A SEXTIC PRIMAL IN FIVE DIMENSIONS

By E. M. HARTLEY

Communicated by J. A. Todd

Received 11 May 1949; *and in revised form* 30 May 1949.

Introduction

In a recent tract (Baker (1)) there is described in considerable detail a configuration of forty-five points which are nodes of a quartic primal in four dimensions. The geometry of this primal is very fascinating; among its interesting properties is the fact that a number of well-known geometrical configurations, which usually arise as unrelated phenomena, here all appear in connexion with the one figure. The interest of the primal, of course, lies chiefly in the large number of collineations which leave it invariant. The group G^* of these collineations is considered in a paper by Burkhardt (2), in which are given explicit expressions for five algebraically independent functions of the five variables, which are left invariant by the operations of the group. The simplest of these invariants is of the fourth order, and when equated to zero represents the quartic primal which is the subject of Baker's tract.

The group G^* is mentioned again by Mitchell (5), in a paper in which he lists those primitive collineation groups in more than three dimensions which contain homologies. It arises in a series of groups in which there are certain sets of nine homologies whose centres are coplanar, and Mitchell shows that the largest group containing such a set is a collineation group G in [5], of order 6,531,840. He proves that G contains 126 homologies, and points out that the fixed prime of any homology of G contains forty-five centres of homologies, these homologies generating in the prime the group G^*. In a recent paper (3) these 126 homologies (termed *projections*, following Baker) are investigated further; the configuration of the centres of the projections (termed *vertices*) is described, and all the operations of G are found as the product of six or fewer projections.

The general method of obtaining the orders of the independent invariants of a collineation group is discussed by Todd (7), and the method is illustrated by solving the problem for the group G. The order of the lowest invariant of G is six, and Todd gives the equation of this sextic primal in several different systems of coordinates. It is this sextic primal which is our main concern in this paper; after a preliminary description of the configuration formed by the vertices the equation of the sextic primal is given and its properties are investigated. On a general sextic primal in [5] there is a single infinity of lines, which form a ruled surface. The order of this ruled surface can be calculated from a general formula ((6), p. 68) as 60,480. A finite number, 34,020, of the lines on the sextic primal under consideration bear a special relationship to the configuration of vertices, and the geometry of these lines, *g-lines*, will in particular be described.

https://doi.org/10.1017/S0305004100025500 Published online by Cambridge University Press

As far as possible the notation of previous papers has been retained, but in one particular a change has been made. The projection, vertex P, say, leaves fixed both P and a prime, associated with P, termed the *polar prime* of P by Baker. This polarity is with respect to a certain Hermitian form (in Baker's first notation, for example, the form is $\Sigma x\bar{x} = 0$). This terminology becomes confusing when there is also need to refer to the polar prime $\Sigma x_i(\partial S/\partial x_i)_{\xi_i} = 0$ of a point ξ with respect to a primal $S(x) = 0$, a concept always denoted by the word polar. We shall therefore retain the word polar to refer to polarity with respect to a primal (usually, though not always, the sextic primal), and when there is any doubt as to which primal is meant, it will be explicitly stated. For the prime associated with a vertex as explained above, the term *conjugate prime* will be used. The conception will also be extended so that any point in [5] will have a conjugate prime, its polar with respect to the Hermitian form; and in general, conjugate to any S_r ($r = 0, \ldots, 4$) in [5] there will be an S_{4-r}, being the common part of all the conjugate primes of the points of S_r. Further, two spaces will be said to be conjugate if each lies in the conjugate space of the other. Since a projection from a vertex leaves fixed the conjugate prime, it is clear that any S_r is left fixed by projections from each vertex in the conjugate S_{4-r}.

1. THE CONFIGURATION OF THE VERTICES

The 126 vertices form a most interesting configuration, which is described in detail in Hamill (3). As this configuration is fundamental in what follows, it will be described briefly in this section; for proofs and more details the reader is referred to Miss Hamill's paper.

There are only two sorts of lines joining vertices in the figure. A vertex is joined to a conjugate vertex by an *e-line*, which contains no further vertices. The join of two vertices which are not conjugate is a *κ-line*, and contains one further vertex. We note that projection from a vertex P (written $p(P)$ as in Baker) interchanges the remaining vertices on a κ-line through P, and leaves fixed the vertices on e-lines through P. Two types of operation arise as the product of projections from two collinear vertices: (i) if U and V are the vertices on an e-line, the operation $p(U)\,p(V)$ is of period two, and leaves fixed every point on UV; (ii) if L, M and N are the vertices on a κ-line, the operation $p(M)\,p(N) = p(N)\,p(L) = p(L)\,p(M)$ is of period three, and leaves fixed just two points on LMN, the Hessian points of the triad of vertices. Such Hessian points are hereafter referred to as *H-points*.

There are four types of plane in the configuration: (i) an *f-plane* containing three mutually conjugate vertices; (ii) a *d-plane*, the join of a κ-line to a conjugate vertex; (iii) a *c-plane*, containing six vertices at the angular points of a complete quadrilateral (the diagonal lines being e-lines intersecting in three points called *B-points*); (iv) a *j-plane*, containing nine vertices lying by threes on twelve κ-lines which form four triangles whose angular points are H-points.

The remaining essentially new elements arise in three sequences:

(a) A sequence based on six mutually conjugate H-points, forming an *α-hexahedron*, such that any two of the points define a κ-line, any three define a j-plane, any four define an *S-solid*, and any five an *α-prime*.

https://doi.org/10.1017/S0305004100025500 Published online by Cambridge University Press

(*b*) A sequence based on six mutually conjugate B-points, forming a *β-hexahedron*, such that any two of the points define an e-line, any three define a c-plane, any four define a *J-solid* (which contains twelve vertices in a desmic set), and any five define a *β-prime*.

(*c*) A sequence of vertex, κ-line, c-plane, *N-solid*, *γ-prime* and *γ-heptahedron*, such that a γ-heptahedron contains seven γ-primes, a γ-prime contains six N-solids, an N-solid contains five c-planes, and so on.

We note that conjugate to a vertex is a *π-prime*, which contains forty-five vertices, conjugate to a vertex in a π-prime is a J-solid, and conjugate to a vertex in a J-plane is an f-plane.

