^1(\mu)$.

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PROPOSITION 5.1. *Let ϕ be a potential of WBV and let ν be an* $\exp[P_{\text{top}}(T,\phi)-\phi]$-*conformal measure on X which is positive on open sets. Suppose that $\sharp\mathcal{D}_1<\infty$ and, for $X_d\in\mathcal{D}_1$, $\text{int}(TX_d\cap B_1)\neq\emptyset$. If $\phi\in L^1(\nu)$ and*

$$C'=\sup_{n\geq 1}\sup_{X_{a_1\dots a_n}\in\bigvee_{i=0}^{n-1}T^{-i}Q}\sup_{x,y\in B_1\cap T^nX_{a_1\dots a_n}}\frac{\exp\left[\sum_{i=0}^{n-1}\phi T^i(\psi_{a_1\dots a_n}(x))\right]}{\exp\left[\sum_{i=0}^{n-1}\phi T^i(\psi_{a_1\dots a_n}(y))\right]}<\infty, \quad (4)$$

then $H_\mu(Q)<\infty$ and $\phi\in L^1(\mu)$. (Cf. [**24, 25, 27**].*)*

LEMMA 5.1. [**34**, Lemma 17] *Let ϕ be a potential of WBV and ν be an* $\exp[P_{\text{top}}(T,\phi)-\phi]$-*conformal measure on X. Then $\exists 1\leq C_1<\infty$ such that $H_\nu(Q)<-\int_X\phi d\nu+\log C_1$.*

LEMMA 5.2. (Cf. [**32**]) *Suppose that all conditions in Proposition 5.1 are satisfied. Then $\exists 0<G<\infty$ such that $(d\mu/d\nu)|_{B_1}(x)\leq G$ and $\exists 0<M<\infty$ such that $\mu(X_a)/\nu(X_a)\leq M(\forall X_a\in Q)$.*

Proof of Lemma 5.2. Since we know a formula of the invariant density,

$$\frac{d\mu}{d\nu}(x)=\sum_{n=0}^{\infty}\sum_{X_{d(n)}\in\mathcal{D}_n}\frac{d(\nu\psi_{d(n)})}{d\nu}(x)1_{T^nX_{d(n)}}(x)\frac{d\mu^*}{d\nu}(\psi_{d(n)}x),$$

and $\exists 0<G'<\infty$ such that $G'^{-1}\leq d\mu^*/d\nu<G'$, condition (4) allows one to have for $x\in B_1$

$$\frac{d\mu}{d\nu}(x)\leq GC'\sum_{n=0}^{\infty}\sum_{X_{d(n)}\in\mathcal{D}_n,x\in T^nX_{d(n)}}\frac{\int_{T^nX_{d(n)}\cap B_1}(d(\nu\psi_{d(n)})/d\nu)(x)\,d\nu}{\nu(T^nX_{d(n)}\cap B_1)}.$$

By the Markov property of Q, we have

$$\begin{aligned}W&:=\inf\{\nu(T^nX_{d(n)}\cap B_1):T^nX_{d(n)}\cap B_1\neq\emptyset,X_{d(n)}\in\mathcal{D}_n,\forall n\geq 1\}\\&=\min\{\nu(TX_d\cap B_1):TX_d\cap B_1\neq\emptyset,X_d\in\mathcal{D}_1\}>0\end{aligned}$$

and so the desired upper bound is $G:=W^{-1}G'C'$. □

Combining Lemmas 5.1 and 5.2 gives the following.

LEMMA 5.3.
(i) *$H_\nu(Q)<\infty$ implies $H_\mu(Q)<\infty$.*
(ii) *$\phi\in L^1(\nu)$ implies $\phi\in L^1(\mu)$.*

Finally, Proposition 5.1 follows immediately from the above lemmas. If T is a piecewise C^1-conformal map on $X\subset\mathbb{R}^d$ and $|\det DT^*|$ satisfies the bounded distortion property (the local bounded distortion property of $|\det DT|$ is sufficient for this), then by [**32**, Corollary 5.2] the condition

$$\{x\in X:R(x)=1\}\neq\emptyset \quad\text{and}\quad R(Tx)=R(x)-1\ (R(x)\geq 2) \quad (5)$$

(equivalently $B_{n+1}=D_n\cap T^{-n}B_1$) allows one to establish μ-a.e. $x\in X$

$$d_\mu(x)=\overline{d}_\mu(x)=\underline{d}_\mu(x)=\frac{h_\mu(T)}{\int_X\log\|DT\|\,d\mu}. \quad (6)$$

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We shall discuss formula (6) in more detail in §6 and will see in §8 a higher-dimensional example of a piecewise C^1-invertible conformal Markov system (T, X, Q) satisfying (6) which is related to a complex continued fraction and possesses the dual system (T_-, X_-, Q_-) which is a piecewise C^1-invertible conformal system for which there exists a T_--invariant ergodic measure μ_- satisfying (6).

6. *Pointwise dimension for weak Gibbs measures*

Let (T, X, Q) be a piecewise C^0-invertible Markov system with FRS and $X \subset \mathbb{R}^d$. In this section, we establish an analogy of (2) for the invertible extension $(\overline{T}, \overline{\mu})$ of T-invariant ergodic weak Gibbs measure μ when T is a piecewise conformal map with $\log \|DT\| \in L^1(\mu)$ and $H_\mu(Q) < \infty$. Ledrappier and Young proved in [**12**] that the stable and unstable pointwise dimensions exist a.e. for any hyperbolic ergodic invariant measure of $C^{1+\alpha}$-diffeomorphisms of a smooth compact Riemannian manifold. In our situation, we impose all conditions in §5 not only on (T, X, Q) but also on its dual system (T_-, X_-, Q_-), so that d_μ and d_{μ_-} exist a.e. with respect to μ and μ_-, respectively. More specifically, let $B_{1,-} := \bigcup_{a\in I, X_a\subset B_1} Y_a$. Since $\mu_-(B_{1,-}) = \mu(B_1) > 0$, by the conservativity we can define the induced map $T_{B_{1,-}}$ over $B_{1,-}$ and the T_--invariant ergodic probability measure μ_- is given by Kac's formula as follows: $\forall E$ a Borel subset of X_-,

