Canad. J. Math. 2026, pp. 1–18
http://dx.doi.org/10.4153/S0008414X2610220X
© The Author(s), 2026. Published by Cambridge University Press on behalf of Canadian Mathematical Society. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (https://creativecommons.org/licenses/by/4.0), which permits unrestricted re-use, distribution and reproduction, provided the original article is properly cited.

Theta cycles of modular forms modulo p^2

Scott Ahlgren, Martin Raum, and Olav K. Richter

Abstract. The theta cycle of a modular form modulo a prime $p \geq 5$ is well understood. By contrast, the theta cycle modulo a power of p is still mysterious and experimentally erratic. Here, we completely determine the theta cycle of a weight $k < p$ modular form modulo p^2 on the initial segment of length p and we prove exact values or nontrivial bounds for the weight filtrations on $p-2$ further segments of length $p-k+1$. In particular, asymptotically as $p \to \infty$, we establish 50% of the theta cycle exactly, and we provide nontrivial bounds for 100% of it. We determine the first two low points exactly and $\left\lfloor \frac{p-k+1}{2} \right\rfloor$ further low points at regular positions. Moreover, we detect low points at *exceptional positions* which solve a quadratic equation modulo p, and which disturb the otherwise regular structure in the segments that we exhibit.

1 Introduction

Throughout, let $p \geq 5$ be a prime. Given a quasi-modular form f on $\mathrm{SL}_2(\mathbb{Z})$ with p-integral rational coefficients and an integer $m \geq 1$, let $\omega_{p^m}(f)$ be the weight filtration of f modulo p^m as defined in (2.1). Iteration of the theta operator θ (defined in (2.7)) yields the extended *theta cycle of f modulo p^m*:

$$\Omega_{p^m} f := \left(\omega_{p^m}(f),\ \omega_{p^m}(\theta^1 f),\ \ldots,\ \omega_{p^m}\left(\theta^{(p-1)p^{m-1}+m-1} f\right)\right),$$

and the periodic subsequence starting with the weight filtration of $\theta^m f$ is called the (Tate) *theta cycle*. In the case $m = 1$, theta cycles of modular forms are completely understood. As an example of a central application which leverages this detailed understanding we mention Edixhoven's work [5] on the weight in Serre's conjecture on modular forms. In a different direction, this understanding leads to the classification [1] of Ramanujan congruences for the partition function.

An integer i, or by extension the quasi-modular form $\theta^i f$, is a *low point* of the cycle if

$$\omega_{p^m}\left(\theta^{i-1} f\right) > \omega_{p^m}\left(\theta^i f\right) \quad \text{and} \quad \omega_{p^m}\left(\theta^i f\right) < \omega_{p^m}\left(\theta^{i+1} f\right).$$

Further, we say that we have a *rise* (resp. a *fall*) at i if

$$\omega_{p^m}\left(\theta^i f\right) < \omega_{p^m}\left(\theta^{i+1} f\right) \qquad \left(\text{resp.} \quad \omega_{p^m}\left(\theta^i f\right) > \omega_{p^m}\left(\theta^{i+1} f\right)\right).$$

Received by the editors July 31, 2025; revised February 23, 2026; accepted April 6, 2026.
Published online on Cambridge Core April 13, 2026.
S.A. was partially supported by a grant from the Simons Foundation (Grant No. 963004). M.R. was partially supported by Vetenskapsrådet Grant 2023-04217. O.K.R. was partially supported by a grant from the Simons Foundation (Grant No. 835652).
AMS subject classification: **11F33**, 11F11.
Keywords: Theta operator, theta cycle, low points.

https://doi.org/10.4153/S0008414X2610220X Published online by Cambridge University Press

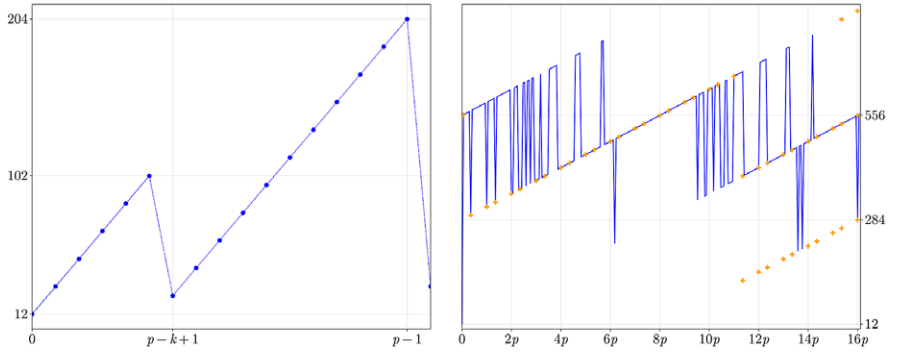


Figure 1: The weight filtrations modulo $p = 17$ (on the left) and p^2 (on the right) of $\theta^i \Delta$, where Δ is the normalized cusp form of weight $k = 12$. Filtration values modulo p given for $0 \le i \le p$ on the x-axis are connected by a dashed line. Filtrations modulo p^2 given for $0 \le i \le p(p-1)$ are represented by a blue line connecting the values. Orange crosses indicate previously known values (in some cases not uniquely determined) for $i = 1$ [4], $i \in p\mathbb{Z}$ [7], and $i \in p - k + 1 + p\mathbb{Z}$ [7].

The theta cycles modulo p are determined by the positions and weight filtrations of their low points [6]. There are either one or two low points in a theta cycle modulo p and the remaining part of $\Omega_p(f)$ is highly regular, as illustrated on the left in Figure 1. The behavior of the theta cycle modulo p^2 is by contrast more intricate and seemingly erratic, as illustrated on the right. Indeed, only partial results on the isolated points $i = 1$ [4] and $i \in p\mathbb{Z}$ or $i \in p - k + 1 + p\mathbb{Z}$ [7] are available. In particular, not a single location of a low point is known.

As one example of the complicated structure which appears, we mention that the theta cycle modulo p^2 can have successive falls; this is not possible in the cycle modulo p. An example of this phenomenon occurs for the normalized weight 12 cusp form Δ when $p = 13$: we have

$$\omega_{13^2}\left(\theta^{133}\,\Delta\right) = 434, \quad \omega_{13^2}\left(\theta^{134}\,\Delta\right) = 280, \quad \text{and} \quad \omega_{13^2}\left(\theta^{135}\,\Delta\right) = 126.$$

Here, we determine exact values for weight filtrations in long segments of theta cycles modulo p^2. Our first result provides the exact weight filtration for all $\theta^i f$ with $0 \le i \le p$. A closer inspection of Figure 1 reveals the alignment of the first two low points in the theta cycles modulo p and p^2. Our theorems in particular imply that these low points always occur and that they are the only ones in this range.

To state the result, recall that $p \ge 5$ and denote by M_k the space of weight k modular forms on $\mathrm{SL}_2(\mathbb{Z})$ with p-integral rational Fourier coefficients (recall that $\mathrm{M}_k = \{0\}$ unless $k = 0$ or $k \ge 4$ is even). We will often assume that $f \in \mathrm{M}_k$ with $0 < k < p$ has $\omega_p(f) = k$. This assumption simply ensures that f is not a constant multiple of the Eisenstein series $E_{p-1} \equiv 1 \pmod{p}$.

https://doi.org/10.4153/S0008414X2610220X Published online by Cambridge University Press

Theorem A *Suppose that $f \in M_k$ with $0 < k < p$ has $\omega_p(f) = k$. Then we have the following exact weight filtrations:*

$$\begin{aligned}
&\omega_{p^2}\left(\theta^i f\right) = k, && \text{if } i = 0;\\
&\omega_{p^2}\left(\theta^i f\right) = k + 2i + 2p(p-1), && \text{if } 0 < i < p - k + 1;\\
&\omega_{p^2}\left(\theta^i f\right) = k + 2i + p(p-1), && \text{if } i = p - k + 1;\\
&\omega_{p^2}\left(\theta^i f\right) = k + 2i + 2p(p-1), && \text{if } p - k + 1 < i < p;\\
&\omega_{p^2}\left(\theta^i f\right) = k + 2i + p(p-1), && \text{if } i = p.
\end{aligned}$$

The next corollary will follow from Theorems A and C.

