NOTE ON NON-HOMOGENEOUS LINEAR FORMS

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Let θ be an algebraic number of degree n, with conjugates $\theta^{(1)}, \ldots, \theta^{(n)}$, where $\theta^{(1)}, \ldots, \theta^{(r)}$ are real and $\theta^{(r+j)}$, $\theta^{(r+s+j)}$ are complex conjugates for $j = 1, \ldots, s$. [Here $r \geqslant 0, s \geqslant 0$, $r + 2s = n$.] Let $\omega_1, \ldots, \omega_n$ be a basis for the integers of $k(\theta)$, and set

$$\xi_i = \omega_1^{(i)} u_1 + \ldots + \omega_n^{(i)} u_n \quad (i = 1, \ldots, n).$$

Then, for integral u_i, $\xi_1 \xi_2 \ldots \xi_n$ is the norm of the general integer of $k(\theta)$, and so

$$|\xi_1 \ldots \xi_n| \geqslant 1 \tag{1}$$

for integral u_i not all zero.

Davenport (2) has recently proved the following theorem:

Let $c_1, \ldots, c_n$ be any numbers such that $c_1, \ldots, c_r$ are real and c_{r+j}, c_{r+s+j} are complex conjugates for $j = 1, \ldots, s$. Then there exist integers $u_1, \ldots, u_n$ satisfying

$$|(\xi_1 - c_1) \ldots (\xi_n - c_n)| < C_n |d|^{n/2(n-s)}, \tag{2}$$

where d is the discriminant of $k(\theta)$ and C_n is a constant depending on n only.

This theorem represents an improvement on a theorem of Clarke (1), who obtained (2) with an exponent $\frac{1}{2}n$ in place of $n/2(n-s)$. The inequality (1) was the only property of the forms ξ_i used in the proofs, though for prime n Davenport has shown that the exponent may be improved to

$$\frac{n}{2(n-s)} - \frac{s-1}{(n-1)(n-s)},$$

by using the fact that $k(\theta)$ has no proper subfields.

I give here a very simple proof of Davenport's theorem, using only the property (1); it is based on the well-known proof of Tschebotareff's theorem given in Hardy and Wright (3).

Let M be the lower bound of $|(\xi_1 - c_1) \ldots (\xi_n - c_n)|$ for integral $u_1, \ldots, u_n$, so that the theorem is equivalent to the assertion that

$$M < C_n |d|^{n/2(n-s)}. \tag{3}$$

We may suppose that $M > 0$, or else there is nothing to prove. Then, for arbitrary $\epsilon_0 > 0$, we can choose a set of integers u_i^* such that for the corresponding value of the product we have

$$|(\xi_1^* - c_1) \ldots (\xi_n^* - c_n)| = \frac{M}{1-\epsilon}, \quad \text{where} \quad 0 \leqslant \epsilon < \epsilon_0.$$

We set

$$L_i = \frac{\xi_i - \xi_i^*}{\xi_i^* - c_i} \quad (i = 1, \ldots, n).$$

Then the L_i are homogeneous linear forms in the variables $v_i = u_i - u_i^*$, and have determinant $(1-\epsilon)|d|^{\frac{1}{2}}/M$. Also, by (1) and the definition of M, we have

$$|L_1 \ldots L_n| \geqslant \frac{1-\epsilon}{M} \tag{4}$$

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for integral v_i not all zero, and

$$|(L_1+1)\dots(L_n+1)| = \left|\frac{(\xi_1-c_1)\dots(\xi_n-c_n)}{(\xi_1^*-c_1)\dots(\xi_n^*-c_n)}\right| \geqslant 1-\epsilon \tag{5}$$

for all integral v_i.

We now define the *real* linear forms $x_1, \dots, x_n$ by

$$x_i = L_i \quad (i = 1, \dots, r), \qquad x_{r+j} + ix_{r+s+j} = L_{r+j} \quad (j = 1, \dots, s).$$

The points $(x_1, \dots, x_n)$ of Euclidean n-dimensional space corresponding to integral v_i form a lattice Λ of determinant

$$d(\Lambda) = \frac{(1-\epsilon)\,2^{-s}\,|d|^{\frac{1}{2}}}{M}.$$

Also, by (4) and (5), every point of Λ satisfies

$$|x_1 \dots x_r|\,(x_{r+1}^2 + x_{r+s+1}^2)\dots(x_{r+s}^2 + x_{r+2s}^2) \geqslant \frac{1-\epsilon}{M} \quad (x_i \text{ not all zero}), \tag{6}$$

$$|(x_1+1)\dots(x_r+1)|\,\{(x_{r+1}+1)^2 + x_{r+s+1}^2\}\dots\{(x_{r+s}+1)^2 + x_{r+2s}^2\} \geqslant 1-\epsilon. \tag{7}$$

We shall show that Λ can have no point other than the origin in a certain parallelepiped, when the result will follow by an application of Minkowski's theorem.