$$\begin{aligned}\mu_-(E) &= \int_{B_{1,-}} \sum_{i=0}^{R_{B_{1,-}}(y)-1} 1_E T_-^i(y)\, d\mu_-|_{B_{1,-}}(y)\\ &= \sum_{n=0}^{\infty} \mu_-|_{B_{1,-}}(\{y \in X_- : R_{B_{1,-}}(y) \ge n\} \cap T_-^{-n}E),\end{aligned}$$

where $R_{B_{1,-}}$ denotes the first return function over $B_{1,-}$. Let $R_- : X_- \to \mathbb{N} \cup \{\infty\}$ be a function defined by $R_-(y) = \inf\{n \ge 0 : T_-^n y \in B_{1,-}\} + 1$ and, for each $n > 1$, define inductively

$$B_{n,-} := \{y \in X_- : R_-(y) = n\}, \quad D_{n,-} := \{y \in X_- : R_-(y) > n\}.$$

Then we define the jump transformation $T_-^* : \bigcup_{n=1}^\infty B_{n,-} \to X_-$ by $T_-^*(y) = T_-^{R_-(y)}y$. Denote $X_-^* := X_-\backslash\big(\bigcup_{i=0}^\infty T_-^{*-i}(\{R_-(y) = \infty\})\big)$ and $I_-^* = \bigcup_{n\ge1}\{(a_1 \dots a_n) \in I^n : Y_{a_1\dots a_n} \subset B_{n,-}\}$. Then $(T_-^*, X_-^*, Q_-^* = \{Y_\gamma : \gamma \in I_-^*\})$ is a piecewise C^0-invertible Markov system with FRS. We can easily verify the following facts.

LEMMA 6.1.

(i) $R_{B_{1,-}} = R \circ T_-|_{B_{1,-}}$.

(ii) $R_-(T_-y) = R_-(y) - 1$.

(iii) $T_-|_{B_{1,-}} \circ T_{B_{1,-}} = T_-^* \circ T_-|_{B_{1,-}}$.

By Lemma 6.1, we see that $\mu_-^* := \mu|_{B_{1,-}}(T_-|_{B_{1,-}})^{-1}$ gives a T_-^*-invariant measure and we can establish Schweiger's formula (3) via Kac's formula. If T_-^* is uniformly expanding and $|\det DT_-|$ satisfies $\log|\det DT_-| \in L^1(\mu_-)$ and the local bounded distortion so that $|\det DT_-^*|$ satisfies the bounded distortion, then we can establish (6) for $(T_-, X_-, Q_-; \mu_-)$. By the definition of μ_-, $H_{\mu_-}(Q_-) < \infty$ follows from $H_\mu(Q) < \infty$. Then we have the following theorem.

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THEOREM 6.1. *Let $\overline{\mu}$ possess the weak local product structure. Then*

$$\overline{\mu}\text{-a.e. } (x, y) \in \overline{X}, \quad d_{\overline{\mu}}(x, y) = d_{\mu}(x) + d_{\mu_-}(y). \tag{2}^*$$

For the proof of the above theorem, we need a sequence of lemmas. First we remark that the pointwise dimension can be described in terms of cylinders.

LEMMA 6.2. *For μ-a.e. $x \in X$,*

$$d_{\mu}(x) = \lim_{n\to\infty} \frac{\log \mu(X_{a_1\dots a_n}(x))}{\log \operatorname{diam} X_{a_1\dots a_n}(x)} = \frac{h_{\mu}(T)}{\int_{X^*} \log \|DT^*\|\, d\mu^*}.$$

In particular, if (5) is valid then

$$d_{\mu}(x) = \frac{h_{\mu_0^*}(T^*)}{\int_{X^*} \log \|DT^*\|\, d\mu_0^*} = \frac{h_{\mu}(T)}{\int_X \log \|DT\|\, d\mu},$$

where $\mu_0^* := \mu^*/\mu^*(X^*)$.

To see this, define

$$N_n(x) := \max\left\{l \geq 0 : \sum_{i=0}^{l} R(T^{*i}x) \leq n\right\}, \quad L_n(x) := \min\left\{l \geq 0 : \sum_{i=0}^{l} R(T^{*i}x) \geq n\right\}.$$

We have the following fact.

LEMMA 6.3. *For μ^*-a.e. $x \in X^*$, $\lim_{n\to\infty} n/N_n(x) = \mu^*(X^*)^{-1}$.*

Proof. Since

$$\sum_{i=0}^{N_n+1} R(T^{*i}x)/N_n > n/N_n \geq \sum_{i=0}^{N_n} R(T^{*i}x)/N_n$$

and $\int_{X^*} R\, d\mu^* = 1$, the result follows from the ergodicity of (T^*, μ_0^*). □

LEMMA 6.4. *For μ^*-a.e. $x \in X^*$,*

$$\lim_{n\to\infty} \frac{\log \operatorname{diam} X_{\alpha_1\dots\alpha_{N_n}}(x)}{N_n(x)} = -\int_{X^*} \log \|DT^*\|\, d\mu_0^*.$$

Proof. First we note that

$$\inf_{z\in T^{*n}X_{\alpha_1\dots\alpha_{N_n}}} \|D\psi^*_{\alpha_1\dots\alpha_{N_n}}(z)\| \leq \operatorname{diam} X_{\alpha_1\dots\alpha_{N_n}} \leq \sup_{z\in T^{*n}X_{\alpha_1\dots\alpha_{N_n}}} \|D\psi^*_{\alpha_1\dots\alpha_{N_n}}(z)\|.$$