Corollary B *Suppose that $f \in M_k$ with $0 < k < p$ has $\omega_p(f) = k$. The first low point of the theta cycle of f modulo p^2 occurs at $i = p - k + 1$ and the second at $i = p$. Moreover, if $0 < i < p - 1$, then*

$$\omega_{p^2}\left(\theta^{i+1} f\right) = \omega_{p^2}\left(\theta^i f\right) + 2 \quad \text{unless } i = p - k.$$

Remark

(1) The weight filtrations at the first and second low points $i = p - k + 1$, $i = p$ were determined in [7], but it was not proved that these are low points.
(2) The positions of the first low point modulo p and p^2 agree, but unlike the modulo p case, the modulo p^2 weight filtration of the first low point is independent of whether or not f has a U_p-congruence modulo p.
(3) The position of the second low point modulo p^2 matches the position of the second low point for modular forms which do not have a U_p-congruence modulo p.
(4) The analysis in [6] shows that $\omega_p(\theta^{i+1} f) \neq \omega_p(\theta^i f) + 2$ for all $i \geq 1$, which is in contrast to the situation modulo p^2, where a rise of 2 is the common case.

Our next result determines a substantial part of the weight filtration modulo p^2 of $\theta^i f$ for general i and identifies families of low points. Some of them appear at regular intervals, and some appear at what we call exceptional positions. We will consider indices i with

$$np \leq i \leq np + p - k + 1, \qquad 1 \leq n < p. \tag{1.1}$$

We call such an index *exceptional* if it is a solution of the congruence

$$i^2 + (k-1)i - n^2 \equiv 0 \pmod{p}. \tag{1.2}$$

In Corollary D, we will show that these exceptional indices often correspond to low points which disrupt regularity in the theta cycle.

Theorem C *Suppose that $f \in M_k$ with $0 < k < p$ has $\omega_p(f) = k$. Let i be as in (1.1).*

(1) *If i is exceptional, then*

$$\omega_{p^2}\left(\theta^i f\right) \leq k + 2i + p(p-1).$$

(2) *If i is not exceptional, then we have the following exact values and nontrivial bounds, where $i' = i - np$ is the least nonnegative residue of $i \pmod{p}$:*

$$\omega_{p^2}\left(\theta^i f\right) \leq k + 2i + p(p-1), \qquad \text{if } i' = 0 \text{ (proved in [7]);}$$

https://doi.org/10.4153/S0008414X2610220X Published online by Cambridge University Press

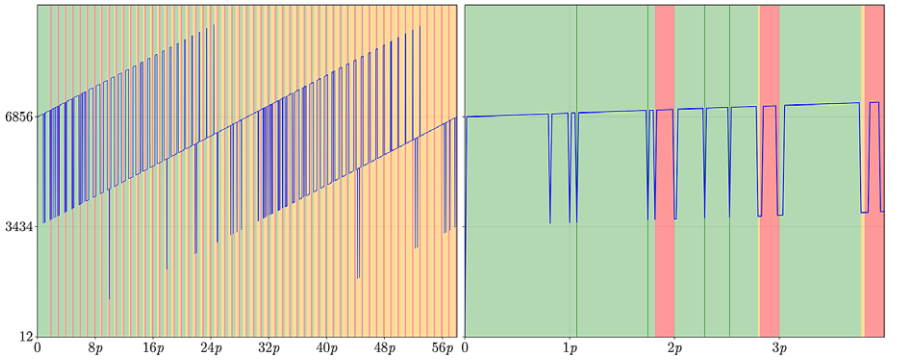


Figure 2: The weight filtrations modulo $p^2 = 59^2$ of $\theta^i \Delta$, where Δ is the normalized cusp form of weight 12. Filtration values are given for $0 \le i \le p(p-1)$ (on the left) and $0 \le i \le 4p$ (on the right) on the x-axis, and are represented by a blue line connecting them. Vertical green lines on the right indicate exceptional low points (see Corollary D). Green shaded areas represent ranges of i for which we prove exact filtration values. Orange shaded areas correspond to ranges of i for which we establish nontrivial upper bounds for the weight filtrations. We do not provide information on filtrations for i in the red shaded areas.

$$\begin{aligned}
&\omega_{p^2}\left(\theta^i f\right) = k + 2i + p(p-1), && \text{if } 0 < i' < n \text{ and } i' \le p - k + 1 - n;\\
&\omega_{p^2}\left(\theta^i f\right) = k + 2i + 2p(p-1), && \text{if } n \le i' \le p - k + 1 - n;\\
&\omega_{p^2}\left(\theta^i f\right) \le k + 2i + p(p-1), && \text{if } p - k + 1 - n < i' < p - k + 1;\\
&\omega_{p^2}\left(\theta^i f\right) \le k + 2i + p(p-1), && \text{if } i' = p - k + 1 \text{ (proved in [7]).}
\end{aligned}$$

For fixed weight and asymptotically as $p \to \infty$, our results provide exact filtration values for 50% of the theta cycle modulo p^2 and nontrivial bounds for 100% of it (see Remark 4.2). We illustrate the range of our results for $f = \Delta$ and $p = 59$ in Figure 2.

In this case, our results give exact filtration values for more than 33% of the theta cycle modulo p^2 and nontrivial bounds for more than 81.6%. If $f = \Delta$ and $p > 1,100$, then the range increases to more than 49% of exact weight filtrations and more than 99% of nontrivial filtration bounds.

In the next corollary, we describe some of the structure of the theta cycle which is forced by Theorem C.

Corollary D *Suppose that $f \in \mathrm{M}_k$ with $0 < k < p$ has $\omega_p(f) = k$.*

(1) *For $1 \le n \le \frac{p-k+1}{2}$, there is a low point at $i = np + p - k + 2 - n$ unless we have $\omega_{p^2}(\theta^i f) = k + 2i + p(p-1)$ and $\omega_{p^2}(\theta^{i+1} f) = k + 2(i+1)$, in which case there is a low point at $i + 1$.*
(2) *Assume that $np + n \le i < np + p - k + 1 - n$ and that i is an exceptional index. Then there is a low point at i (which we call an exceptional low point).*
(3) *Assume that $np < i \le np + p - k + 1 - n$ and that i is not an exceptional index. Then i is not a low point.*
(4) *Assume that $np < i < np + p - k + 1 - n$, that $i \ne np + n - 1$, and that neither i nor $i + 1$ is an exceptional index. Then there is a rise by 2 at i.*

https://doi.org/10.4153/S0008414X2610220X Published online by Cambridge University Press

Remark

(1) In the range $0 \le i < 2p - k + 1$, the two theorems give exact weight filtrations for $\theta^i f$ apart from possible exceptional indices. By part (2) of Corollary D and the fact that the index $i = 2p - k$ is not exceptional, it follows that there are exceptional low points at all exceptional indices in this range.
(2) There are two boundary cases $i' = 0$ and $i' = p - k + 1$ in Theorem C; these indices cannot be exceptional. These cases were studied in detail in [7], and in many cases, exact values for the filtration are given in [7, Theorem 1.8]. Here, we give a bound which follows from this result and which applies to all cases.
(3) The exact filtration values in Theorem C for $n \le i' \le p - k + 1 - n$ are independent of whether f has a U_p-congruence or not. Thus, information on U_p-congruences must be encoded in the part of the theta cycle for which we do not have exact filtration values.
(4) The filtration bounds in Theorem C are quadratic in p, while the bounds in Lemma 2.4 and Remark 2.5 below are cubic in p, and the bounds which follow from Inequality (2.19) are quartic in p.
(5) Let f be a modular form as in Theorems A and C. Then Corollary B identifies two low points occurring at regular positions, and the first part of Corollary D identifies $\left\lfloor \frac{p-k+1}{2} \right\rfloor$ such low points. All of these occur in the green-shaded areas of Figure 2.
(6) It would be interesting to determine the number of exceptional low points, which would amount to counting solutions to the congruence $i^2 + (k-1)i - n^2 \equiv 0 \pmod{p}$ with $n \le i < p - k + 1 - n$. Heuristically, the number of such solutions should be proportional to p on average as p varies.

