Teaching Notes

A small note on small angle approximations

Now that the topic of small angle approximations to trigonometric functions is back on the latest A-level specifications, I offer the following three examples for classroom discussion. The point they make – that the form of an approximate answer is not unique – is not one that I have seen explicitly made in textbooks, but it is one to look out for when marking (or setting) questions on this topic. For the last two examples, we shall have need of the following inequalities, valid for $0 < x < \frac{1}{2}\pi$,

$$\frac{\sin x + \tan x}{2} > x, \qquad \sqrt{\sin x \tan x} > x, \qquad \frac{2 \sin x \tan x}{\sin x + \tan x} > x.$$

A neat way of proving these is to use the $t = \tan \frac{1}{2}x$ substitution: it is not just an integration device! By the inequality of the arithmetic/geometric/harmonic means, it suffices just to check the third inequality:

$$\frac{2 \sin x \tan x}{\sin x + \tan x} = \frac{2\dfrac{2t}{1+t^2}\dfrac{2t}{1-t^2}}{\dfrac{2t}{1+t^2} + \dfrac{2t}{1-t^2}} = \frac{8t^2}{4t} = 2t > x,$$

since $t = \tan \frac{1}{2}x > \frac{1}{2}x$.

Example 1

For small $\varepsilon > 0$, the equation $f(x) = x \sin x - \varepsilon \cos x = 0$ has a positive root near $x = 0$.

- Using $\sin x \approx x$ and $\cos x \approx 1 - \frac{1}{2}x^2$, the root satisfies $x^2 - \varepsilon\left(1 - \frac{1}{2}x^2\right) \approx 0$ so that $x \approx \sqrt{\dfrac{2\varepsilon}{2+\varepsilon}}$.
- The root also satisfies $x \tan x - \varepsilon = 0$. Using $\tan x \approx x$, we obtain $x \approx \sqrt{\varepsilon}$.

We can say a bit more about these two approximations. Since $\sin x < x < \tan x$ and $\cos x > 1 - \frac{1}{2}x^2$, we have

$$f(x) = x \sin x - \varepsilon \cos x < x^2 - \varepsilon\left(1 - \tfrac{1}{2}x^2\right)$$

and

$$f(x) = \cos x (x \tan x - \varepsilon) > \cos x \left(x^2 - \varepsilon\right).$$

Thus $f\left(\sqrt{\varepsilon}\right) > 0 > f\left(\sqrt{\dfrac{2\varepsilon}{2+\varepsilon}}\right)$ so our two approximations actually straddle the root:

$$\sqrt{\frac{2\varepsilon}{2+\varepsilon}} < x < \sqrt{\varepsilon}.$$

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Example 2

For small $\varepsilon > 0$, the equation $g(x) = \sin x + \tan x - \varepsilon = 0$ has a positive root near $x = 0$.

- Using $\sin x \approx x$ and $\tan x \approx x$, we obtain $x \approx \frac{1}{2}\varepsilon$.
- The root also satisfies $\frac{1}{2}\sin 2x + \sin x - \varepsilon \cos x = 0$. Using $\sin x \approx x$ and $\cos x \approx 1 - \frac{1}{2}x^2$, the root satisfies $2x - \varepsilon\left(1 - \frac{1}{2}x^2\right) \approx 0$ from which $x \approx \dfrac{\sqrt{4 + 2\varepsilon^2} - 2}{\varepsilon} = \dfrac{\varepsilon}{1 + \sqrt{1 + \frac{1}{2}\varepsilon^2}}$.

Again we can say more. Using the first inequality above, $\frac{1}{2}(\sin x + \tan x) > x$, we have

$$g(x) = \sin x + \tan x - \varepsilon > 2x - \varepsilon.$$

And, since $\sin x < x$ and $\cos x > 1 - \frac{1}{2}x^2$,

$$g(x) = \sec x\left(\tfrac{1}{2}\sin 2x + \sin x - \varepsilon \cos x\right) < \sec x\left[2x - \varepsilon\left(1 - \tfrac{1}{2}x^2\right)\right]$$

from which we deduce that the root satisfies $\dfrac{\varepsilon}{1 + \sqrt{1 + \frac{1}{2}\varepsilon^2}} < x < \frac{1}{2}\varepsilon$.

