Mathematical Proceedings of the Cambridge Philosophical Society

VOL. 90 NOVEMBER 1981 PART 3

Math. Proc. Camb. Phil. Soc. (1981), **90**, 385
Printed in Great Britain

Fibonacci and Lucas polynomials

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(*Received* 17 *March* 1981)

The Fibonacci and Lucas polynomials $F_n(z)$ and $L_n(z)$ are defined. These reduce to the familiar Fibonacci and Lucas numbers when $z = 1$. The polynomials are shown to satisfy a second order linear difference equation. Generating functions are derived, and also various simple identities, and relations with hypergeometric functions, Gegenbauer and Chebyshev polynomials.

An examination of the theory of a one dimensional Ising chain with randomly ferro and antiferromagnetic bonds in a magnetic field of random sign (2) suggested the truth of certain identities. An examination of these identities led to the definition of the Fibonacci polynomials and an examination of their properties.

We define the Fibonacci and Lucas polynomials by

$$F_{n+1}(z) = \sum_{m=0}^{[n/2]} \binom{n-m}{m} z^m, \tag{1}$$

$$L_n(z) = \sum_{m=0}^{[n/2]} \frac{n}{n-m} \binom{n-m}{m} z^m. \tag{2}$$

For $n > 0$ and $F_0(z) = 0$, $F_1(z) = 1$, $L_0(z) = 2$. In (1) and (2) [ ] means the integer part of, and $\binom{p}{q}$ is a binomial coefficient. When $z = 1$, the polynomials reduce to the Fibonacci and Lucas numbers (1). Using the recurrence of the binomial coefficients

$$\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1}, \tag{3}$$

it is easy to show that the Fibonacci and Lucas polynomials both satisfy the second order linear difference equation

$$P_{n+1}(z) = P_n(z) + zP_{n-1}(z). \tag{4}$$

0305-0041/81/0000-8920 $03.50 © 1981 Cambridge Philosophical Society

https://doi.org/10.1017/S0305004100058850 Published online by Cambridge University Press

The general solution of this equation is $\alpha a^n + \beta b^n$, where a and b satisfy $x^2 = x + z$ and are given by

$$a = \tfrac{1}{2}(1 + (1+4z)^{\frac{1}{2}}), \quad b = \tfrac{1}{2}(1 - (1+4z)^{\frac{1}{2}}). \tag{5}$$

Equating this general solution of (4) to $AF_n(z) + BL_n(z)$ and using the expressions for $F_0(z)$, $F_1(z)$, $L_0(z)$, $L_1(z)$ leads to the expressions

$$F_n(z) = (a^n - b^n)/(a-b), \tag{6}$$

$$L_n(z) = a^n + b^n. \tag{7}$$

The general second order linear difference equation

$$G_{n+1} = cG_n + dG_{n-1}, \tag{8}$$

reduces simply to (4) on writing $G_n = c^n P_n$, $z = d/c^2$.

The identities (6) and (7) may be obtained alternatively by using generating functions which turn out to be analogous to those for Fibonacci and Lucas polynomials

$$F(z,t) = \sum_{n=0}^{\infty} t^n F_n(z), \tag{9}$$

$$L(z,t) = \sum_{n=0}^{\infty} t^n L_n(z). \tag{10}$$

The recurrence relations (4) lead to the expressions

$$F(z,t) = t/(1-t-zt^2), \tag{11}$$

$$L(z,t) = (2-t)/(1-t-zt^2). \tag{12}$$

We can write the denominator of (11) and (12) in the form $(1-at)(1-bt)$ where a and b are given by (5). Expansion of the right hand sides of (11) and (12) in powers of t using partial fractions and equating equal powers of t with (9) and (10) leads to the expressions (6) and (7) for $F_n(z)$ and $L_n(z)$. Dividing (10) and (12) by t and integrating, or straightforward expansion leads to the alternative generating function

$$-\ln(1-t-zt^2) = \sum_{n=1}^{\infty} \frac{t^n}{n} L_n(z). \tag{13}$$

Using the relations (6) and (7) it is easy to prove a number of simple identities.

$$F_n(z) L_n(z) = F_{2n}(z) \tag{14}$$

$$F_n^2(z) - F_{n+k}(z) F_{n-k}(z) = (-z)^{n-k} F_k^2(z), \tag{15}$$

$$L_n^2(z) - L_{n+k}(z) L_{n-k}(z) = (-z)^{n-k}(1+4z) F_k^2(z), \tag{16}$$

$$L_n^2(z) = (1+4z) F_n^2(z) + 4(-z)^n, \tag{17}$$

$$L_{2n+1}(z) = \sum_{k=0}^{n} z^{n-k} L_{2k}(z), \tag{18}$$

$$F_{2n}(z) = \sum_{k=0}^{n-1} (-z)^{n-k} L_{2k+1}(z) \tag{19}$$

$$F_{2n+1}(z) = \sum_{k=0}^{n-1} (-z)^{n-k-1} L_{2k}(z) + (-z)^n. \tag{20}$$

https://doi.org/10.1017/S0305004100058850 Published online by Cambridge University Press

It is identity (15) with $k = 1$ which is of importance in the one-dimensional Ising problem (2).

The identities above imply further identities between certain generalised hypergeometric functions of the form ${}_4F_3(a,b,c,d;e,f,g;1)$ obtained by equating the coefficients of z^p on both sides of the above identities. In particular from (15) with $k = 1$ and $p < n-1$ we have

$$(n-p)\,{}_4F_3\left(-\frac{n}{2},\frac{1}{2}-\frac{n}{2},n-p+1,-p;\ -n,\frac{n}{2}+\frac{1}{2}-p,\frac{n}{2}+1-p;\ 1\right) =$$

$$(n-2p)\,{}_4F_3\left(-\frac{n}{2},-\frac{1}{2}-\frac{n}{2},n-p,-p;\ -n-1,\frac{n}{2}-p,\frac{n}{2}+\frac{1}{2}-p;\ 1\right). \quad (21)$$

The Fibonacci and Lucas polynomials are related to the hypergeometric functions ${}_2F_1(a,b;c;-4z)$ by

$$F_{n+1}(z) = {}_2F_1\left(-\frac{n}{2},-\frac{n}{2}+\frac{1}{2};\ -n;\ -4z\right), \quad (22)$$

$$L_n(z) = {}_2F_1\left(-\frac{n}{2},-\frac{n}{2}+\frac{1}{2},1-n;-4z\right). \quad (23)$$

The Gegenbauer and Chebyshev polynomials are related to the Fibonacci and Lucas polynomials by the relations

$$C_n^1(z) = (2z)^n F_n(-1/4z^2), \quad (24)$$

$$T_n(z) = 2^{n-1} z^n L_n(-1/4z^2). \quad (25)$$

In view of the widespread application and use of Fibonacci and Lucas numbers, the number of simple identities we have proved for Fibonacci and Lucas polynomials, their relation to the hypergeometric functions and use in proving many hypergeometric identities, it seems probable that these polynomials will find application in numerous fields other than the one dimensional Ising problem.

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