We briefly describe some of the new ideas and input which allow us to access a large part of the theta cycle. In the next section, we introduce the *factor filtration*, a refinement of the weight filtration which plays a key role (all of the results above will follow from our study of the corresponding factor filtrations). Roughly speaking, the factor filtration involves dividing out as many powers of $E_{p-1} \equiv 1 \pmod{p}$ as possible before considering the weight filtration. This idea was inspired in part by recent work of the authors with Hanson [2] in which we prove that every Eisenstein series on $\mathrm{SL}_2(\mathbb{Z})$ has uniformly low factor filtration modulo p^2. In the same paper, we determine a precise expression (2.13) for $E_2 \pmod{p^2}$; this is another important input here. The starting point for our results is the expansion (2.11) of $\theta^i f$ in terms of E_2. Our ability to determine the precise factor filtration depends on isolating a group of terms in this expansion which combine to give a unique term of highest factor filtration (we often use the properties of the theta cycle modulo p in order to isolate these terms). When we are unable to isolate such a group of terms, the method still provides a nontrivial bound for the factor filtration. Finally, the weight filtrations are easily determined using Lemma 2.1.

After some preliminaries in Section 2, we prove Theorem A in Section 3. The proofs of Theorem C and Corollaries B and D are more involved and occupy Section 4. In Section 2.4, we discuss the quality of the general bounds which are obtained for filtrations modulo p^2.

https://doi.org/10.4153/S0008414X2610220X Published online by Cambridge University Press

2 Preliminaries

Let $p \geq 5$ be prime, and let M_k be the space of holomorphic weight k modular forms on $\mathrm{SL}_2(\mathbb{Z})$ with p-integral coefficients. Every $f \in \mathrm{M}_k$ has a Fourier expansion $\sum a(n)q^n$, and we identify f with this expansion. For even $k \geq 2$, let

$$E_k := 1 - \frac{2k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{k-1}(n)\, q^n$$

be the weight k Eisenstein series, which for $k = 2$ is quasi-modular and for $k > 2$ is modular.

Let m be a positive integer, and let $\mathcal{M}_{\bullet,m} \subseteq (\mathbb{Z}/p^m\mathbb{Z})[\![q]\!]$ be the set of reductions modulo p^m of all elements of all M_k, that is, the reductions of their Fourier expansions. If $\overline{f} = \sum \overline{a(n)} q^n \in \mathcal{M}_{\bullet,m}$, then we define the weight filtration

$$\omega_{p^m}(\overline{f}) := \inf\left\{k \,:\, \overline{f} \equiv g \pmod{p^m} \text{ for some } g \in \mathrm{M}_k\right\}. \tag{2.1}$$

Recall that

$$E_{p-1} \equiv 1 \pmod{p} \quad \text{and therefore} \quad E_{p-1}^{p^{m-1}} \equiv 1 \pmod{p^m}. \tag{2.2}$$

As a refinement of the weight filtration, we introduce the factor filtration, which is defined by

$$\widetilde{\omega}_{p^m}(\overline{f}) := \inf\left\{k \,:\, \overline{f} \equiv E_{p-1}^n g \pmod{p^m} \text{ for some } n \geq 0 \text{ and some } g \in \mathrm{M}_k\right\}. \tag{2.3}$$

By a standard abuse of notation, we will write

$$\omega_{p^m}(f) = \omega_{p^m}(\overline{f}), \qquad \widetilde{\omega}_{p^m}(f) = \widetilde{\omega}_{p^m}(\overline{f})$$

when $f \in \mathbb{Z}_{(p)}[\![q]\!]$ has $\overline{f} = f \pmod{p^m} \in \mathcal{M}_{\bullet,m}$. From [9, Lemma 5], we have $\omega_p(f) = \widetilde{\omega}_p(f)$. If $f_1 \in \mathrm{M}_{k_1}$, $f_2 \in \mathrm{M}_{k_2}$ have $f_1, f_2 \not\equiv 0 \pmod{p}$, then by [8, Theorem 1], we have

$$f_1 \equiv f_2 \pmod{p^m} \implies k_1 \equiv k_2 \pmod{p^{m-1}(p-1)}.$$

It follows that if $\overline{f} \in \mathcal{M}_{\bullet,m}$ is the reduction modulo p^m of a modular form of weight k, and if $\overline{f} \not\equiv 0 \pmod{p}$, then

$$\omega_{p^m}(\overline{f}) \equiv k \pmod{p^{m-1}(p-1)}. \tag{2.4}$$

We also have

$$\omega_{p^m}(f) \equiv \widetilde{\omega}_{p^m}(f) \pmod{p-1},$$

which follows from observing that $\omega_{p^m}(\overline{f}) = \omega_{p^m}(E_{p-1}^n g)$ on the right-hand side of (2.3).

The next lemma, which follows from the definitions and these facts, allows us to determine the weight filtration from the factor filtration.

Lemma 2.1 *Suppose that $\overline{f} \in \mathcal{M}_{\bullet,m}$ has $\overline{f} \not\equiv 0 \pmod{p}$ and that $\overline{f}$ is the reduction modulo p^m of a modular form of weight k. Then $\omega_{p^m}(\overline{f})$ is the smallest integer such that*

https://doi.org/10.4153/S0008414X2610220X Published online by Cambridge University Press

$$\omega_{p^m}(\overline{f}) \geq \widetilde{\omega}_{p^m}(\overline{f}) \quad \textit{and} \quad \omega_{p^m}(\overline{f}) \equiv k \pmod{p^{m-1}(p-1)}. \tag{2.5}$$

Proof Let $k_0 = \widetilde{\omega}_{p^m}(\overline{f})$. By definition, there exists $g \in M_{k_0}$ such that $\overline{f} \equiv E_{p-1}^n g \pmod{p^m}$ for some $n \geq 0$. Letting n' be the least nonnegative residue of n modulo p^{m-1}, we see by (2.2) that $\overline{f} \equiv E_{p-1}^{n'} g \pmod{p^m}$; we also see that $n'(p-1) + k_0$ is the integer characterized by (2.5). If $\omega_{p^m}(\overline{f}) \neq n'(p-1) + k_0$, then by (2.4), we would have the clear contradiction

$$\omega_{p^m}(\overline{f}) \leq n'(p-1) + k_0 - p^{m-1}(p-1) < k_0 = \widetilde{\omega}_{p^m}(\overline{f}). \qquad \blacksquare$$

We will often use the following bounds for the factor filtration. First, for $\overline{f}, \overline{g} \in \mathcal{M}_{\bullet,m}$, it is clear from the definition that

$$\widetilde{\omega}_{p^m}(\overline{f}\,\overline{g}) \leq \widetilde{\omega}_{p^m}(\overline{f}) + \widetilde{\omega}_{p^m}(\overline{g}).$$

Further, if $\overline{f}$ and $\overline{g}$ are the reductions of elements of M_k, then we have

$$\widetilde{\omega}_{p^m}(\overline{f} + \overline{g}) \leq \max\left\{\widetilde{\omega}_{p^m}(\overline{f}), \widetilde{\omega}_{p^m}(\overline{g})\right\} \quad \text{with equality if } \widetilde{\omega}_{p^m}(\overline{f}) \neq \widetilde{\omega}_{p^m}(\overline{g}). \tag{2.6}$$

For the equality in (2.6), we argue as follows: Suppose that $k_0 = \widetilde{\omega}_{p^m}(\overline{f}) > \widetilde{\omega}_{p^m}(\overline{g})$. Then $\overline{f} + \overline{g} \equiv E_{p-1}^a \left(f' + E_{p-1} g'\right) \pmod{p^m}$ for some $f' \in M_{k_0}$, $g' \in M_{k_0-(p-1)}$. If $\widetilde{\omega}_{p^m}(\overline{f} + \overline{g}) < k_0$, then $f' + E_{p-1} g' \equiv E_{p-1} h \pmod{p^m}$ for some $h \in M_{k_0-(p-1)}$, which contradicts $k_0 = \widetilde{\omega}_{p^m}(\overline{f})$.