By using the piecewise conformality of T^* and the bounded distortion property of $|\det DT^*|$ we can observe that

$$C^{-1}\|DT^{*N_n}(x)\|^{-1} \leq \operatorname{diam} X_{\alpha_1\dots\alpha_{N_n}}(x) \leq C\|DT^{*N_n}(x)\|^{-1}.$$

This completes the proof. □

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Proof of Lemma 6.2. Since Q is generating, for each small $r > 0, \exists n(x) > 0$ such that $\exists k > 0$ with

$$X_{a_1 \dots a_{n(x)} a_{n(x)+1} \dots a_{n(x)+k}}(x) \subset B(x, r) \subset X_{a_1 \dots a_{n(x)}}(x),$$

and so

$$\frac{\log \mu(X_{a_1 \dots a_n}(x))}{\log \operatorname{diam} X_{a_1 \dots a_{n+k}}(x) - \log 2} \leq \frac{\log \mu(B(x, r))}{\log r} \leq \frac{\log \mu(X_{a_1 \dots a_{n+k}}(x))}{\log \operatorname{diam} X_{a_1 \dots a_n}(x) - \log 2}.$$

We note that

$$\operatorname{diam} X_{\alpha_1 \dots \alpha_{L_n}}(x) \leq \operatorname{diam} X_{a_1 \dots a_n}(x) \leq \operatorname{diam} X_{\alpha_1 \dots \alpha_{N_n}}(x).$$

Then by the Shannon–McMillan–Breiman Theorem and Lemmas 6.3 and 6.4 we see that for μ^*-a.e. $x \in X^*$

$$\begin{aligned}
&\limsup_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} \\
&\quad \leq \limsup_{n \to \infty} \frac{\log \mu(X_{a_1 \dots a_{n+k}}(x))}{\log \operatorname{diam} X_{a_1 \dots a_n}(x) - \log 2} \\
&\quad \leq \limsup_{n \to \infty} \frac{-\log \mu(X_{a_1 \dots a_{n+k}}(x))}{n+k} \left(\frac{n+k}{n} \right) \left(\frac{n}{N_n(x)} \right) \frac{N_n(x)}{-\log \operatorname{diam} X_{\alpha_1 \dots \alpha_{N_n}}(x)} \\
&\quad = \frac{h_\mu(T)}{\mu^*(X^*) \int_{X^*} \log \|DT^*\| \, d\mu_0^*}.
\end{aligned}$$

Similarly we can show that

$$\begin{aligned}
\liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} &\geq \liminf_{n \to \infty} \frac{\log \mu(X_{a_1 \dots a_n}(x))}{\log \operatorname{diam} X_{a_1 \dots a_{n+k}}(x) - \log 2} \\
&\geq \frac{h_\mu(T)}{\mu^*(X^*) \int_{X^*} \log \|DT^*\| \, d\mu_0^*}.
\end{aligned}$$

Hence we have

$$\lim_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} = \frac{h_\mu(T)}{\mu^*(X^*) \int_{X^*} \log \|DT^*\| \, d\mu_0^*}.$$

A similar argument allows one to establish

$$\lim_{n \to \infty} \frac{\log \mu(X_{a_1 \dots a_n}(x))}{\log \operatorname{diam} X_{a_1 \dots a_n}(x)} = \frac{h_\mu(T)}{\mu^*(X^*) \int_{X^*} \log \|DT^*\| \, d\mu_0^*}.$$

If (5) is satisfied, then by [**30**, Lemma 4.2], we have

$$\int_{B_1} \log |\det DT_{B_1}| \, d\mu_{B_1} = \int_{X^*} \log |\det DT^*| \, d\mu_0^*,$$

where T_{B_1} is the induced map over B_1 and $\mu_{B_1} = \mu|_{B_1}/\mu(B_1)$. Then by Kac's formula, $\mu(B_1)^{-1} = \int_{X^*} R \, d\mu_0^* = \mu(X^*)^{-1}$ and so

$$\int_{X^*} \log |\det DT^*| \, d\mu_0^* = \frac{\int_X \log |\det DT| \, d\mu}{\mu^*(X^*)}.$$

This completes the proof. □

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Proof of Theorem 6.1. Since $\overline{X} \subset X \times X_-$ provides the product metric of $X(\subset \mathbb{R}^d)$ and $X_-(\subset \mathbb{R}^d)$ naturally and $\overline{Q}$ is generating, we have that for each small $r > 0$ and $\forall (x, y) \in \overline{X}$, there exist $n(x, y) > 0$ and $k > 0$ such that

$$X_{a_0 \dots a_{n+k}}(x) \times Y_{a_{-1} \dots a_{-(n+k-1)}}(y) \subset B((x, y), r) \subset X_{a_0 \dots a_n}(x) \times Y_{a_{-1} \dots a_{-(n+1)}}(y).$$

Hence we see that

$$2r \geq \max\{\operatorname{diam} X_{a_0 \dots a_{n+k}}(x), \operatorname{diam} Y_{a_{-1} \dots a_{-(n+k+1)}}(y)\}$$

and

$$2r \leq \min\{\operatorname{diam} X_{a_0 \dots a_n}(x), \operatorname{diam} Y_{a_{-1} \dots a_{-(n+1)}}(y)\}.$$