Example 3

For small $\varepsilon > 0$, the equation $h(x) = \sin x \tan x - \varepsilon = 0$ has a positive root near $x = 0$. Arguing as in Example 1 with

$$\sin x \tan x - \varepsilon = \sec x\left(\sin^2 x - \varepsilon \cos x\right) = 0,$$

we obtain the respective approximations to the root, $x \approx \sqrt{\varepsilon}$ and $x \approx \sqrt{\dfrac{2\varepsilon}{2 + \varepsilon}}$.

Also, using the second inequality above, $\sin x \tan x > x^2$, we can mimic the argument in Example 1 to show that these approximations straddle the root. This time, however, we can also solve for x explicitly: $0 = \varepsilon \cos x - \sin^2 x = \cos^2 x + \varepsilon \cos x - 1$ which gives $x = \cos^{-1}\left(\sqrt{1 + \frac{1}{4}\varepsilon^2} - \frac{1}{2}\varepsilon\right)$ and hence the unusual inequality

$$\sqrt{\frac{2\varepsilon}{2 + \varepsilon}} < \cos^{-1}\left(\sqrt{1 + \tfrac{1}{4}\varepsilon^2} - \tfrac{1}{2}\varepsilon\right) < \sqrt{\varepsilon}$$

or, equivalently,

$$\cos\sqrt{\varepsilon} < \sqrt{1 + \frac{1}{4}\varepsilon^2} - \frac{1}{2}\varepsilon < \cos\sqrt{\frac{2\varepsilon}{2 + \varepsilon}}.$$

Further insight into the different error expressions comes from looking at higher order terms in the approximating equations. To illustrate this using Example 2, if we posit that the root is given by $x = \frac{1}{2}\varepsilon + a\varepsilon^2 + b\varepsilon^3 + \cdots$, then the Maclaurin expansions for $\sin x$ and $\tan x$ in the arrangement $\sin x + \tan x - \varepsilon = 0$ give

$$\left(\tfrac{1}{2}\varepsilon + a\varepsilon^2 + b\varepsilon^3 + \cdots\right) - \tfrac{1}{6}\left(\tfrac{1}{2}\varepsilon + a\varepsilon^2 + b\varepsilon^3 + \cdots\right)^3 + \cdots$$
$$+\left(\tfrac{1}{2}\varepsilon + a\varepsilon^2 + b\varepsilon^3 + \cdots\right) + \tfrac{1}{3}\left(\tfrac{1}{2}\varepsilon + a\varepsilon^2 + b\varepsilon^3 + \cdots\right)^3 + \cdots - \varepsilon = 0,$$

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and, equating the coefficients of ε^2 and ε^3 to 0, we obtain respectively $a = 0$ and

$$b + \frac{1}{24} + b - \frac{1}{48} = 0 \qquad \text{or} \qquad b = -\frac{1}{96}.$$

(We get the same values of a and b using the second arrangement of the equation.)

But the two respective approximations yield $x = \frac{1}{2}\varepsilon\left(+ 0\varepsilon^3\right)$ and

$$x = \frac{\sqrt{4 + 2\varepsilon^2} - 2}{\varepsilon} = \frac{\varepsilon}{2} - \frac{\varepsilon^3}{16} + \cdots,$$

so they only agree as far as terms in ε^2. However, the terms in ε^3 do align with the inequality established at the end of Example 2.

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$\sum_{n=2}^{\infty} \frac{1}{nH_{n-1}}$ diverges while $\sum_{n=2}^{\infty} \frac{1}{nH_n^{1+\varepsilon}}$ converges

The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is an example of a divergent series with positive terms where the general term tends to zero as n tends to infinty. For any integer number $n \geqslant 1$, the nth harmonic number H_n, is defined by $H_n = \sum_{k=1}^{n} \frac{1}{k}$. Then the terms of $\sum_{n=2}^{\infty} \frac{1}{nH_{n-1}}$ grow much more slowly than those of $\sum_{n=1}^{\infty} \frac{1}{n}$ since $\lim_{n\to\infty} \frac{\frac{1}{nH_{n-1}}}{\frac{1}{n}} = 0$. However,

Theorem 1: $\sum_{n=2}^{\infty} \frac{1}{nH_{n-1}}$ is divergent.

Proof: Here we present in Figure 1 a visual proof of this fact, following [1].

Raising the second factor of each summand to the $1 + \varepsilon$ power 'only slightly' shrinks the size of each summand, especially if ε is very small. But this modification is enough to transform divergence into convergence.

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