2.1 The theta operator and Serre derivative

If $f = \sum a(n) q^n \in \mathbb{Z}_{(p)}[\![q]\!]$, then the action of the theta operator is given by

$$\theta f := q \frac{\mathrm{d}f}{\mathrm{d}q} = \sum n a(n) q^n. \tag{2.7}$$

It does not preserve modularity, but it allows for a modular correction, which leads to the Serre derivative: if $f \in \mathrm{M}_k$, then (since $p \geq 5$) we have

$$\partial f := \partial_k f := \theta f - \frac{k}{12} f E_2 \in \mathrm{M}_{k+2}. \tag{2.8}$$

Note that there are two common normalizations; the Serre derivative in [2] corresponds to 12∂ in this article.

From (64) of the first part of [3], we adopt the modified Serre derivative

$$\widehat{\partial}_k^0 f := f, \quad \widehat{\partial}_k^1 f := \partial_k f, \quad \widehat{\partial}_k^{i+1} f := \partial_{k+2i}\left(\widehat{\partial}_k^i f\right) - \frac{i(i+k-1)}{12^2} E_4 \, \widehat{\partial}_k^{i-1} f \quad \text{for } i \geq 1. \tag{2.9}$$

Observe that our notation differs from that in [3] in order to accommodate established notation for the theta operator. The modified Serre derivative preserves modularity

https://doi.org/10.4153/S0008414X2610220X Published online by Cambridge University Press

and p-integrality of Fourier coefficients: if $f \in \mathrm{M}_k$, then we have

$$\widehat{\partial}_k^i f \in \mathrm{M}_{k+2i}. \tag{2.10}$$

The theta cycle modulo p was completely described by Jochnowitz (following Tate). Here, we say that f is non-ordinary (at p) if it has a U_p-congruence modulo p, that is, $a(np) \equiv 0 \pmod{p}$ for all $n \in \mathbb{Z}$, which is equivalent to $\theta^{p-1} f \equiv f \pmod{p}$. Note that the notion of ordinary and non-ordinary modular forms is usually reserved for eigenforms, and here we extend it to all modular forms.

Proposition 2.2 [6] *Suppose that $f \in \mathrm{M}_k$ with $0 < k < p$ has $\omega_p(f) = k$. Then*

$$\theta^i f \equiv \theta^{i-p+1} f \pmod{p} \quad \text{for } i \geq p,$$

and for $0 \leq i < p$, we have the filtration values

$$\omega_p\big(\theta^i f\big) = \begin{cases} k + i(p+1), & \text{if } 0 \leq i < p-k+1; \\ k + i(p+1) - (p-k+1)(p-1), & \text{if } p-k+1 \leq i < p \text{ and } f \text{ is ordinary;} \\ k + i(p+1) - (p-k+2)(p-1), & \text{if } p-k+1 \leq i < p-1 \\ & \text{and } f \text{ is non-ordinary;} \\ k = k + i(p+1) - (p+1)(p-1), & \text{if } i = p-1 \text{ and } f \text{ is non-ordinary.} \end{cases}$$

We recall two basic facts [8, Lemma 1] and [9, Lemma 5]: If $\overline{f} \in \mathcal{M}_{\bullet,1}$, then

$$\omega_p(\overline{f}^a) = a\,\omega_p(\overline{f}),$$
$$\omega_p(\theta\,\overline{f}) \leq \omega_p(\overline{f}) + p + 1 \quad \text{with equality unless } \omega_p(\overline{f}) \equiv 0 \pmod{p}.$$

2.2 Expansion of $\theta^i f$ in E_2

If $f \in \mathrm{M}_k$, then the quasi-modular form $\theta^i f$ has an expansion in powers of the Eisenstein series E_2. In particular, from (65) of [3, Chapter 1], we have the expansion

$$\theta^i f = \sum_{j=0}^{i} \binom{i}{j} \frac{(i+k-1)!}{(j+k-1)!} \big(\widehat{\partial}_k^j f\big) \big(\tfrac{1}{12} E_2\big)^{i-j}, \tag{2.11}$$

where $\widehat{\partial}_k$ is the modified Serre derivative in (2.9). For simplicity, we will write

$$f_j := \widehat{\partial}_k^j f$$

throughout the article when using this formula.

2.3 The weight-2 Eisenstein series modulo p^2

For the duration, we will focus on the case $m = 2$. From [9] (see Theorem 2(iii) and Lemma 5), we know that

$$\omega_p(fE_2^n) = \omega_p(fE_{p+1}^n) = \omega_p(f) + n(p+1), \quad n \geq 0; \tag{2.12}$$

to make full use of (2.11), we need the analog of this fact for factor filtrations modulo p^2. From Theorem 1.2 of [2] and the remarks which follow (recall that the Serre derivative

https://doi.org/10.4153/S0008414X2610220X Published online by Cambridge University Press

is normalized differently there), we have

$$E_2 \equiv 12\partial E_{p-1} E_{p-1}^{2p-1} + pE_{p+1}^{p} E_{p-1}^{p-2} \pmod{p^2} \tag{2.13}$$

(note that the right side is a modular form of weight $2 + 2p(p-1)$). Since

$$12\partial E_{p-1} \equiv E_{p+1} \pmod{p}, \tag{2.14}$$

it follows that

$$E_2^n \equiv \left(12\partial E_{p-1}\right)^n E_{p-1}^{n(2p-1)} + pnE_{p+1}^{n+p-1} E_{p-1}^{(n-1)(2p-1)+p-2} \pmod{p^2}. \tag{2.15}$$

We will need the following result; with $f = 1$ and $k = 0$, it gives the factor filtration of E_2^n.

Lemma 2.3 *Suppose that $k \geq 0$, that $\omega_p(f) = k$, and that $n \geq 0$. Then*

$$\widetilde{\omega}_{p^2}(pfE_2^n) = k + 2n + n(p-1).$$

If, in addition, we have $\widetilde{\omega}_{p^2}(f) = k$, then

$$\widetilde{\omega}_{p^2}(fE_2^n) = \begin{cases} k + 2n + n(p-1) + (p+1)(p-1) & \text{if } p \nmid n; \\ k + 2n + n(p-1) & \text{if } p \mid n. \end{cases}$$

Proof Note that, for any f, we have

$$\widetilde{\omega}_{p^2}(f) \geq \omega_p(f) \quad \text{and} \quad \widetilde{\omega}_{p^2}(pf) = \omega_p(f). \tag{2.16}$$

For $n > 0$, (2.15) gives

$$fE_2^n \equiv f\left(12\partial E_{p-1}\right)^n E_{p-1}^{n(2p-1)} + pnfE_{p+1}^{n+p-1} E_{p-1}^{(n-1)(2p-1)+p-2} \pmod{p^2}. \tag{2.17}$$

If $p \mid n$, then (2.17) gives $\widetilde{\omega}_{p^2}(fE_2^n) \leq k + n(p+1)$, while (2.12) and (2.16) give

$$\widetilde{\omega}_{p^2}(fE_2^n) \geq \omega_p(fE_2^n) = k + n(p+1) = k + 2n + n(p-1),$$

so the lemma follows in this case.