Then the weak local product structure of $\overline{\mu}$ allows one to see that

$$\begin{aligned} \frac{\log \overline{\mu}(B((x, y), r))}{\log r} &\leq \frac{\log \overline{\mu}(X_{a_0 \dots a_{n+k}}(x) \times Y_{a_{-1} \dots a_{-(n+k+1)}}(y))}{\log \min\{\operatorname{diam} X_{a_0 \dots a_n}(x), \operatorname{diam} Y_{a_{-1} \dots a_{-(n+1)}}(y)\} - \log 2} \\ &\leq (7) + (8) + (9), \end{aligned}$$

where

$$(7) := \frac{-\log K_{n+k}}{\log \min\{\operatorname{diam} X_{a_0 \dots a_n}(x), \operatorname{diam} Y_{a_{-1} \dots a_{-(n+1)}}(y)\} - \log 2},$$

$$(8) := \frac{\log \overline{\mu} \pi_+^{-1}(X_{a_0 \dots a_{n+k}}(x))}{\log \min\{\operatorname{diam} X_{a_0 \dots a_n}(x), \operatorname{diam} Y_{a_{-1} \dots a_{-(n+1)}}(y)\} - \log 2},$$

and

$$(9) := \frac{\log \overline{\mu} \pi_-{}^{-1}(Y_{a_{-1} \dots a_{-(n+k+1)}}(y))}{\log \min\{\operatorname{diam} X_{a_0 \dots a_n}(x), \operatorname{diam} Y_{a_{-1} \dots a_{-(n+1)}}(y)\} - \log 2}.$$

Let $N_n^{(-)}$, $L_n^{(-)}$ be defined with respect to the dual system T_- as N_n, L_n with respect to T. Then we have that

$$\begin{aligned} &\min\{\operatorname{diam} X_{\alpha_1 \dots \alpha_{L_{n+1}(x)}}(x), \operatorname{diam} Y_{\gamma_{-1} \dots \gamma_{-L_{n+1}^{(-)}(y)}}(y)\} \\ &\quad \leq \min\{\operatorname{diam} X_{a_0 \dots a_n}(x), \operatorname{diam} Y_{a_{-1} \dots a_{-(n+1)}}(y)\} \\ &\quad \leq \min\{\operatorname{diam} X_{\alpha_1 \dots \alpha_{N_{n+1}(x)}}(x), \operatorname{diam} Y_{\gamma_{-1} \dots \gamma_{-N_{n+1}^{(-)}(y)}}(y)\}. \end{aligned}$$

By the above inequalities we obtain that

$$\begin{aligned} (8) &= \frac{\log \mu(X_{a_0 \dots a_{n+k}}(x))}{\log \min\{\operatorname{diam} X_{a_0 \dots a_n}(x), \operatorname{diam} Y_{a_{-1} \dots a_{-(n+1)}}(y)\} - \log 2} \\ &\leq \min\left\{ \frac{\log \mu(X_{a_0 \dots a_{n+k}}(x))}{\log \operatorname{diam} X_{a_0 \dots a_n}(x) - \log 2}, \frac{\log \mu(X_{a_0 \dots a_{n+k}}(x))}{\log \operatorname{diam} Y_{a_{-1} \dots a_{-(n+1)}}(y) - \log 2} \right\} \\ &\leq \frac{\log \mu(X_{a_0 \dots a_{n+k}}(x))}{\log \operatorname{diam} X_{a_0 \dots a_n}(x) - \log 2} \\ &\leq -\frac{1}{n+k+1} \log \mu(X_{a_0 \dots a_{n+k}}(x)) \frac{n+k+1}{n+1} \frac{n+1}{N_{n+1}(x)} \\ &\quad \times \frac{N_{n+1}(x)}{-\log \operatorname{diam} X_{\alpha_1 \dots \alpha_{N_{n+1}(x)}}(x) + \log 2}. \end{aligned}$$

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Similarly, we have that

$$(9) = \frac{\log \mu_-(Y_{a_{-1}\dots a_{-(n+k+1)}}(y))}{\log \min\{\operatorname{diam} X_{a_0\dots a_n}(x), \operatorname{diam} Y_{a_{-1}\dots a_{-(n+1)}}(y)\} - \log 2}$$
$$\le -\frac{1}{n+k+1} \log \mu_-(Y_{a_{-1}\dots a_{-(n+k+1)}}(y)) \frac{n+k+1}{n+1} \frac{n+1}{N^{(-)}_{n+1}(y)}$$
$$\times \frac{N^{(-)}_{n+1}(y)}{-\log \operatorname{diam} Y_{\gamma_{-1}\dots\gamma_{-N^{(-)}_{n+1}(y)}}(y) + \log 2}.$$

It follows from Lemmas 6.3 and 6.4 that the upper bounds of (8) and (9) converge to

$$h_\mu(T)\mu^*(X^*)^{-1}\left(\int_{X^*} \log \|DT^*\|\, d\mu^*_0\right)^{-1} = \frac{h_\mu(T)}{\int_X \log |DT|\, d\mu}$$

and

$$h_{\mu_-}(T_-)\mu^*_-(X^*_-)^{-1}\left(\int_{X^*_-} \log \|DT^*_-\|\, d\mu^*_{-0}\right)^{-1} = \frac{h_{\mu_-}(T_-)}{\int_{X_-} \log \|DT_-\|\, d\mu_-}$$

as $n \to \infty$, respectively. Hence the lim sup of (8) is bounded from above by $h_\mu(T)/\left(\int_X \log \|DT\|\, d\mu\right)$ and the lim sup of (9) is bounded from above by $h_{\mu_-}(T_-)/\left(\int_{X_-} \log \|DT_-\|\, d\mu_-\right)$. Since $\lim_{n\to\infty} n^{-1} \log K_n = 0$ and (7) ≥ 0, finally we establish

$$\limsup_{r\to 0} \frac{\log \overline{\mu}(B((x,y),r))}{\log r} \le \frac{h_\mu(T)}{\int_X \log \|DT\|\, d\mu} + \frac{h_{\mu_-}(T_-)}{\int_{X_-} \log \|DT_-\|\, d\mu_-}.$$