Space	n	m	Description	Conjugate space
e-line	2,835	2	—	J-solid
κ-line	1,680	3	—	S-solid
f-plane	11,340	3	3 e-lines	f-plane
d-plane	30,240	4	$\kappa \times P$	—
c-plane	11,340	6	4 κ-lines	c-plane
j-plane	560	9	Inflexions of a plane cubic curve	j-plane
V-solid	45,360	5	$\kappa \times e$	—
U-solid	30,240	6	$\kappa \times \kappa$	—
X-solid	68,040	7	$c \times P$	—
W-solid	5,040	10	$j \times P$	—
N-solid	27,216	10	5 c-planes	—
I-solid	2,835	12	Desmic set	e-line
S-solid	1,680	18	4 j-planes	κ-line
δ-prime	45,360	9	$c \times \kappa$	—
χ-prime	27,216	11	$N \times P$	—
γ-prime	18,144	15	6 N-solids	—
λ-prime	5,040	19	$S \times P$	—
β-prime	3,402	20	5 J-solids	B-point
α-prime	672	30	5 S-solids	H-point
π-prime	126	45	—	Vertex
α-hexahedron	112	45	6 α-primes	—
β-hexahedron	567	30	6 β-primes	—
γ-heptahedron	2,592	21	7 γ-primes	—

For reference, a complete list of all the lines, planes, solids and primes of the configuration is now given, noting for each space the number n of them in the whole figure, the number m of vertices in the space, a description of their configuration where possible, and the conjugate space if it has been named. Some of the spaces are described as $a \times b$; this implies that the space is the join of a space a to a space b conjugate to it, and such that the join contains only vertices in a or in b. The e-lines in such a space either lie wholly in a or b, or join a vertex in a to one in b; in the latter case they are termed *struts*.

It will be seen that two conjugate κ-lines may determine a U-solid or an S-solid. Consider a line κ_0 and its conjugate solid S_0, which contains six κ-lines as edges of a

https://doi.org/10.1017/S0305004100025500 Published online by Cambridge University Press

tetrahedron, and thirty-six others, nine in each j-plane. The join of κ_0 to one of the six edges in S_0 is an S-solid, and the join to one of the other thirty-six κ-lines is a U-solid.

With the exception of the γ-primes, which fall into two sets, the group G is transitive on all the lines, planes, solids, primes and subconfigurations listed above. This enables us to deduce that any property holding in a particular subspace holds in every space of that type.

2. The equation of the primal and some of its properties

In Todd (7) the equation of the sextic primal is given in four systems of coordinates. In this paper we shall use a system of coordinates with a β-hexahedron as simplex of reference. In this system, the coordinates of the 126 vertices are all permutations of $(1, \pm 1, 0, 0, 0, 0)$, $(\sigma, 1, 1, 1, 1, 1)$, $(\sigma, 1, 1, 1, -1, -1)$ and $(\sigma, 1, -1, -1, -1, -1)$, where $\sigma = \omega^2 - \omega$, ω being a complex cube root of unity. The conjugate primes are then the polars with respect to $\Sigma x\bar{x} = 0$, and the equation of the sextic primal Ψ is

$$\Sigma x_i^6 + 15\Sigma x_i^4 x_j^2 - 30\Sigma x_i^2 x_j^2 x_k^2 - 240\sigma x_0 x_1 x_2 x_3 x_4 x_5 = 0.$$

A primal which is left invariant by projections from the vertices in the configuration described will be expected to exhibit properties related to the configuration. The sextic primal Ψ does not pass through the vertices; nor, indeed, does it contain points conjugate to any of the other six types of primes of the configuration. However, the polar prime of a vertex, say $(1, 1, 0, 0, 0, 0)$, with respect to Ψ, is the prime $x_0 + x_1 = 0$, that is, the conjugate prime of the vertex. Further, the polar quadric of a vertex is the conjugate prime taken twice, and the polar cubic is the conjugate prime taken three times.

An e-line meets Ψ in three pairs of points which are of interest. Consider, for example, the e-line containing the vertices $U(1, 1, 0, 0, 0, 0)$ and $V(1, -1, 0, 0, 0, 0)$. The point $U + \alpha V$ lies on Ψ when $1 + \alpha^6 = 0$; that is, when $\alpha = \pm i, \pm i\omega, \pm i\omega^2$. These values give three pairs of mutually conjugate points on the e-line. Consider the pair of points given by $\alpha = \pm i$; they are $(1 \pm i, 1 \mp i, 0, 0, 0, 0)$, that is, $(1, \pm i, 0, 0, 0, 0)$. These two points are joined to a similar pair $(0, 0, 1, \pm i, 0, 0)$ on a conjugate e-line, by four lines $(1, \pm i, \lambda, \pm \lambda i, 0, 0)$, which lie entirely on Ψ. Now since the group is transitive on e-lines, there are similar lines linking every pair of conjugate e-lines; these are the g-lines mentioned in the introduction.

The fundamental property of the g-lines is that each one lies in a unique J-solid. For consider the g-line $(1, i, \lambda, \lambda i, 0, 0)$. This is conjugate to just two vertices $(0, 0, 0, 0, 1, 1)$ and $(0, 0, 0, 0, 1, -1)$, lying on an e-line. It follows that the g-line lies in the unique J-solid conjugate to this e-line; and, moreover, that a g-line does not lie in an S-solid, nor in one of the four types of plane of the configuration, since any one of these is conjugate to three or more vertices. It is thus reasonable, before investigating the situation in [5], to examine the g-lines which lie in a J-solid. This we do in the next section.

3. The section of Ψ by a J-solid

A new configuration arises at this point, which we describe briefly. It is the Klein 60_{15} configuration in [3], consisting of sixty points which are the angular points of a set of fifteen tetrahedra; for a full account the reader is referred to Hudson ((4), chapter IV).

https://doi.org/10.1017/S0305004100025500 Published online by Cambridge University Press