Suppose that $p \nmid n$. The factor filtration of the first term on the right side of (2.17) is at most $k + n(p+1)$, while by (2.12) and (2.16), the factor filtration of the second term is

$$\omega_p(fE_2^{n+p-1}) = k + (n+p-1)(p+1) = k + 2n + n(p-1) + (p+1)(p-1).$$

By (2.6), we see that

$$\widetilde{\omega}_{p^2}(fE_2^n) = k + 2n + n(p-1) + (p+1)(p-1).$$

Finally, if $\omega_p(f) = k \geq 0$, then

$$\widetilde{\omega}_{p^2}(pfE_2^n) = \omega_p(fE_2^n) = k + n(p+1). \qquad \blacksquare$$

2.4 First upper bounds for the theta cycle filtrations

Suppose that $f \in \mathrm{M}_k$ with $0 < k < p$ has $\omega_p(f) = k$. From Proposition 2.2, we see that $\theta^i f \not\equiv 0 \pmod{p}$ for any i. Relation (2.8) together with (2.4) and the expression (2.13)

https://doi.org/10.4153/S0008414X2610220X Published online by Cambridge University Press

for E_2 yield

$$\omega_{p^2}\left(\theta^i f\right) \equiv k + 2i \pmod{p(p-1)}. \tag{2.18}$$

Furthermore, (2.8) and (2.13) imply that if $f \in \mathrm{M}_k$, then

$$\widetilde{\omega}_{p^2}\left(\theta^i f\right) \leq \omega_{p^2}\left(\theta^i f\right) \leq k + 2i + 2ip(p-1). \tag{2.19}$$

This also follows from the work of Chen–Kiming [4, Theorem 1] on the theta operator modulo p^m for general m. The bound (2.19) is however far from optimal, and Lemma 2.3 already yields much improved bounds.

Lemma 2.4 *Assume that $f \in \mathrm{M}_k$. Then we have the following bounds on the factor and weight filtrations:*

$$\widetilde{\omega}_{p^2}\left(\theta^i f\right) \leq k + 2i + (i+p+1)(p-1),$$

$$\omega_{p^2}\left(\theta^i f\right) \leq k + 2i + \left(\left\lceil \tfrac{i+1}{p} \right\rceil + 1\right)p(p-1).$$

Proof We estimate the factor filtration of each term in the expansion of $\theta^i f$ given in (2.11). By (2.10), we have $\widetilde{\omega}_{p^2}(f_j) \leq k + 2j$. From Lemma 2.3, we obtain

$$\begin{aligned}\widetilde{\omega}_{p^2}\left(\left(\widetilde{\partial}_k^j f\right)\left(\tfrac{1}{12}E_2\right)^{i-j}\right) &\leq k + 2j + 2(i-j) + (i-j+p+1)(p-1)\\ &= k + 2i + (i-j+p+1)(p-1),\end{aligned}$$

and the stated bound for the factor filtration follows from (2.6). The weight filtration is determined using Lemma 2.1 and (2.18). ∎

Remark 2.5 The bounds in Lemma 2.4 are monotone in i. Since

$$\theta^{i+p(p-1)} f \equiv \theta^i f \pmod{p^2} \quad \text{for } i \geq 2,$$

we immediately obtain upper bounds for all i which are cubic in p:

$$\begin{aligned}\widetilde{\omega}_{p^2}\left(\theta^i f\right) &\leq k + 2(p(p-1)+1) + (p^2+2)(p-1),\\ \omega_{p^2}\left(\theta^i f\right) &\leq k + 2(p(p-1)+1) + (p+1)p(p-1).\end{aligned}$$

This improves the upper bound (2.19), which is quartic in p if $i \asymp p^2$. Our results replace this by a quadratic upper bound for asymptotically 100% of the theta cycle (see Remark 4.2 below).

3 Filtrations in the first *p*-interval

In this section, we prove Theorem A, which will follow from a series of propositions which determine the factor filtrations for all $\theta^i f$ with $0 \leq i \leq p$.

3.1 Filtrations up to the first low point

Recall that the first low point of the theta cycle modulo p occurs at $i = p - k + 1$ (we will show that the same is true modulo p^2).

https://doi.org/10.4153/S0008414X2610220X Published online by Cambridge University Press

Proposition 3.1 *Suppose that $f \in \mathrm{M}_k$ with $0 < k < p$ has $\omega_p(f) = k$. In the range $0 < i < p - k + 1$, we have the factor filtration*

$$\widetilde{\omega}_{p^2}\left(\theta^i f\right) = k + 2i + (i + p + 1)(p - 1).$$

Proof We investigate the factor filtrations of the terms in (2.11). From Lemma 2.3 and the fact that $f_0 = f$, we have

$$\widetilde{\omega}_{p^2}\left(f_0 E_2^i\right) = k + 2i + (i + p + 1)(p - 1). \tag{3.1}$$

Since $\widetilde{\omega}_{p^2}(f_j) \le k + 2j$, the factor filtrations of the terms in (2.11) with $j > 0$ are bounded by

$$\widetilde{\omega}_{p^2}\left(f_j E_2^{i-j}\right) \le k + 2j + 2(i - j) + (i - j + p + 1)(p - 1) = k + 2i + (i - j + p + 1)(p - 1).$$

Therefore, the factor filtration of (2.11) is dominated by the term (3.1) and the proposition follows from (2.6). ∎

Proposition 3.2 *Suppose that $f \in \mathrm{M}_k$ with $0 < k < p$ has $\omega_p(f) = k$. Then at $i = p - k + 1$, we have the factor filtration*

$$\widetilde{\omega}_{p^2}\left(\theta^i f\right) = k + 2i + i(p - 1) = k + i(p + 1).$$

Proof If $j < i$, then (since $i + k - 1 = p$) we have the following bound for the j-th term in (2.11):

$$\widetilde{\omega}_{p^2}\left(\binom{i}{j} \frac{(i + k - 1)!}{(j + k - 1)!} f_j E_2^{i-j}\right) = \omega_p\left(f_j E_2^{i-j}\right) \le k + 2j + (i - j)(p + 1) = k + 2i + (i - j)(p - 1).$$

For $j = 0$, this is an equality by (2.12). On the other hand, the term with $j = i$ in (2.11) gives

$$\widetilde{\omega}_{p^2}(f_i) \le k + 2i.$$

Thus, the factor filtration of $\theta^i f$ is determined again by the term with $j = 0$. ∎

3.2 Filtrations between the first and second low points

Here, we determine the factor filtration of $\theta^i f$ for $p - k + 1 < i \le p$. (We will show that there is a second low point of the theta cycle at $i = p$.) We again use (2.11), but the analysis involved is more subtle.

Proposition 3.3 *Suppose that $f \in \mathrm{M}_k$ with $0 < k < p$ has $\omega_p(f) = k$ and let $p - k + 1 < i < p$. Then we have the factor filtrations*

$$\widetilde{\omega}_{p^2}\left(\theta^i f\right) = \begin{cases} k + 2i + (i + k)(p - 1), & \text{if } f \text{ is ordinary at } p; \\ k + 2i + (i + k - 1)(p - 1), & \text{if } f \text{ is non-ordinary at } p. \end{cases}$$

Proof Observe that the coefficients in the expansion (2.11) of $\theta^i f$ satisfy

$$p \parallel \frac{(i + k - 1)!}{(j + k - 1)!} \quad \text{if } j < p - k + 1. \tag{3.2}$$

https://doi.org/10.4153/S0008414X2610220X Published online by Cambridge University Press

Using this fact with (2.11), (2.14), and (2.15) gives the decomposition

$$\theta^i f \equiv p\tilde{f}_1 + p\tilde{f}_2 + \tilde{f}_3 \pmod{p^2},$$

where

$$\tilde{f}_1 := \sum_{j=0}^{p-k} \frac{1}{12^{i-j}} \binom{i}{j} \frac{(i+k-1)!}{p(j+k-1)!} f_j E_{p+1}^{i-j} E_{p-1}^{(i-j)(2p-1)},$$

$$\tilde{f}_2 := \sum_{j=p-k+1}^{i} \frac{i-j}{12^{i-j}} \binom{i}{j} \frac{(i+k-1)!}{(j+k-1)!} f_j E_{p+1}^{i-j+p-1} E_{p-1}^{(i-j-1)(2p-1)+p-2},$$

$$\tilde{f}_3 := \sum_{j=p-k+1}^{i} \binom{i}{j} \frac{(i+k-1)!}{(j+k-1)!} f_j \left(\partial E_{p-1}\right)^{i-j} E_{p-1}^{(i-j)(2p-1)}.$$