Define a number δ by the equation

$$2\epsilon + (4s+1)\,\delta = \left(\frac{1-\epsilon}{M}\right)^{2/n}. \tag{8}$$

We may clearly suppose ϵ_0 so small that $\delta > 0$. Suppose now that $(x_1, \dots, x_n)$ is a point of Λ in the parallelepiped defined by

$$|x_i| < 1 \quad (i = 1, \dots, r), \qquad |x_{r+j}| < 1, \quad |x_{r+s+j}| < \delta^{\frac{1}{2}} \quad (j = 1, \dots, s). \tag{9}$$

On multiplying (7) by the corresponding inequality for the lattice-point $(-x_1, \dots, -x_n)$ we obtain

$$\prod_{i=1}^{r} |1-x_i^2| \prod_{j=1}^{s} \{(1-x_{r+j}^2)^2 + 2x_{r+s+j}^2(1+x_{r+j}^2) + x_{r+s+j}^4\} \geqslant (1-\epsilon)^2. \tag{10}$$

By (9) we have

$$0 \leqslant 1-x_1^2 \leqslant 1 \quad (i = 1, \dots, r),$$

$$0 \leqslant (1-x_{r+j}^2)^2 + 2x_{r+s+j}^2(1+x_{r+j}^2) + x_{r+s+j}^4 \leqslant 1 + 4\delta + \delta^2 < (1+2\delta)^2 \quad (j = 1, \dots, s).$$

From these inequalities and (10) we deduce that

$$(1-x_1^2)\,(1+2\delta)^{2s} \geqslant (1-\epsilon)^2, \tag{11}$$

$$\{(1-x_{r+j}^2)^2 + 2x_{r+s+j}^2(1+x_{r+j}^2) + x_{r+s+j}^4\}\,(1+2\delta)^{2s-2} \geqslant (1-\epsilon)^2. \tag{12}$$

From (11) we obtain

$$1-x_1^2 \geqslant (1-\epsilon)^2\,(1-4s\delta),$$

$$x_1^2 \leqslant 2\epsilon - \epsilon^2 + 4s\delta(1-\epsilon)^2 \leqslant 2\epsilon + 4s\delta; \tag{13}$$

and from (12) we obtain

$$(1-x_{r+j}^2)^2 + 4\delta + \delta^2 \geqslant (1-\epsilon)^2\{1-(4s-4)\,\delta\},$$

which we may replace by

$$x_{r+j}^2 \leqslant 2\epsilon + 4s\delta. \tag{14}$$

For if (14) does not hold, we have

$$(1-2\epsilon-4s\delta)^2 + 4\delta + \delta^2 \geqslant (1-\epsilon)^2\{1-(4s-4)\,\delta\},$$

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whence $$\delta \geqslant \frac{4s}{16s^2+1} + O(\epsilon);$$

then (since here we may assume that $s \geqslant 1$) (8) gives $M < 1$, which is much stronger than (3), since trivially $|d| > 1$.

Using (13), (14) and the last inequalities of (9), we now have

$$|x_1 \dots x_r| (x_{r+1}^2 + x_{r+s+1}^2) \dots (x_{r+s}^2 + x_{r+2s}^2)$$
$$\leqslant (2\epsilon + 4s\delta)^{\frac{1}{2}r} \{2\epsilon + (4s+1)\delta\}^s < \{2\epsilon + (4s+1)\delta\}^{\frac{1}{2}n} = \frac{1-\epsilon}{M},$$

by (8). Now (6) shows that all x_i must be zero.

We have therefore shown that Λ has no point other than the origin in the region (9), whence, by Minkowski's theorem, $d(\Lambda) \geqslant \delta^{\frac{1}{2}s}$. Hence, using (8),

$$\frac{(1-\epsilon)\, 2^{-s} |d|^{\frac{1}{2}}}{M} \geqslant \delta^{\frac{1}{2}s} = \left[\left(\frac{1-\epsilon}{M}\right)^{2/n} - 2\epsilon\right]^{\frac{1}{2}s} (4s+1)^{-\frac{1}{2}s}.$$

Since M and d are independent of ϵ, this gives

$$\frac{2^{-s} |d|^{\frac{1}{2}}}{M} \geqslant M^{-s/n} (4s+1)^{-\frac{1}{2}s},$$

and so $$M \leqslant (\tfrac{1}{4} + s)^{ns/[2(n-s)]} |d|^{n/2(n-s)}.$$

This proves (3), and so the theorem is established.

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REMARKS ON THE PROBLEM OF SLOW MOTIONS IN A ROTATING FLUID

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This note is concerned with the problem of the slow motions of an inviscid, incompressible rotating fluid, and in particular with the motion of a sphere along the axis of rotation. This problem was studied recently by Stewartson (2), who overcomes the principal mathematical difficulty, viz. that of formulating the problem in a coordinate system in which the appropriate differential equation can be solved simply and in which the sphere is a coordinate surface, by a very elegant transformation of independent variables. Stewartson, however, uses inappropriate initial conditions.

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