The fifteen tetrahedra have in all thirty edges, each edge being common to three tetrahedra. From these can be selected (in ten ways) two *associated* sets of three desmic tetrahedra, having the same eighteen lines as edges, and such that each tetrahedron is self-polar with respect to a quadric which contains the remaining twelve lines of the figure. The tetrahedra can be denoted by the symbols $\{ab\}$, $\{ac\}$, ..., $\{ef\}$, in such a way that a set of the type $\{bc\}\{ca\}\{ab\}$ is desmic, the associated set being $\{ef\}\{fd\}\{de\}$. A set of fifteen tetrahedra of this sort can be derived from one desmic set. If we consider the J-solid $x_4 = x_5 = 0$, the vertices form the desmic set $\{bc\}\{ca\}\{ab\}$, say, where the coordinates of the vertices (omitting the last two zeros) are

$$\begin{array}{llll}
\{bc\} & P_0 = (1, -1, 0, 0) & \{ca\} \; Q_0 = (1, 0, -1, 0) & \{ab\} \; R_0 = (0, 1, -1, 0) \\
& P_1 = (1, 1, 0, 0) & \quad\; Q_1 = (1, 0, 1, 0) & \quad\; R_1 = (0, 1, 1, 0) \\
& P_2 = (0, 0, 1, -1) & \quad\; Q_2 = (0, 1, 0, -1) & \quad\; R_2 = (1, 0, 0, -1) \\
& P_3 = (0, 0, 1, 1) & \quad\; Q_3 = (0, 1, 0, 1) & \quad\; R_3 = (1, 0, 0, 1)
\end{array}$$

then the associated desmic set is

$$\begin{array}{llll}
\{ef\} & (1, 0, 0, 0) & \{fd\} \; (1, 1, 1, 1) & \{de\} \; (-1, 1, 1, 1) \\
& (0, 1, 0, 0) & \quad\; (-1, -1, 1, 1) & \quad\; (1, -1, 1, 1) \\
& (0, 0, 1, 0) & \quad\; (-1, 1, -1, 1) & \quad\; (1, 1, -1, 1) \\
& (0, 0, 0, 1) & \quad\; (-1, 1, 1, -1) & \quad\; (1, 1, 1, -1)
\end{array}$$

and the other nine tetrahedra completing the Klein configuration are

$$\begin{array}{llll}
\{ad\} & (1, i, 0, 0) & \{ae\} \; (1, 1, i, i) & \{af\} \; (-1, 1, i, i) \\
& (i, 1, 0, 0) & \quad\; (-1, -1, i, i) & \quad\; (1, -1, i, i) \\
& (0, 0, 1, i) & \quad\; (-1, 1, -i, i) & \quad\; (1, 1, -i, i) \\
& (0, 0, i, 1) & \quad\; (-1, 1, i, -i) & \quad\; (1, 1, i, -i),
\end{array}$$

three tetrahedra $\{bd\}\{be\}\{bf\}$ obtained from $\{ad\}\{ae\}\{af\}$ by interchanging x_1 and x_2, and three tetrahedra $\{cd\}\{ce\}\{cf\}$ obtained from $\{ad\}\{ae\}\{af\}$ by interchanging x_1 and x_3.

The twelve lines which are the edges, other than the e-lines, of the tetrahedra $\{ae\}\{be\}\{ce\}\{af\}\{bf\}\{cf\}$ are seen from the tetrahedra $\{ad\}\{bd\}\{cd\}$ to be the permutations of $(1, i, \lambda, \lambda i)$; that is, they are the g-lines defined in the last section. We have seen that the angular points of the nine tetrahedra lie in pairs on the eighteen e-lines, and since each g-line is the edge of three tetrahedra, each is the transversal of six e-lines. For example, there is a g-line containing the points

$$(1, i, 0, 0), \quad (0, 0, 1, i), \quad (1, i, -1, -i), \quad (1, i, 1, i), \quad (1, i, -i, 1) \quad \text{and} \quad (1, i, i, -1),$$

which lie respectively on the e-lines P_0P_1, P_2P_3, Q_0Q_2, Q_1Q_3, R_0R_3 and R_1R_2. Moreover, these are three pairs of conjugate e-lines, so that each g-line, which we showed in § 2 to be the transversal of one pair of conjugate e-lines, actually meets three such pairs. It is easy to see that the twelve g-lines in the J-solid are equivalent, by projections from the vertices. Thus, since the group is transitive on J-solids, there are in all $12 \times 2835 = 34{,}020$ g-lines in the figure, and they are all equivalent under G.

The quadric $\mathcal{Q}$ with respect to which the tetrahedra $\{bc\}\{ca\}\{ab\}\{ef\}\{fd\}$ and $\{de\}$ are all self-polar, has equation $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$, and the g-lines are generators of this quadric, six of each system. Projection from the vertices leave $\mathcal{Q}$ unaltered, and so

https://doi.org/10.1017/S0305004100025500 Published online by Cambridge University Press

$\mathcal{Q}$ is invariant under the group of order 96 which is generated by the projections. The operations of this group may be listed in their six conjugate sets, as follows (cf. the analysis of G in Hamill (3)):

Type	Typical operation	Period	No. of operations
I	E	1	1
II	$p(P_0)$	2	12
III	$p(P_0)p(P_1)$	2	9
IV	$p(P_0)p(Q_0)$	3	32
V	$p(P_0)p(P_1)p(Q_0)$	4	36
VI	$p(P_0)p(P_1)p(Q_0)p(Q_2)$	2	6
			96

Now consider one of the sixteen κ-lines in the solid. If the vertices on it are denoted by L, M, N, we know (§ 1) that the operation $p(M)p(N)$ leaves fixed just two H-points, H and H', say, on the line. Moreover, since $p(M)p(N)$ is an operation of period 3, no pair of points on the line is interchanged. Since $\mathcal{Q}$ is invariant under the group G the two points in which the κ-line meets it must therefore be H and H'. Further, there are just thirty-two distinct H-points in the J-solid. For no two κ-lines in a J-solid intersect in an H-point, since two intersecting κ-lines lie in a plane, and the only type of plane in a J-solid containing more than one κ-line is a c-plane, in which the κ-lines meet in vertices. So sixteen distinct pairs of H-points lie on $\mathcal{Q}$.

Now the operations $p(M)p(N)$ generate a group of order 48, comprising the operations of types I, III, IV and VI of the list above, that is, those which are the product of an even number of projections. It is clear that G_{48} is not transitive on the thirty-two H-points; however, it is transitive on the κ-lines, and so the H-points fall into two sets, each set containing one H-point from each κ-line. The sixteen κ-lines are

$$x_i + x_j + x_k = 0, \quad x_l = 0 \quad (i, j, k, l = 0, 1, 2, 3),$$

and so the coordinates of the H-points are the permutations of $(0, 1, \pm\omega^2, \pm\omega)$ with all possible arrangements of sign. Consider the set **H** which contains $(0, 1, \omega^2, \omega)$. This point is sent to $(1, 0, \omega, \omega^2)$ by $p(P_0)p(P_2)$ (an operation of G_{48}, since it is the product of two projections), to $(\omega^2, \omega, 0, 1)$ by $p(R_0)p(P_0)$, and to $(0, -1, \omega^2, \omega)$ by $p(P_0)p(P_1)$. Thus the set **H** contains the points

$$(0, 1, \pm\omega^2, \pm\omega), \quad (1, 0, \pm\omega, \pm\omega^2), \quad (\pm\omega^2, \pm\omega, 0, 1) \text{ and } \quad (\pm\omega, \pm\omega^2, 1, 0),$$

sixteen points in all.