By Lemma 6.2, we obtain $\overline{d}_{\overline{\mu}}(x,y) \le d_\mu(x) + d_{\mu_-}(y)$. On the other hand, using L_n, $L^{(-)}_n$ instead of N_n, $N^{(-)}_n$ allows one to establish

$$\liminf_{r\to 0} \frac{\log \overline{\mu}(B((x,y),r))}{\log r} \ge \frac{h_\mu(T)}{\int_X \log \|DT\|\, d\mu} + \frac{h_{\mu_-}(T_-)}{\int_{X_-} \log \|DT_-\|\, d\mu_-}$$

similarly to the above statements. This completes the proof. □

Remark D. In order to establish Lemma 6.2, we used essentially the existence of the jump transformation which satisfies the bounded distortion property. On the other hand, for conformal maps T of which $-\log \|DT\|$ satisfies the WBV property, without any derived systems we can establish, for any ergodic T-invariant probability measure μ with $H_\mu(Q) < \infty$ and $\log \|DT\| \in L^1(\mu)$, that

$$\liminf_{n\to\infty} \frac{\log \mu(X_{a_1\dots a_n}(x))}{\log \operatorname{diam}(X_{a_1\dots a_n}(x))} \ge \frac{h_\mu(T)}{\int_X \log \|DT\|\, d\mu}.$$

7. *Proofs of the theorems in §3*

Proof of Theorem 3.1. First we note the following equalities:

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$$\frac{\overline{\mu}\big(\big(\bigcap_{i=-n}^{n}\overline{T}^{-i}Z_{(a_{i-1},a_i)}\big)(x,y)\big)}{\exp\big[\sum_{i=-(n+1)}^{n}\pi_+^*\phi\circ\overline{T}^i(x,y)\big]}=\frac{\overline{\mu}(\overline{T}^{-(n+1)}(X_{a_0\dots a_n}(x)\times Y_{a_{-1}\dots a_{-(n+1)}}(y)))}{\exp\big[\sum_{j=0}^{2n+1}\pi_+^*\phi\circ\overline{T}^j\circ\overline{T}^{-(n+1)}(x,y)\big]}$$

$$=\frac{\overline{\mu}(X_{a_{-(n+1)}\dots a_{-1}a_0\dots a_n}(\psi_{a_{-(n+1)}\dots a_{-1}}x)\times T_-Y_{a_{-(n+1)}}(T_-^{(n+1)}y))}{\exp\big[\sum_{j=0}^{2n+1}\pi_+^*\phi\circ\overline{T}^j(\psi_{a_{-(n+1)}\dots a_{-1}}x,T_-^{(n+1)}y)\big]}$$

$$=\frac{\overline{\mu}\pi_+^{-1}(X_{a_{-(n+1)}\dots a_{-1}a_0\dots a_n}(\psi_{a_{-(n+1)}\dots a_{-1}}x))}{\exp\big[\sum_{j=0}^{2n+1}\phi\pi_+\circ\overline{T}^j(\psi_{a_{-(n+1)}\dots a_{-1}}x,T_-^{(n+1)}y)\big]}$$

$$=\frac{\mu(X_{a_{-(n+1)}\dots a_{-1}a_0\dots a_n}(\psi_{a_{-(n+1)}\dots a_{-1}}\circ\psi_{a_0\dots a_n}\circ T^{n+1}x))}{\exp\big[\sum_{j=0}^{2n+1}\phi\circ T^j(\psi_{a_{-(n+1)}\dots a_{-1}}\circ\psi_{a_0\dots a_n}\circ T^{n+1}x)\big]}$$

$$=\frac{\mu(X_{a_{-(n+1)}\dots a_{-1}a_0\dots a_n}(\psi_{a_{-(n+1)}\dots a_{-1}a_0\dots a_n}(T^{n+1}x)))}{\exp\big[\sum_{j=0}^{2n+1}\phi\circ T^j(\psi_{a_{-(n+1)}\dots a_{-1}a_0\dots a_n}(T^{n+1}x))\big]}.$$

Then the result follows from the weak Gibbs property of μ associated to ϕ. □

Proof of Theorem 3.2. Let $\{K_n'\}_{n\geq 1}$ be the positive sequence satisfying $\lim_{n\to\infty}(1/n)\log K_n'=0$ for the weak Gibbs measure μ for ϕ. Then we have observed the following facts in the proof of Theorem 3.1:

$$K_{2(n+1)}'^{-1}\leq\frac{\overline{\mu}\big(\big(\bigcap_{i=-n}^{n}\overline{T}^{-i}Z_{(a_{i-1},a_i)}\big)(x,y)\big)}{\exp\big[\sum_{i=-(n+1)}^{n}\pi_+^*\phi\circ\overline{T}^i(x,y)\big]}\leq K_{2(n+1)}'$$

and

$$K_{n+1}'^{-1}\leq\frac{\overline{\mu}\big(\big(\bigcap_{i=-n}^{-1}\overline{T}^{-i}Z_{(a_{i-1},a_i)}\big)(x,y)\big)}{\exp\big[\sum_{i=-(n+1)}^{-1}\pi_+^*\phi\circ\overline{T}^i(x,y)\big]}=\frac{\overline{\mu}(TX_{a_{-1}}\cap Y_{a_{-1}\dots a_{-(n+1)}}(y))}{\exp\big[\sum_{i=-(n+1)}^{-1}\pi_+^*\phi\circ\overline{T}^i(x,y)\big]}$$

$$=\frac{\overline{\mu}\pi_-^{-1}(Y_{a_{-1}\dots a_{-(n+1)}}(y))}{\exp\big[\sum_{i=-(n+1)}^{-1}\pi_+^*\phi\circ\overline{T}^i(x,y)\big]}\leq K_{n+1}'.$$