For $\tilde{f}_3$, we have (recalling that $k < p$) the factor filtration bound

$$\widetilde{\omega}_{p^2}(\tilde{f}_3) \le \max\left\{k + 2j + (i-j)(p+1) \ : \ p-k+1 \le j \le i\right\} \le k + 2i + (i-2)(p-1).$$

For $p\tilde{f}_1$, we have

$$\widetilde{\omega}_{p^2}(p\tilde{f}_1) = \omega_p(\tilde{f}_1) \le \max\left\{k + 2j + (i-j)(p+1) \ : \ 0 \le j \le p-k\right\} = k + 2i + i(p-1).$$

It remains to examine $p\tilde{f}_2$. Recalling (2.11) and (3.2), we obtain the following congruence involving modular forms of different weights:

$$\begin{aligned}\tilde{f}_2 &\equiv \sum_{j=0}^{i} \frac{i-j}{12^{i-j}} \binom{i}{j} \frac{(i+k-1)!}{(j+k-1)!} f_j E_{p+1}^{i-j+p-1} \\ &\equiv E_{p+1}^p \sum_{j=0}^{i-1} \frac{i(i+k-1)}{12^{i-j}} \binom{i-1}{j} \frac{(i-1+k-1)!}{(j+k-1)!} f_j E_{p+1}^{i-1-j} \equiv \frac{i(i+k-1)}{12} E_{p+1}^p \theta^{i-1} f \pmod{p}.\end{aligned}$$

Since $p-k+1 \le i-1 < p-1$, Proposition 2.2 gives

$$\omega_p\left(\theta^{i-1} f\right) = \begin{cases} k + (i-1)(p+1) - (p-k+1)(p-1), & \text{if } f \text{ is ordinary at } p; \\ k + (i-1)(p+1) - (p-k+2)(p-1), & \text{if } f \text{ is non-ordinary at } p. \end{cases}$$

Since the assumptions ensure that $i(i+k-1) \not\equiv 0 \pmod{p}$, we find using (2.12) that

$$\widetilde{\omega}_{p^2}(p\tilde{f}_2) = \omega_p(\tilde{f}_2) = p(p+1) + \omega_p\left(\theta^{i-1} f\right).$$

This gives the values

$$\widetilde{\omega}_{p^2}(p\tilde{f}_2) = \begin{cases} k + 2i + (i+k)(p-1), & \text{if } f \text{ is ordinary at } p; \\ k + 2i + (i+k-1)(p-1), & \text{if } f \text{ is non-ordinary at } p. \end{cases}$$

Since these exceed the bounds which we obtained for the factor filtrations of $p\tilde{f}_1$ and $\tilde{f}_3$, we have $\widetilde{\omega}_{p^2}\left(\theta^i f\right) = \widetilde{\omega}_{p^2}(p\tilde{f}_2)$, which confirms the claimed factor filtrations. ∎

Proposition 3.4 *Suppose that $f \in \mathrm{M}_k$ with $0 < k < p$ has $\omega_p(f) = k$. At $i = p$, we have the factor filtration*

$$\widetilde{\omega}_{p^2}\left(\theta^p f\right) = k + 2p + p(p-1).$$

https://doi.org/10.4153/S0008414X2610220X Published online by Cambridge University Press

Proof The coefficients in formula (2.11) satisfy

$$p^2 \,\Big\|\, \binom{p}{j} \frac{(p+k-1)!}{(j+k-1)!}, \qquad \text{if } 1 \le j \le p-k;$$
$$p \,\Big\|\, \binom{p}{j} \frac{(p+k-1)!}{(j+k-1)!}, \qquad \text{if } j = 0 \text{ or } p-k+1 \le j \le p-1.$$

It follows that for some p-adic units λ_0 and λ_j, we have

$$\theta^p f \equiv p\lambda_0 f_0 E_2^p + p\tilde{f}_1 + f_p \pmod{p^2}, \quad \text{where } \tilde{f}_1 := \sum_{j=p-k+1}^{p-1} \lambda_j f_j E_2^{p-j}. \tag{3.3}$$

We have $\widetilde{\omega}_{p^2}(f_p) \le k + 2p$. For the terms appearing in $\tilde{f}_1$, we have

$$\omega_p\big(f_j E_2^{p-j}\big) \le k + 2j + (p-j)(p+1) = k + p(p+1) - j(p-1),$$

from which

$$\widetilde{\omega}_{p^2}(p\tilde{f}_1) \le k + p(p+1) - (p-k+1)(p-1) = k + 2p + (k-1)(p-1).$$

Finally, we have

$$\widetilde{\omega}_{p^2}\big(p\lambda_0 f_0 E_2^p\big) = \omega_p\big(f_0 E_2^p\big) = k + p(p+1) = k + 2p + p(p-1).$$

Since this exceeds the other filtrations, it determines the factor filtration of $\widetilde{\omega}_{p^2}(\theta^p f)$. ∎

Proof of Theorem A Once the factor filtration of $\theta^i f$ is known, the weight filtration is determined using (2.18) and Lemma 2.1. For example, in the range $0 < i < p-k+1$, we have shown that the factor filtration of $\widetilde{\omega}_{p^2}(\theta^i f)$ is $k + 2i + (i + p + 1)(p-1)$. Since $p < i + p + 1 < 2p$, we conclude that the weight filtration must be $k + 2i + 2p(p-1)$. The analysis is similar in the other ranges and we omit the details. ∎

4 Filtrations in the first part of each *p*-interval

In this section, we prove Theorem C and Corollaries B and D. We will always write

$$i = np + i', \qquad 0 < n < p, \quad 0 < i' < p. \tag{4.1}$$

Recall the definition (1.2) of an exceptional index i. The results will follow from Theorem 4.1, which gives exact values and bounds for factor filtrations when $i' < p-k+1$.

Theorem 4.1 *Suppose that $f \in \mathrm{M}_k$ with $0 < k < p$ has $\omega_p(f) = k$, that i and i' are as in (4.1), and that $i' < p-k+1$.*

https://doi.org/10.4153/S0008414X2610220X Published online by Cambridge University Press

(1) *If $i' \leq (p-k+1) - n$ and i is not exceptional, then we have*

$$\widetilde{\omega}_{p^2}\left(\theta^i f\right) = k + 2i + (i' + p - n + 1)(p-1).$$

(Note that this includes all non-exceptional i with $p < i < 2p - k + 1$.)

(2) *If $i' > (p-k+1) - n$ or if i is exceptional, then we have the bounds*

$$\tilde{\omega}_{p^2}\left(\theta^i f\right) \leq \begin{cases} k + 2i + (i' + k - 1)(p-1), & \text{if } n = 1; \\ k + 2i + (i' + k - 2)(p-1), & \text{if } n > 1. \end{cases}$$

Proof Analyzing the factorials (using, for example, Lucas' theorem for the binomial coefficients), we find that

$$p \,\Big|\, \frac{(i+k-1)!}{(j+k-1)!}, \quad \text{if } j < np - k + 1; \qquad p^2 \,\Big|\, \frac{(i+k-1)!}{(j+k-1)!}, \quad \text{if } j < (n-1)p - k + 1;$$

$$p \,\Big|\, \binom{i}{j}, \quad \text{if } (n-1)p + i' < j < np \quad \text{or} \quad (n-2)p + i' < j < (n-1)p. \tag{4.2}$$

Together with (2.11), (2.15), and the inequality $i' < p - k + 1$, this gives

$$\theta^i f \equiv p\tilde{f}_1 + p\tilde{f}_2 + p\tilde{f}_3 + \tilde{f}_4 \pmod{p^2} \tag{4.3}$$

with weight $k + 2i + 2(i-j)p(p-1)$ modular forms

$$\tilde{f}_1 := \sum_{j=(n-1)p}^{(n-1)p+i'} \frac{1}{12^{i-j}} \binom{i}{j} \frac{(i+k-1)!}{p\,(j+k-1)!} f_j E_{p+1}^{i-j} E_{p-1}^{(i-j)(2p-1)},$$

$$\tilde{f}_2 := \sum_{j=np-k+1}^{np-1} \frac{1}{12^{i-j}} \binom{i}{j} \frac{(i+k-1)!}{p\,(j+k-1)!} f_j E_{p+1}^{i-j} E_{p-1}^{(i-j)(2p-1)},$$

$$\tilde{f}_3 := \sum_{j=np}^{i} \frac{i-j}{12^{i-j}} \binom{i}{j} \frac{(i+k-1)!}{(j+k-1)!} f_j E_{p+1}^{i-j+p-1} E_{p-1}^{(i-j-1)(2p-1)+p-2},$$

$$\tilde{f}_4 := \sum_{j=np}^{i} \binom{i}{j} \frac{(i+k-1)!}{(j+k-1)!} f_j \left(\partial E_{p-1}\right)^{i-j} E_{p-1}^{(i-j)(2p-1)}.$$