Now the four points

$$(0, 1, \omega^2, \omega), \quad (1, 0, \omega, -\omega^2), \quad (\omega^2, -\omega, 0, 1) \quad \text{and} \quad (-\omega, -\omega^2, 1, 0)$$

are collinear, and so are

$$(0, 1, \omega^2, \omega), \quad (1, 0, -\omega, \omega^2), \quad (-\omega^2, -\omega, 0, 1) \quad \text{and} \quad (\omega, -\omega^2, 1, 0).$$

These lines, since they contain four points of $\mathcal{Q}$, are generators of $\mathcal{Q}$, and so, as G_{48} is transitive on the points of **H**, there are two generators through each of the sixteen points, each generator containing four points. Thus the points of **H** lie by fours on

https://doi.org/10.1017/S0305004100025500 Published online by Cambridge University Press

eight generators of $\mathcal{Q}$, four of each system; and the same is true for the other set $\mathbf{H}'$. Two properties of these H-points are worth noting; the proof by elementary algebra is immediate.

(i) A set of four κ-lines through four collinear H-points are skew (and so among the sixteen κ-lines there are eight sets of four skew lines, two sets containing a given line, as can be otherwise verified).

(ii) A set of four κ-lines through four H-points of the same set which lie on different generators are either concurrent (in one of the twelve vertices) or coplanar (lying in one of the twelve c-planes).

Of the generators of one system, four contain by fours the points of $\mathbf{H}$ and the other four contain the points of $\mathbf{H}'$. These two sets of generators therefore give rise to two binary quartic forms, each of which is seen to be equianharmonic. Moreover, each form is the Hessian of the other. Now a binary quartic form and its Hessian have a sextic covariant, the Jacobian of the two forms, which in this case determines a set of six generators. Algebraically, the points in which the two sets of generators meet a generator of the opposite system are

$$\begin{aligned} H_0 &= (\ 0,\ 1,\ \omega^2,\ \omega), & L_0 &= (\ \sigma,\ 1,\ 1,\ 1),\\ H_1 &= (\ 1,\ 0,\ \omega,\ -\omega^2), & L_1 &= (-1,\ \sigma,\ -1,\ 1),\\ H_2 &= (\ \omega^2,\ -\omega,\ 0,\ 1), & L_2 &= (-1,\ 1,\ \sigma,\ -1),\\ H_3 &= (-\omega,\ -\omega^2,\ 1,\ 0), & L_3 &= (-1,\ -1,\ 1,\ \sigma), \end{aligned}$$

and so

$$\begin{aligned} &H_1 = H_0 - L_0, \quad H_2 = H_0 - \omega^2 L_0, \quad H_3 = H_0 - \omega L_0,\\ &L_1 = 2H_0 + L_0, \quad L_2 = 2\omega H_0 + L_0, \quad L_3 = 2\omega^2 H_0 + L_0. \end{aligned}$$

It follows, on taking (x, y) as coordinates on the line referred to H_0 and L_0, that the two binary quartic forms are

$$y(x+y)(x+\omega y)(x+\omega^2 y), \qquad \text{that is} \quad x^3y + y^4,$$

and

$$x(x-2y)(x-2\omega^2 y)(x-2\omega y), \qquad \text{that is} \quad x^4 - 8xy^3,$$

each of which is readily verified to be the Hessian of the other. The sextic covariant is

$$x^6 + 20x^3y^3 - 8y^6,$$

the roots of which, when equated to zero, give the six generators of $\mathcal{Q}$.

To locate these generators, we examine the section of the sextic primal Ψ by the J-solid $x_4 = x_5 = 0$ under consideration. The equation of the sextic surface ψ in this solid is

$$\Sigma x_i^6 + 15\Sigma x_i^4 x_j^2 - 30\Sigma x_i^2 x_j^2 x_k^2 = 0 \quad (i, j, k = 0, 1, 2, 3;\ i \neq j \neq k).$$

We have seen that the twelve g-lines in the J-solid are generators of $\mathcal{Q}$, and that they also lie on ψ. Thus, since ψ does not contain $\mathcal{Q}$, we see that the twelve g-lines form the curve of order twelve which is the complete intersection of $\mathcal{Q}$ and ψ. Now the line $xH_0 + yL_0$, that is $(y\sigma, x+y, \omega^2 x+y, \omega x+y)$, meets ψ in points given by the equation

$$\begin{aligned} &[-27y^6 + 3(x^6 + 20x^3y^3 + y^6)] + 15[9y^4(3y^2) - 3y^2 \,.\, 3(4x^3y + y^4) + (-3x^6 - 24x^3y^3 + 6y^6)]\\ &\qquad - 30[(x^3+y^3)^2 - 3y^2(-6x^3y + 3y^4)] = 0, \end{aligned}$$

which reduces to

$$x^6 + 20x^3y^3 - 8y^6 = 0.$$

https://doi.org/10.1017/S0305004100025500 Published online by Cambridge University Press

Comparison with the sextic covariant obtained above shows that the generators obtained from the sextic covariant are the six g-lines of one system of generators on $\mathcal{Q}$. The other set of six g-lines are found similarly from among the opposite system of generators.

The points and lines on $\mathcal{Q}$ which have been described in this section are conjugate to primes and solids which are of interest. We therefore next proceed to investigate them.

4. Points and lines on $\mathcal{Q}$ and their conjugate spaces

In this section we shall be considering a particular J-solid, which we continue to take as $x_4 = x_5 = 0$, and its conjugate e-line. We shall refer to this solid as J_0, and to its conjugate e-line as e_0, with vertices $E(0, 0, 0, 0, 1, 1)$ and $E'(0, 0, 0, 0, 1, -1)$. Now any point, line or plane in J_0 has a conjugate prime, solid or plane which contains e_0. When this correspondence is established, we have a type of representation of spaces through e_0, and so incidence relations of elements in J_0, and more particularly of elements on $\mathcal{Q}$, can be interpreted as results about the corresponding spaces which pass through e_0.