We also see that

$$K_{n+1}'^{-1}\leq\frac{\overline{\mu}\big(\big(\bigcap_{i=1}^{n}\overline{T}^{-i}Z_{(a_{i-1},a_i)}\big)(x,y)\big)}{\exp\big[\sum_{i=0}^{n}\pi_+^*\phi\circ\overline{T}^i(x,y)\big]}=\frac{\overline{\mu}\pi_+^{-1}(X_{a_0\dots a_n}(x))}{\exp\big[\sum_{i=0}^{n}\pi_+^*\phi\circ\overline{T}^i(x,y)\big]}$$

$$=\frac{\mu(X_{a_0\dots a_n}(x))}{\exp\big[\sum_{i=0}^{n}\phi\circ T^i(x)\big]}\leq K_{n+1}'.$$

Combining these inequalities and noting the equality

$$\frac{\overline{\mu}\big(\big(\bigcap_{i=-n}^{n}\overline{T}^{-i}Z_{(a_{i-1},a_i)}\big)(x,y)\big)}{\overline{\mu}\big(\big(\bigcap_{i=-n}^{-1}\overline{T}^{-i}Z_{(a_{i-1},a_i)}\big)(x,y)\big)\overline{\mu}\big(\big(\bigcap_{i=1}^{n}\overline{T}^{-i}Z_{(a_{i-1},a_i)}\big)(x,y)\big)}$$

$$=\frac{\overline{\mu}\big(\big(\bigcap_{i=-n}^{n}\overline{T}^{-i}Z_{(a_{i-1},a_i)}\big)(x,y)\big)}{\overline{\mu}\pi_-^{-1}(Y_{a_{-1}\dots a_{-(n+1)}}(y))\overline{\mu}\pi_+^{-1}(X_{a_0\dots a_n}(x))}$$

allows one to put $K_n=K_{2(n+1)}'K_{n+1}'^2$. □

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Proof of Corollary 3.1. The assertion follows from Theorem 3.2 immediately. □

Proof of Corollary 3.2. For each large $n > 0$, we choose a sufficiently small $r := r(x, y) > 0$ such that

$$\operatorname{diam}[X_{a_0 \dots a_{n-1}}(x) \times Y_{a_{-1} \dots a_{-n}}(y)] \geq r \geq \operatorname{diam}[X_{a_0 \dots a_n}(x) \times Y_{a_{-1} \dots a_{-(n+1)}}(y)].$$

Hence we have that

$$\underline{C_1}\theta_1^{n+1} \leq \operatorname{diam} X_{a_0 \dots a_n}(x) \leq r \quad \text{and} \quad \underline{C_2}\theta_2^{n+1} \leq \operatorname{diam} Y_{a_{-1} \dots a_{-(n+1)}}(y) \leq r.$$

These inequalities imply that

$$\overline{C}_1\theta_1^{n+1} \leq \underline{C_1}^{-1}\overline{C}_1 r \quad \text{and} \quad \overline{C}_2\theta_2^{n+1} \leq \underline{C_2}^{-1}\overline{C}_2 r.$$

We note that

$$B((x, y), r) \subset \bigcup_{X_{b_0 \dots b_n} \times Y_{b_{-1} \dots b_{-(n+1)}} \cap B((x,y),r) \neq \emptyset} X_{b_0 \dots b_n} \times Y_{b_{-1} \dots b_{-(n+1)}}.$$

By the weak Gibbs property of $\overline{\mu}$, we have that

$$\begin{aligned}
\overline{\mu}(B((x, y), r)) &\leq \sum_{X_{b_0 \dots b_n} \times Y_{b_{-1} \dots b_{-(n+1)}} \cap B((x,y),r) \neq \emptyset} \overline{\mu}(X_{b_0 \dots b_n} \times Y_{b_{-1} \dots b_{-(n+1)}}) \\
&\leq K_{n+1} \sum_{X_{b_0 \dots b_n} \times Y_{b_{-1} \dots b_{-(n+1)}} \cap B((x,y),r) \neq \emptyset} \mu(X_{b_0 \dots b_n}) \mu_-(Y_{b_{-1} \dots b_{-(n+1)}}) \\
&\leq K_{n+1} \sum_{X_{b_0 \dots b_n} \cap B(x,r) \neq \emptyset} \mu(X_{b_0 \dots b_n}) \sum_{Y_{b_{-1} \dots b_{-(n+1)}} \cap B(y,r) \neq \emptyset} \mu_-(Y_{b_{-1} \dots b_{-(n+1)}})
\end{aligned}$$

and so

$$\overline{\mu}(B((x, y), r)) \leq K_{n+1}\mu(B(x, (1 + \underline{C_1}^{-1}\overline{C}_1)r))\mu_-(B(y, (1 + \underline{C_2}^{-1}\overline{C}_2)r)).$$

On the other hand, since we have

$$B((x, y), r) \supset X_{a_0 \dots a_n}(x) \times Y_{a_{-1} \dots a_{-n+1}}(y),$$

we see that

$$\overline{\mu}(B((x, y), r)) \geq K_{n+1}^{-1}\mu(X_{a_0 \dots a_n}(x))\mu_-(Y_{a_{-1} \dots a_{-(n+1)}}(y)).$$

Since both Q and Q_- are generating and each cylinder has a non-empty interior, we have $l > 0$ such that $X_{a_0 \dots a_n}(x) \supset B(x, l^{-1}r)$ and $Y_{a_{-1} \dots a_{-(n+1)}}(y) \supset B(y, l^{-1}r)$. From the fact that $r \leq 2 \max\{\overline{C_1}\theta_1^n, \overline{C_2}\theta_2^n\}$ and K_n^{-1} decays subexponentially (i.e. slower than any exponential decay), we have that for $\overline{\mu}$-a.e. $(x, y) \in \overline{X}$, for all $r > 0$ sufficiently small (depending on (x, y)), there exists $k > 0$ satisfying (1) for all $\delta > 0$. Then along the line of [**3**], we have asymptotically almost local product structure (cf. [**15, 16**]). □