For $\tilde{f}_4$, we have the factor filtration bound

$$\widetilde{\omega}_{p^2}(\tilde{f}_4) \leq \max\left\{k + 2j + (i-j)(p+1) \;:\; np \leq j \leq i\right\} = k + 2i + i'(p-1). \tag{4.4}$$

Recall that $\widetilde{\omega}_{p^2}(p\tilde{f}_2) = \omega_p(\tilde{f}_2)$. We first estimate the contribution to the filtration from the term $j = np - k + 1$. From (2.11), we obtain

$$f_{np-k+1} \equiv \theta^{np-k+1} f \equiv \theta^{n-1+p-k+1} f \pmod{p}.$$

If $0 < n < k$, then we have $p - k + 1 \leq n - 1 + p - k + 1 < p$, and Proposition 2.2 (with $i = n - 1 + p - k + 1$, while still $j = np - k + 1$) gives

$$\begin{aligned} \omega_p\left(f_j E_{p+1}^{i-j}\right) &\leq k + (n - 1 + p - k + 1)(p+1) - (p - k + 1)(p-1) + (i-j)(p+1) \\ &= k + 2i + (i' + k - n)(p-1). \end{aligned} \tag{4.5}$$

https://doi.org/10.4153/S0008414X2610220X Published online by Cambridge University Press

If, however, $k \le n < p$, we have $f_{np-k+1} \equiv \theta^{n-k+1} f \pmod p$, so Proposition 2.2 (with $i = n-k+1$ and still $j = np-k+1$) gives

$$\omega_p\big(f_j E_{p+1}^{i-j}\big) \le k + (n-k+1)(p+1) + (i-j)(p+1),$$

which is less than the bound in (4.5). So the contribution from the term with $j = np - k + 1$ is bounded by (4.5).

For the terms with $np - k + 2 \le j \le np - 1$, each term contributing to $\tilde{f}_2$ has

$$\begin{aligned}\omega_p\big(f_j E_{p+1}^{i-j}\big) &\le k + 2j + (i-j)(p+1) = k + 2i + (i-j)(p-1)\\ &\le k + 2i + (i'+k-2)(p-1).\end{aligned} \tag{4.6}$$

The estimate (4.5) is larger than the estimate (4.6) if and only if $n = 1$. Therefore, we obtain

$$\tilde{\omega}_{p^2}(p\tilde{f}_2) \le \begin{cases} k + 2i + (i'+k-1)(p-1), & \text{if } n = 1;\\ k + 2i + (i'+k-2)(p-1), & \text{if } n > 1.\end{cases} \tag{4.7}$$

Next, we inspect $\tilde{f}_1$ and $\tilde{f}_3$. We omit powers of E_{p-1} and hence work with congruences involving modular forms of different weights. We employ Wilson's theorem, Lucas' theorem, and the fact that $i' < p - k + 1$ to find that

$$\begin{aligned}\tilde{f}_1 &\equiv \sum_{j=(n-1)p}^{i-p} \frac{1}{12^{i-j}} \binom{i}{j} \frac{(i+k-1)!}{p\,(j+k-1)!} f_j E_{p+1}^{i-j}\\ &\equiv \frac{-n}{12^p} E_{p+1}^p \sum_{j=(n-1)p}^{i-p} \frac{1}{12^{i-p-j}} \binom{i}{j} \frac{(i-p+k-1)!}{(j+k-1)!} f_j E_{p+1}^{i-p-j}\\ &\equiv \frac{-n^2}{12^p} E_{p+1}^p \sum_{j=(n-1)p}^{i-p} \frac{1}{12^{i-p-j}} \binom{i-p}{j} \frac{(i-p+k-1)!}{(j+k-1)!} f_j E_{p+1}^{i-p-j} \pmod p.\end{aligned}$$

Arguing as in (4.2) and using (2.11) gives

$$\tilde{f}_1 \equiv \frac{-n^2}{12} E_{p+1}^p \theta^{i-p} f \equiv \frac{-n^2}{12} E_{p+1}^p \theta^{i'+n-1} f \pmod p.$$

For $\tilde{f}_3$, we observe that there is no contribution from $j = i$ and use the identity $(i-j)\binom{i}{j} = i\binom{i-1}{j}$ to obtain

$$\begin{aligned}\tilde{f}_3 &\equiv \frac{i(i+k-1)}{12} E_{p+1}^p \sum_{j=np}^{i-1} \frac{1}{12^{i-1-j}} \binom{i-1}{j} \frac{(i-1+k-1)!}{(j+k-1)!} f_j E_{p+1}^{i-1-j}\\ &\equiv \frac{i(i+k-1)}{12} E_{p+1}^p \theta^{i-1} f \equiv \frac{i(i+k-1)}{12} E_{p+1}^p \theta^{i'+n-1} f \pmod p.\end{aligned}$$

We conclude that

$$\begin{aligned}\tilde{f}_1 + \tilde{f}_3 &\equiv \frac{-n^2}{12} E_{p+1}^p \theta^{i'+n-1} f + \frac{i(i+k-1)}{12} E_{p+1}^p \theta^{i'+n-1} f\\ &\equiv \tfrac{1}{12}\big(i^2 + (k-1)i - n^2\big) E_{p+1}^p \theta^{i'+n-1} f \pmod p.\end{aligned} \tag{4.8}$$

https://doi.org/10.4153/S0008414X2610220X Published online by Cambridge University Press

From (4.4) and (4.7), we have

$$\widetilde{\omega}_{p^2}\big(p\tilde{f}_2+\tilde{f}_4\big) \le \begin{cases} k+2i+(i'+k-1)(p-1), & \text{if } n=1;\\ k+2i+(i'+k-2)(p-1), & \text{if } n>1. \end{cases} \tag{4.9}$$

If i is exceptional (i.e., $i^2+(k-1)i-n^2 \equiv 0 \pmod{p}$), then the term (4.8) vanishes modulo p. Together with (4.3) and (4.9), this gives the claimed bounds in part (2) of the theorem.

Now assume that i is not exceptional. If f is ordinary at p, then Proposition 2.2 gives

$$\omega_p\big(\theta^{i'+n-1}f\big) = \begin{cases} k+2i+(i'-n-1)(p-1)-2, & \text{if } i'+n-1<p-k+1;\\ k+2i+(i'-n-p+k-2)(p-1)-2, & \text{if } p-k+1\le i'+n-1<p. \end{cases}$$

Furthermore, when $p \le i'+n-1$, the same proposition (note that $i'+n-p < p-k+1$) gives

$$\omega_p\big(\theta^{i'+n-1}f\big) = \omega_p\big(\theta^{i'+n-p}f\big) = k+2i+(i'-n-p-2)(p-1)-2.$$

Adding $p(p+1) = (p+2)(p-1)+2$ in each case gives

$$\widetilde{\omega}_{p^2}\left(p(\tilde{f}_1+\tilde{f}_3)\right) = \begin{cases} k+2i+(i'+p-n+1)(p-1), & \text{if } i'+n-1<p-k+1;\\ k+2i+(i'-n+k)(p-1), & \text{if } p-k+1\le i'+n-1<p;\\ k+2i+(i'-n)(p-1), & \text{if } p\le i'+n-1. \end{cases}$$

Similarly, if f is non-ordinary at p, then we obtain

$$\widetilde{\omega}_{p^2}\left(p(\tilde{f}_1+\tilde{f}_3)\right) = \begin{cases} k+2i+(i'+p-n+1)(p-1), & \text{if } i'+n-1<p-k+1;\\ k+2i+(i'-n+k-1)(p-1), & \text{if } p-k+1\le i'+n-1<p-1;\\ k+2i+(i'-n)(p-1), & \text{if } p-1\le i'+n-1. \end{cases}$$

If $i'+n-1<p-k+1$, then $i'+k-1<p-n+1$ and the bound in (4.9) is strictly less than the values above, and we find that

$$\widetilde{\omega}_{p^2}\big(\theta^i f\big) = \widetilde{\omega}_{p^2}\big(p(\tilde{f}_1+\tilde{f}_3)\big) = k+2i+(i'+p-n+1)(p-1),$$

which confirms part (1) of the theorem.