It is a simple matter to calculate the number of spaces through one of the 2835 e-lines in the configuration. For example, there are in all 11,340 f-planes, each containing three e-lines, and so through an e-line there are $(11{,}340 \times 3)/2835 = 12$ f-planes. Solids and primes which contain e-lines both in one of the component parts and as struts, can be dealt with separately. For example, a V-solid, $e \times \kappa$, has one e-line as a component part and six e-lines as struts; hence since there are 45,360 V-solids in the figure, there are through e_0 $(45{,}360 \times 1)/2835 = 16$ V-solids of the form $e_0 \times \kappa$, and $(45{,}360 \times 6)/2835 = 96$ V-solids in which e_0 appears as a strut. All the numbers of spaces required subsequently can be calculated in this way, or ascertained more rapidly from the table of incidences which appears in Hamill (3). The numbers of spaces through e_0 will, in particular, be required below, and so they are listed; where e_0 appears in a space both in one of the components and as a strut, the number of spaces which contain e_0 in a component part is given first, and then the number which contain e_0 as a strut.

Through e_0 there are:

12 f-planes, 32 d-planes, and 12 c-planes;

16 and 96 V-solids, 96 U-solids, 72 and 144 X-solids, 16 W-solids, 144 N-solids, 18 J-solids, and 16 S-solids;

48 and 288 δ-primes, 144 and 96 χ-primes, 288 γ-primes, 48 and 32 λ-primes, 84 β-primes, 32 α-primes, and 12 π-primes.

Of the 84 β-primes, twelve contain e_0 as an edge joining two of the five B-points in the prime.

When considering the quadric $\mathcal{Q}$ in the last section, we saw that it contains thirty-two H-points, the points in which the κ-lines meet $\mathcal{Q}$, and that they fall into two sets $\mathbf{H}$ and $\mathbf{H}'$. These H-points, as we noted in the first section, are conjugate to α-primes. Through e_0 there are sixteen S-solids, each having a conjugate κ-line lying in J_0. Suppose a particular S-solid S_1 is conjugate to a κ-line on which the Hessian points are H_1 and H_1'. These points are conjugate to each other, and so the prime $[S_1, H_1']$ is the α-prime conjugate to H_1, and similarly $[S_1, H_1]$ is the α-prime conjugate to H_1'. We are thus led

https://doi.org/10.1017/S0305004100025500 Published online by Cambridge University Press

to expect thirty-two α-primes through e_0, as is otherwise found to be the case; and the thirty-two H-points are their conjugates.

We also found in the last section that the two generators of the quadric $\mathcal{Q}$ through each of the H-points each contain three other H-points, so that there are in all sixteen of these generators, eight of each system on $\mathcal{Q}$. The generators of each system fall into two sets, four passing through the points of the set **H** and four through the points of **H'**. Suppose the generators of one system are a_i and a_i', the generators of the other system being b_i and b_i', and let the notation be arranged so that the sixteen points (a_i, b_j') compose the set **H**, and the other points (a_i', b_j) compose the set **H'**. The other points in which the generators meet are (a_i, b_j) and (a_i', b_j'), two sets of sixteen points which we shall call **L** and **L'** respectively.

We gave in the last section the coordinates of four of the points of **L**, and it is easily verified that the coordinates in the solid $x_4 = x_5 = 0$ of the points of **L** are the permutations of $(\sigma, \pm 1, \pm 1, \pm 1)$ with an even number of negative signs, the other set **L'** containing points whose coordinates are the permutations of $(\sigma, \pm 1, \pm 1, \pm 1)$ with an odd number of negative signs. Consider one point $(\sigma, 1, 1, 1, 0, 0)$. This is conjugate to nineteen vertices, three like $(0, 1, -1, 0, 0, 0)$, two like $(-\sigma, 1, 1, 1, 1, -1)$, twelve like $(1, \sigma, 1, -1, 1, -1)$, and the pair E, E'. These nineteen vertices lie in a λ-prime, and as we see that the eighteen vertices other than E are all conjugate to E, it is a λ-prime having the line e_0 as a strut. It is evident that, of the thirty-two λ-primes which contain e_0 as a strut, sixteen consist of $E \times (S$-solid through $E')$, and are conjugate to points of the set **L**, the remaining sixteen being of the form $E' \times (S$-solid through $E)$ and being conjugate to points of **L'**.

The tangent planes to $\mathcal{Q}$ at H_1 and H_1' (where H_1 and H_1' lie on the same κ-line) intersect in a line which meets $\mathcal{Q}$ in the two points L_1 and L_1' say. Now since each coordinate of H_1' is the complex conjugate of the corresponding coordinate of H_1 it is clear that the conjugate plane of H_1 (being the polar plane with respect to $\Sigma x\bar{x} = 0$) is the tangent plane at H_1' to $\mathcal{Q}$, whose equation is $\Sigma x^2 = 0$. Hence $L_1 L_1'$ is the conjugate line of $H_1 H_1'$ in J_0. But J_0 is conjugate to e_0, and so the solid conjugate to the line $L_1 L_1'$ is $e_0 \times (\kappa$-line $H_1 H_1')$, that is, a V-solid having e_0 as one component. In J_0 there are clearly sixteen such lines, corresponding as they should to the sixteen V-solids which contain e_0 as one component.

The tangent plane to $\mathcal{Q}$ at one of the points of **L** contains two generators, on which there are a pair of H-points on a κ-line. For suppose H_1 and H_1' are the points (a_1, b_1') and (a_1', b_1), L_1 and L_1' being (a_1, b_1) and (a_1', b_1'). Then the tangent plane to $\mathcal{Q}$ at L_1 is the plane $[a_1, b_1]$ which contains both H_1 and H_1'. The tangent plane at the point $(\sigma, 1, 1, 1, 0, 0)$, for example, contains the points $(0, 1, \omega^2, \omega, 0, 0)$ and $(0, 1, \omega, \omega^2, 0, 0)$, and these three points are conjugate to just two vertices $(-\sigma, 1, 1, 1, 1, -1)$ and $(-\sigma, 1, 1, 1, -1, 1)$, as well as E and E'. The tangent plane is thus conjugate to a d-plane, $E \times (\kappa$-line through $E')$; and similarly the tangent planes at points of **L'** are conjugate to sixteen d-planes consisting of κ-lines through E conjugate to E'.