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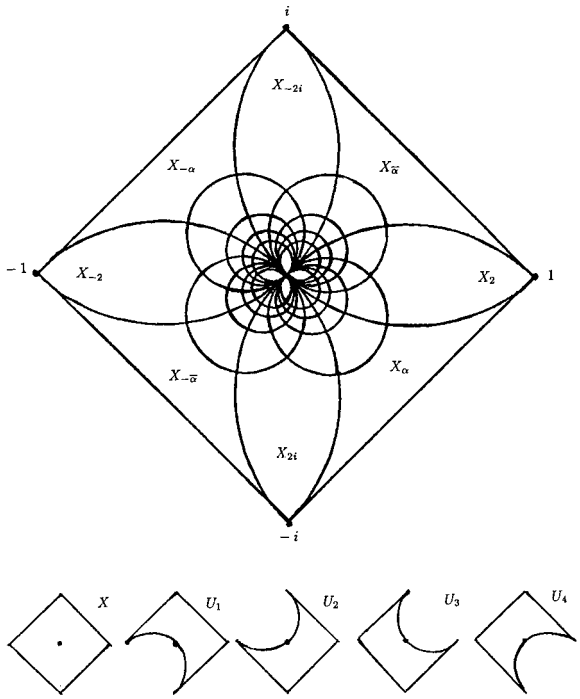


FIGURE 1. The Markov partition $Q = \{X_a\}_{a\in I}$ of X and $\mathcal{U}$ which consists of five subsets of X.

8. *A higher-dimensional example*

Let $\alpha = 1 + i$. We can define the complex continued fraction transformation $T : X \to X$ on the diamond-shaped region

$$X = \{z = x_1\alpha + x_2\overline{\alpha} : -1/2 \le x_1, x_2 \le 1/2\}$$

by $T(z) = 1/z - [1/z]_{(+)}$. Here $[z]_{(+)}$ denotes $[x_1 + 1/2]\alpha + [x_2 + 1/2]\overline{\alpha}$, where z is written in the form $z = x_1\alpha + x_2\overline{\alpha}$, $[x] = \max\{n \in \mathbb{Z} \mid n \le x\}$ $(x \in \mathbb{N})$ and $[x] = \max\{n \in \mathbb{Z} \mid n < x\}$ $(x \in \mathbb{Z} - \mathbb{N})$. The index set is $I = \{m\alpha + n\overline{\alpha} : (m, n) \in \mathbb{Z}^2 - (0, 0)\}$. For each $n\alpha + m\overline{\alpha} \in I$, we define

$$X_{n\alpha+m\overline{\alpha}} := \{z \in X : [1/z]_1 = n\alpha + m\overline{\alpha}\}.$$

Then we have a countable partition $Q = \{X_a\}_{a\in I}$ of X which is a topologically mixing Markov partition with FRS (Figure 1). In fact, $\mathcal{U}$ consists of five subsets of X. The map T induces a continued fraction-like expansion of $z \in X$,

$$z = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots \cfrac{1}{a_n + \cdots}}}},$$

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where each a_i is contained in I. This transformation has an indifferent periodic orbit $\{1, -1\}$ of period two and two indifferent fixed points at i and $-i$. All conditions (O1)–(O3) are easily verified [**22, 24, 25**]. The inverse branches to T take the form $\psi_a(z) = 1/(a + z)$ for $a \in I$. Therefore, the inverse branches of the nth iterate of the transformation T^n take the form

$$\psi_{a_1,\dots,a_n}(z) = \frac{p_n + zp_{n-1}}{q_n + zq_{n-1}} \quad \text{and} \quad |\psi'_{a_1,\dots,a_n}(z)| = \frac{1}{|q_n + zq_{n-1}|^2}$$

where $p_n = a_n p_{n-1} + p_{n-2}$ and $q_n = a_n q_{n-1} + q_{n-2}$, $n \geq 1$, and $p_{-1} = \alpha$, $p_0 = 0 = q_{-1}$ and $q_0 = \alpha$. We call p_n/q_n the nth approximant of z with respect to T. For $X_{a_1\dots a_n}$ such that X_{a_n} does not touch the indifferent periodic points, we can see that $|q_{n-1}/q_n| < 2/3$. From this fact, letting D_n be a union of cylinders of rank n touching the indifferent periodic points allows one to define a uniformly expanding jump transformation T^* and obtain the LBD property of $-\log|\det DT|(= -\log\|DT\|^2)$. In fact, we can define the jump system (T^*, X^*, Q^*) over $R(5^4, T, \nu)$ as in §5 for ν the normalized Lebesgue measure of X [**24, 25**]. Although (T, X, Q) fails (5), $(T^2, X, Q \vee T^{-1}Q)$ satisfies it. Hence, if ϕ satisfies the LBD, then we can establish for all T-invariant ergodic weak Gibbs measures μ for ϕ which satisfy $\log\|DT\| \in L^1(\mu)$ and $H_\mu(Q) < \infty$ that

$$d_\mu(x) = \frac{h_\mu(T^2)}{\int_X \log\|DT^2\|\,d\mu} = \frac{h_\mu(T)}{\int_X \log\|DT\|\,d\mu} \quad (\mu\text{-a.e. } x \in X).$$

In particular, $\log|\det DT|$ satisfies the WBV property and $\log|\det DT| \in L^1(\nu)$ for ν the normalized Lebesgue measure of X [**24**]. Furthermore, by using results in [**32**], we can show that the T-invariant ergodic probability measure equivalent to ν is a weak Gibbs measure for $-\log|\det DT|$ with $P_{\text{top}}(T, -\log|\det DT|) = 0$. We can easily verify all conditions in Proposition 5.1 and so also (6) for the SBR measure. We should remark that the SBR measure is hyperbolic since $|T'(z)| > 1$ for $z \notin \{\pm 1, \pm i\}$.