The next investigation concerns the solids through e_0 which are conjugate to the generators a_i, a_i', b_i, b_i', in J_0. The particular generator, eight points of which were listed in the last section, contains the points $(\sigma, 1, 1, 1, 0, 0)$ and $(0, 1, \omega^2, \omega, 0, 0)$. These

https://doi.org/10.1017/S0305004100025500 Published online by Cambridge University Press

are conjugate to E, E', two vertices like $(-\sigma, 1, 1, 1, 1, -1)$ and six vertices like $(1, \sigma, 1, -1, 1, -1)$; that is, to ten vertices which lie in a W-solid, $E \times (j$-plane through $E')$. We thus find that the eight generators a_i, b_i are conjugate to W-solids in which the component j-plane is conjugate to E, while the remaining eight generators a_i', b_i', correspond to W-solids $E' \times (j$-plane through $E)$. The j-planes in the W-solids conjugate to the lines a_1 and b_1 say, determine an S-solid which is a component of the λ-prime conjugate to the point (a_1, b_1); the κ-line in which the two j-planes meet is a component of the d-plane conjugate to the plane $[a_1, b_1]$. Two generators of the same system on $\mathcal{Q}$, say a_1 and a_2, are conjugate to W-solids of which the component j-planes meet only in E'; the j-planes therefore do not lie together in a [3], and so the corresponding generators a_1 and a_2 have no common point.

The fact that these generators are conjugate to the sixteen W-solids through e_0 could have been foreseen from the fact that there are four α-primes through a W-solid, so that the conjugate line contains four H-points. Since the H-points all lie on $\mathcal{Q}$, the conjugate line, containing four points of $\mathcal{Q}$, is a generator. Since these generators also each contain four points of either L or L', we see also that the W-solids each lie in four λ-primes which have e_0 as a strut, and a little reflexion shows that a W-solid through e_0 cannot lie in a λ-prime in which e_0 appears in a component, so that there are just four λ-primes through a W-solid.

We pass now to an important property of the g-lines. Consider a particular generator w_0 conjugate to a W-solid W_0. The six g-lines which are generators of $\mathcal{Q}$ and of the opposite system to w_0 meet w_0 in six distinct points. Now W_0 contains nine e-lines in all, among them e_0. It follows that w_0 lies on the quadrics in eight other J-solids, as well as the one we have been considering. In each of these solids there will be six g-lines meeting w_0, and they will all meet w_0 in the same six points, since, as w_0 does not lie on Ψ, it contains only six points of Ψ. We have thus arrived at six points, through each of which pass nine g-lines. There are six of these points on each line conjugate to a W-solid, and so there are $6 \times 5040 = 30{,}240$ of them in the whole configuration. Further, since through each point there pass nine of the 34,020 g-lines, there are

$$(30{,}240 \times 9)/34{,}020 = 8$$

of these points on each g-line; this can also be seen from the fact that a g-line which is a generator of a quadric $\mathcal{Q}$ meets each of the eight lines which are generators, of the opposite system on $\mathcal{Q}$, and which are conjugate to W-solids.

We shall in future refer to the lines which are conjugate to W-solids as w-lines. They have one further interesting property. Since a W-solid consists of a j-plane joined to a vertex conjugate to it, the corresponding w-line lies in the conjugate j-plane, and is conjugate to one of the vertices in that plane. Now we have seen that the twelve κ-lines in a j-plane form four triangles, whose angular points are the twelve H-points. These four triangles are such that any three are in perspective from each angular point of the fourth. The nine perspective lines which arise in this way are each conjugate to one of the nine vertices in the plane, and so they are identified with the w-lines, which thus clearly each contain four H-points. Now the eight w-lines which are generators of one system on $\mathcal{Q}$ arise in four pairs of conjugate lines (for example, if we select the line a_1

https://doi.org/10.1017/S0305004100025500 Published online by Cambridge University Press

which contains H_1, the line a_1' which contains the other point H_1' on the κ-line through H_1 is conjugate to a_1). A pair of conjugate w-lines therefore lie each in one of two conjugate j-planes, and so the g-lines link pairs of conjugate j-planes. Consider such a pair of conjugate planes j_1 and j_1' say, and let w_1 be a w-line in j_1. Through one of the six points of Ψ on w_1 there are nine g-lines each meeting j_1' on one of the nine w-lines in it. The fifty-four g-lines through points on w_1 pass one through each of the fifty-four points of Ψ on w-lines in j_1'. So we find that there are altogether $6 \times 9 \times 9 = 486$ g-lines linking j_1 and j_1', and a particular g-line is a transversal of four pairs of conjugate j-planes.

A set of points in J_0 whose conjugate primes have not so far been located are the twelve angular points of the set of desmic tetrahedra associated with the desmic set of vertices. These twelve points are the points of intersection of e-lines which do not meet in vertices. When the β-hexahedron was described in § 1, it was noted that the angular points, B-points, are points in which the e-lines in a c-plane intersect, and so these B-points can be identified with the angular points of the associated set of desmic tetrahedra. Through e_0 there are three β-hexahedra, each of which has four of its angular points lying in J_0; in this way the three tetrahedra of B-points are formed.

The remaining conjugate points of primes through e_0 have no obvious bearing on the geometry of Ψ, nor do they lie on $\mathit{2}$. The conjugate lines of solids through e_0 which have not yet been mentioned, though interesting in themselves, are also not immediately relevant; we shall therefore conclude this section with a brief list of these remaining lines, and verification of the facts stated will be left to the reader.

(i) Consider the tangent plane $[a_1, b_1]$ to $\mathit{2}$ at $L_1(a_1, b_1)$. This plane contains a κ-line H_1H_1', on which the vertices are P, Q, R, say. Let P' be the harmonic conjugate of P with respect to Q and R, and define Q' and R' similarly. Then the three lines L_1P', L_1Q' and L_1R' are conjugate to three V-solids, having e_0 as a strut, and the ninety-six V-solids of this type all arise in this way.

(ii) The line joining two H-points in different sets which are conjugate but not on the same κ-line is conjugate to one of the ninety-six U-solids through e_0.

(iii) One of the twelve vertices in J_0 is joined to six of the twelve angular points of the associated desmic set by e-lines. It is joined to the other six by lines which are conjugate to X-solids in which e_0 lies in the component c-plane. There are in all seventy-two of these lines in J_0.

(iv) The other X-solids, of which e_0 is a strut, are conjugate to lines joining two points of L (or of L'), such that each is not in the tangent plane of the other. There are clearly seen to be $\frac{1}{2}(16 \times 9 \times 2) = 144$ of these lines, corresponding to the remaining 144 X-solids, and each line passes through one of the vertices, conjugate to the unique π-prime in which the appropriate X-solid lies.

(v) Finally, there are 144 N-solids, which are conjugate to lines joining two H-points in opposite sets which are not mutually conjugate. These lines also each pass through one vertex.

5. Quadrics in β-hexahedra

When investigating the ways in which g-lines meet, it is simplest to examine the way in which two J-solids intersect. Two J-solids which do not lie in a prime meet in a line, and it is not hard to show in each case that the only lines (other than e-lines) in which

https://doi.org/10.1017/S0305004100025500 Published online by Cambridge University Press

J-solids meet are skew to the g-lines in those solids, and so do not yield any fresh information. The case of two J-solids which intersect in an e-line will be considered below. If two J-solids lie in a prime their conjugate e-lines determine a plane, and so three cases arise, according as the plane is a d-plane, a c-plane or an f-plane. The first case has already been considered, for then the J-solids meet in a plane conjugate to the d-plane, which, as we showed in the last section, is a tangent plane to the quadric surface in each of the J-solids. The g-lines in the J-solids through this plane meet it in the same twelve points, six on each of the generators in it. The second case will be dealt with in this section, and since in this case the two J-solids lie in a β-prime, we shall now consider the configuration of g-lines in a β-prime and in a β-hexahedron.

In the J-solid $x_4 = x_5 = 0$, the twelve g-lines all lie on the quadric surface $\sum_{i=0}^{3} x_i^2 = 0$. This solid lies with four other J-solids $x_i = x_5 = 0$ $(i = 0, 1, 2, 3)$ in the β-prime $x_5 = 0$, and it is clear that the five quadric surfaces in the J-solids in this prime are all sections of the quadric $\sum_{i=0}^{4} x_i^2 = 0, x_5 = 0$. Moreover, any β-prime lies with five others in a β-hexahedron, and in this case the hexahedron is the simplex of reference. The six quadrics in the β-primes $x_i = 0$ $(i = 0, \ldots, 5)$ are evidently all sections of a quadric $\Omega \equiv \sum_{i=0}^{5} x_i^2 = 0$. Thus, since the group is transitive on β-hexahedra, there is associated with each β-hexahedron a quadric primal, the sections of which by each of the fifteen J-solids in the hexahedron are the quadric surfaces on which the g-lines are generators. We conclude that on Ω there are $12 \times 15 = 180$ g-lines. Incidentally, since there are altogether 34,020 g-lines, and 567 β-hexahedra, a g-line must lie on the quadric primal associated with $(180 \times 567)/34020 = 3$ β-hexahedra, which agrees with the fact that a J-solid lies in three β-hexahedra.

Some of the planes of the quadric primals whose existence has just been proved bear a special relation to the sextic primal Ψ. Consider, for example, the plane $(\lambda, \lambda i, \mu, \mu i, \nu, \nu i)$. This is clearly a plane of Ω; it meets Ψ in the sextic curve given by $\lambda^2\mu^2\nu^2 = 0$, that is, in three lines taken twice. The three lines in the plane, given by $\lambda = 0$, $\mu = 0$, $\nu = 0$, are g-lines. Their points of intersection are $(1, i, 0, 0, 0, 0)$, $(0, 0, 1, i, 0, 0)$ and $(0, 0, 0, 0, 1, i)$, three points which are mutually conjugate. We shall refer to points of this type as *D-points*; they have already arisen in § 3, as angular points of Klein tetrahedra. The planes we shall call Δ-*planes*; there are $6!/3! = 120$ of them lying on Ω.

Each Δ-plane is conjugate to another Δ-plane; for example the plane $(\lambda, \lambda i, \mu, \mu i, \nu, \nu i)$ is conjugate to the plane $(\lambda i, \lambda, \mu i, \mu, \nu i, \nu)$. In each of these planes there are three D-points, which together form a set of six mutually conjugate points. The points may be named

$$L = (1, i, 0, 0, 0, 0), \quad M = (0, 0, 1, i, 0, 0), \quad N = (0, 0, 0, 0, 1, i),$$
$$L' = (i, 1, 0, 0, 0, 0), \quad M' = (0, 0, i, 1, 0, 0), \quad N' = (0, 0, 0, 0, i, 1).$$

The three lines LL', MM' and NN' are e-lines, while the other twelve lines joining two of the points are g-lines. Through each g-line there are clearly two Δ-planes, and so there are eight Δ-planes in the figure, namely the planes containing one point from each

https://doi.org/10.1017/S0305004100025500 Published online by Cambridge University Press

of the pairs LL', MM' and NN'. Each of these eight planes is conjugate to one of the others, and any pair, such as LMN and $L'M'N'$, contains all six points of the set. As two conjugate planes have no common point, the planes of a conjugate pair do not meet, and so belong to opposite systems on Ω. The set of D-points which forms the basis of this figure will be called an *octic set*, as there are eight Δ-planes defined by its points.

Now consider the pair of points N, N'. Each is joined by g-lines to the four points L, L', M, M', which lie in the J-solid $x_4 = x_5 = 0$ conjugate to NN'. Reference to the list given in § 3 shows these points to be the angular points of the tetrahedron $\{ad\}$. The joins of N and N' to the angular points of the tetrahedra $\{bd\}$ and $\{cd\}$ are also evidently g-lines. However, the line joining N, or N', to one of the angular points of any of the remaining tetrahedra $\{ae\} \ldots \{cf\}$ does not lie on Ψ, and so we conclude that through each of N, N' there are just twelve g-lines, which meet the J-solid conjugate to the e-line NN' in the same twelve points. These twelve points are the angular points of three of the Klein tetrahedra in the solid; the tetrahedra are not a desmic set, but any two of them are in fourfold perspective from one of the tetrahedra of vertices. Since there are six D-points on each g-line (from the geometry of the Klein configuration) and twelve lines through each D-point, there are $(34{,}020 \times 6)/12 = 17{,}010$ D-points altogether in the configuration.