Next we construct the dual algorithm to the complex continued fraction. Define $X_- := \{w \in \mathbb{C} : |w| \leq 1\}$ and $U_1 := \{w \in X_- : |w + \alpha| \geq 1\}$. We put $U_2 := -i \times U_1, U_3 := -i \times U_2, U_4 := -i \times U_3, U_5 := U_1 \cap U_2, U_6 := -i \times U_5, U_7 := -i \times U_6$ and $U_8 := -i \times U_7$. In particular, we put $U_0 = X_-$. Next we define the following subsets of I: $J_1 := \{n\alpha : n > 0\}, J_2 := -i \times J_1, J_3 := -i \times J_2, J_4 := -i \times J_3, J_5 := \{n\alpha + m\overline{\alpha} : n, m > 0\}, J_6 := -i \times J_5, J_7 := -i \times J_6$ and $J_8 := -i \times J_7$. Then $U_0 \cup \bigcup_{k=1}^8 \left(\bigcup_{a\in J_k} a + U_k\right)$ gives a countable partition of $\mathbb{C}$. Now we define $T_- : X_- \to X_-$ by $T_- w = 1/w - [1/w]_{(-)}$, where $[w]_{(-)} = a$ if $w \in a + U_k$ for some $a \in J_k$. Define, for each $a \in I, Y_a := \{w \in X_- : [1/w]_{(-)} = a\}$ and $Q_- := \{Y_a : a \in I\}$. Then (T_-, X_-, Q_-) is a piecewise C^1-invertible Markov system with FRS (Figure 2). In fact, $\mathcal{U} = \{U_i : i = 1, 2, \dots, 8\}$. For each $n \geq 1$ and $w \in X_-$, we define $a_{-n}(w) := [1/T_-^{n-1} w]_{(-)}$. Then we can easily see that $(a_{-1}, a_{-2}, \dots, a_{-n}) \in I^n$ is T_--admissible iff $(a_{-n}, a_{-(n-1)}, \dots, a_{-1}) \in I^n$ is T-admissible. We remark that the dual algorithm T_- also admits an indifferent periodic orbit $\{1, -1\}$ of period two and two indifferent fixed points at i and $-i$. The indices of cylinders touching these periodic points are the same as those of cylinders with respect to T. T_- induces a continued fraction-like expansion of $w \in X_-$,

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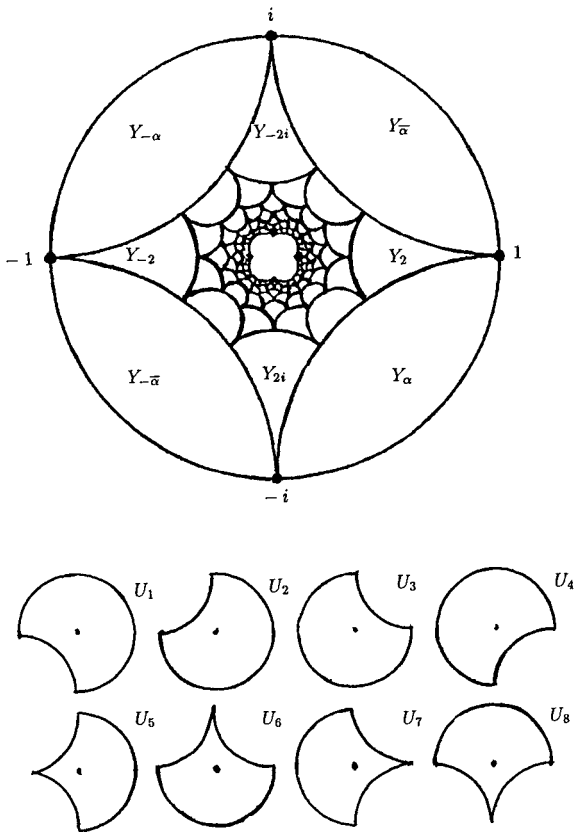


FIGURE 2. The Markov partition $Q_- = \{Y_a\}_{a\in I}$ of X_- and $\mathcal{U}$ which consists of eight subsets of X_-.

$$w = \cfrac{1}{a_{-1} + \cfrac{1}{a_{-2} + \cfrac{1}{\cdots \cfrac{1}{a_{-n} + \ldots}}}},$$

where each a_{-i} is contained in I. Therefore, we can see that $|q_{n-1}/q_n| \le 1$ for all $n > 0$ because

$$q_{n-1}/q_n = \cfrac{1}{a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{\cdots \cfrac{1}{a_1}}}} \in X_-.$$

Furthermore, the inverse branches to T_- take the same form as ψ_a, i.e. $\psi_{a,-}(w) = 1/(a + w)$ for $a \in I$ and

$$\psi_{-,a_{-1},\ldots,a_{-n}}(w) = \frac{r_n + zr_{n-1}}{s_n + zs_{n-1}} \quad \text{and} \quad |\psi'_{a_{-1},\ldots,a_{-n}}(w)| = \frac{1}{|s_n + ws_{n-1}|^2},$$

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where r_n/s_n is the nth approximant of w with respect to T_-. Let $D_{n,-}$ be a union of cylinders of rank n touching the indifferent periodic points. Then we can define (T_-^*, X_-^*, Q_-^*) similarly to (T^*, X^*, Q^*) and T_-^* satisfies the uniformly expanding property. Although (T_-, X_-, Q_-) fails (5), $(T_-^2, X_-, Q_- \vee T_-^{-1}Q_-)$ satisfies it. Hence if the dual μ_- of the T-invariant ergodic weak Gibbs measure μ satisfies $\log \|DT_-\| \in L^1(\mu_-)$ ($H_{\mu_-}(Q_-) < \infty$ follows from $H_\mu(Q) < \infty$), then we can establish for μ_--a.e. $y \in X_-$ that

$$d_{\mu_-}(y) = \frac{h_{\mu_-}(T_-^2)}{\int_{X_-} \log \|DT_-^2\| \, d\mu_-} = \frac{h_{\mu_-}(T_-)}{\int_{X_-} \log \|DT_-\| \, d\mu_-}.$$

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