The octic sets obtained above are related in ways which are of interest. We have shown that there are three such sets containing the pairs of points N, N'. Now these points are one pair of conjugate points out of three such pairs on an e-line (§ 2). They are joined to the angular points of the tetrahedra $\{ad\}$, $\{bd\}$ and $\{cd\}$ by g-lines, similarly, points of a second pair are joined by g-lines to the angular points of $\{ae\}$, $\{be\}$ and $\{ce\}$, and each of the remaining two points to the angular points of the last three tetrahedra $\{af\}$, $\{bf\}$ and $\{cf\}$. So we see that an e-line is the diagonal line of nine octic sets. But in each such set there are three diagonal lines (LL', MM' and NN' in the example given), and so there are $\frac{1}{3} \times 2835 \times 9 = 8505$ such sets altogether. On a quadric primal there are fifteen octic sets (corresponding to the partitions of three numbers into three pairs), and so each octic set lies on $(567 \times 15)/8505 = 1$ quadric primal. Further, since each set is determined when one plane of it is given, each Δ-plane lies on a unique quadric, and so there are $(567 \times 120) = 68{,}040$ Δ-planes in all.

From the numbers we have now established, it is clear that a g-line is common to three octic sets. Since a J-solid lies in three β-hexahedra, it follows that a g-line lies on the three quadric primals associated with these β-hexahedra, and each of the three octic sets is associated with one of the primals.

An e-line is an edge of three β-hexahedra, and so on an e-line there are two B-points from each of these hexahedra. From each pair of B-points we derive a pair of D-points (the double points of the involution determined by the B-points and the two vertices), and so we arrive at the three pairs of D-points on the e-line, which are the six points in which it meets Ψ, and we see that one pair of the points is associated with each hexahedron through the e-line. Now through an e-line there are eighteen J-solids, six lying in each of the β-hexahedra containing it. Consider a particular e-line, e_0, and two J-solids through it, J_1 and J_2. Of the twelve g-lines in J_1, four meet e_0, two passing

https://doi.org/10.1017/S0305004100025500 Published online by Cambridge University Press

through each of the D-points associated with the β-hexahedron containing J_2. First suppose J_1 and J_2 meet only in e_0. Then there are two cases to consider.

(i) If J_1 and J_2 lie in the same β-hexahedron, four g-lines of J_1 meet four g-lines of J_2 in the two D-points associated with that β-hexahedron.

(ii) If J_1 and J_2 lie in different β-hexahedra through e_0, the g-lines which meet e_0 do so in different points, and so the g-lines of J_1 are skew to those of J_2.

Now suppose that J_1 and J_2 meet in a c-plane through e_0. Then they lie together in a β-prime and so are in the same β-hexahedron. The twelve g-lines of J_1 meet the c-plane in six D-points, one pair on each of the three e-lines in the c-plane, and the twelve g-lines of J_2 also pass in pairs through the same six points.

From the point of view of the octic sets, consider two g-lines, g_1 and g_2 say, through N, meeting the J-solid conjugate to NN' in the points A_1 and A_2. Then if A_1 and A_2 belong to the same tetrahedron and are joined by an e-line, it is easy to see that g_1 and g_2 lie in the same J-solid. If A_1 and A_2 belong to the same tetrahedron but are joined by a g-line, g_1 and g_2 lie in J-solids which meet in an e-line; while if A_1 and A_2 belong to different tetrahedra, the J-solids containing g_1 and g_2 meet in a c-plane.

6. The section of Ψ by a π-prime

We still have to consider the way in which g-lines meet when the J-solids in which they lie intersect in an f-plane. In this case the two solids lie together in a π-prime. We saw in the last section that each e-line in a J-solid is met by four g-lines in that solid, the g-lines passing in pairs through each of two D-points on the e-line. Now in an f-plane there are three e-lines, and so all twelve g-lines of a J-solid through the f-plane meet the plane on these three e-lines. Thus the g-lines of two J-solids through the f-plane, if they meet at all, can meet only in D-points. But we saw in the last section that two g-lines through a D-point belong to J-solids which meet only in an e-line or in a c-plane, and so the g-lines lying in two J-solids which intersect in an f-plane do not meet.

We are thus led to the conclusion that g-lines intersect in only two types of points:

(i) There are eight points on each g-line (the points where it meets w-lines), through each of which there pass nine g-lines.

(ii) There are six D-points on each g-line, where it meets e-lines, through each of which there pass twelve g-lines.

The way in which g-lines link pairs of conjugate e-lines in a π-prime is as follows. In a π-prime there are twenty-seven sets of mutually conjugate vertices, the five vertices of a set being joined by ten e-lines. Consider one e-line e_0; it is conjugate to an f-plane f_0. On e_0 there are six D-points, in three conjugate pairs. The selection of one pair of D-points from e_0 determines a particular e-line in f_0, and a pair of D-points on that e-line, such that the four lines joining these two pairs are all g-lines.

The section of Ψ by a π-prime is a sextic primal Ψ^* in [4], which is an invariant of the group G^*. Its properties are easily deduced from those of Ψ which have already been described in this paper. It contains 540 g-lines, which are the generators, six of each system, on forty-five quadrics, one in each of forty-five J-solids. On one such quadric there are also eight generators, four of each system, conjugate to the eight j-planes through the vertex conjugate to the J-solid. These are the lines which in the

https://doi.org/10.1017/S0305004100025500 Published online by Cambridge University Press

figure in [5] were called w-lines. So we find that in a π-prime there are only four points on a g-line through each of which pass eight further g-lines. Also on a g-line there are six D-points, but through each pass only five further g-lines in the π-prime.

As none of the Δ-planes which arose in connexion with Ψ lie in a π-prime, there are no planes of this type touching Ψ^*. In fact, the properties of Ψ^* are in general less interesting and less complete than those of the sextic primal in [5] of which it is a section.

In my investigation of the geometry of these invariant primals I have had the benefit of constant encouragement and advice from Dr J. A. Todd, to whom I should like to express my thanks.

REFERENCES

(1) Baker, H. F. *A locus with* 25920 *linear self-transformations* (Cambridge, 1946).
(2) Burkhardt, H. *Math. Ann.* 38 (1890), 161–224.
(3) Hamill, C. M. To appear in *Proc. London Math. Soc.*
(4) Hudson, R. W. H. T. *Kummer's quartic surface* (Cambridge, 1905).
(5) Mitchell, H. H. *American J. Math.* 36 (1914), 1–12.
(6) Schubert, H. *Math. Ann.* 26 (1886), 52–73.
(7) Todd, J. A. *Proc. Cambridge Phil. Soc.* 46 (1950), 73–90.

Newnham College
Cambridge

https://doi.org/10.1017/S0305004100025500 Published online by Cambridge University Press