It remains to consider the cases when i is not exceptional and $i'+n-1 \ge p-k+1$. In these cases, the factor filtration of $p(\tilde{f}_1+\tilde{f}_3)$ does not dominate the factor filtration in (4.9), and we can deduce only that

$$\tilde{\omega}_{p^2}\big(\theta^i f\big) \le \begin{cases} k+2i+(i'+k-1)(p-1), & \text{if } n=1;\\ k+2i+(i'+k-2)(p-1), & \text{if } n>1. \end{cases}$$

This completes the proof of the theorem. ∎

Proof of Theorem C As mentioned above, the statements when $i'=0$ and $i'=p-k+1$ are a consequence of [7, Theorem 1.8]. For the rest, we use Theorem 4.1, Lemma 2.1, and (2.18).

By assumption, we have $4 < i'+k < p+1$. In the cases covered by part (2) of Theorem 4.1, this gives the weight filtration bound $k+2i+p(p-1)$. In the cases covered by part (2), we see that

https://doi.org/10.4153/S0008414X2610220X Published online by Cambridge University Press

$$0 < i' + p - n + 1 < p + 1, \quad \text{if } 0 < i' < n \text{ and } i' \le p - k + 1 - n;$$
$$p < i' + p - n + 1 < 2p, \quad \text{if } n \le i' \le p - k + 1 - n,$$

from which the exact weight filtrations follow. ∎

Proof of Corollary B To show that the second low point occurs at index p, we use Theorems A and C together with the fact that $p+1$ is not an exceptional index. All other assertions follow immediately from Theorem A. ∎

Proof of Corollary D Part (4) follows from Theorem C since the hypotheses ensure that i and $i+1$ are both contained in one of the two ranges where the filtrations are exactly determined. For part (3), note that under the hypotheses, the filtration at i is exactly determined by the theorem, while the filtration at $i-1$ is bounded to be less than the filtration at i.

To prove part (1), let $i = np + p - k + 2 - n$ be as in the statement. Using both parts of Theorem C, we see that we always have $\omega_{p^2}(\theta^i f) \le k + 2i + p(p-1)$. Since $2n \le p - k + 1$ by hypothesis and $k \ge 4$, it follows that

$$k + 2i - p(p-1) \le (4-k)(p+1) - 2n < 0 \tag{4.10}$$

from which we conclude using (2.18) that

$$\omega_{p^2}(\theta^i f) = k + 2i \quad \text{or} \quad k + 2i + p(p-1).$$

The index $i-1$ is not exceptional since

$$(i-1)^2 + (k-1)(i-1) - n^2 \equiv n(k-1) \not\equiv 0 \pmod{p}.$$

Since the hypotheses give $n \le i' - 1 = p - k + 1 - n$, it follows from Theorem C that

$$\omega_{p^2}(\theta^{i-1} f) = k + 2(i-1) + 2p(p-1).$$

From (4.10), we have $k + 2(i+1) - p(p-1) \le 0$. Note that $\omega_{p^2}(\theta^{i+1} f) \ne 0$ (equality would imply that $\theta^{i+2} f \equiv 0 \pmod{p}$ which contradicts Proposition 2.2). It follows that

$$\omega_{p^2}(\theta^{i+1} f) \ge k + 2(i+1).$$

If $\omega_{p^2}(\theta^i f) = k + 2i$ or $\omega_{p^2}(\theta^{i+1} f) \ge k + 2(i+1) + p(p-1)$, then there is a low point at i as claimed. In the remaining case, we have $\omega_{p^2}(\theta^i f) = k + 2i + p(p-1)$ and $\omega_{p^2}(\theta^{i+1} f) = k + 2(i+1)$. By (4.10), we have $k + 2(i+2) - p(p-1) \le 2$. Since we cannot have $\omega_{p^2}(\theta^{i+2} f) = 0$ or 2, it follows that $\omega_{p^2}(\theta^{i+2} f) \ge k + 2(i+2)$, which establishes that there is a low point at $i+1$ in this case.

We turn to the proof of part (2). We must show that there is a low point at i whenever i is an exceptional index and $n \le i' \le p - k + 1 - n$. It is straightforward to check that indices with $i' = n$ or $i' = p - k + 1 - n$ are not exceptional. It follows that

$$n \le i' - 1 < i' + 1 \le p - k + 1 - n.$$

The filtration values and bounds in Theorem C show that there is a low point at i unless one of $i-1$ or $i+1$ is exceptional and it therefore suffices to show that there are no consecutive solutions i', $i'+1$ to the congruence (1.2). This follows since

https://doi.org/10.4153/S0008414X2610220X Published online by Cambridge University Press

$$\left((i'+1)^2+(k-1)(i'+1)-n^2\right)-\left(i'^2+(k-1)i'-n^2\right)=2i'+k,$$

and by the assumptions, the right side is positive, even, and less than $2p$. ∎

Remark 4.2 Theorem A and the results in Section 3 provide exact values for the weight and factor filtration of $\theta^i f$ at p positions. Theorems C and 4.1 provide exact values for both filtrations at non-exceptional indices when $0 < i' \le p-k+1-n$. Since in each p-interval there are at most two exceptional indices, we have exact values for both filtrations of $\theta^i f$ with $0 \le i \le p(p-1)$ for at least

$$p+\sum_{n=1}^{p-k+1}(p-k-1-n)=p+\frac{(p-k-4)(p-k+1)}{2}$$

positions. This is asymptotically 50% of the $p(p-1)$ total number of positions. Similarly, we have bounds for at least

$$p+\sum_{n=1}^{p-1}(p-k+1)=p+(p-k+1)(p-1)$$

positions, which yields asymptotically 100%.

Acknowledgements We thank Amy Woodall and the referees for many helpful suggestions which improved our exposition.

References

[1] S. Ahlgren and M. Boylan, *Arithmetic properties of the partition function*. Invent. Math. **153**(2003), no. 3, 487–502.
[2] S. Ahlgren, M. Hanson, M. Raum, and O. K. Richter, *Eisenstein series modulo p2*. Forum Math. **38**(2026), no. 1, 93–102. https://doi.org/10.1515/forum-2024-0470
[3] J. H. Bruinier, G. van der Geer, G. Harder, and D. B. Zagier, *The 1-2-3 of modular forms: Lectures from the summer school on modular forms and their applications held in Nordfjordeid, June 2004*, K. Ranestad (ed.), Universitext, Springer, Berlin, 2008, pp. x+266.
[4] I. Chen and I. Kiming, *On the theta operator for modular forms modulo prime powers*. Mathematika **62**(2016), no. 2, 321–336.
[5] B. Edixhoven, *The weight in Serre's conjectures on modular forms*. Invent. Math. **109**(1992), no. 3, 563–594.
[6] N. Jochnowitz, *A study of the local components of the Hecke algebra mod l*. Trans. Amer. Math. Soc. **270**(1982), no. 1, 253–267.
[7] J. Kim and Y. Lee, *The theta cycles for modular forms modulo prime powers*. Forum Math. **35**(2023), no. 3, 591–613.
[8] J.-P. Serre, *Formes modulaires et fonctions zêta p-adiques*. In: W. Kuijk and J.-P. Serre (eds.), Modular functions of one variable, III (proceedings international Summer School, University of Antwerp, Antwerp, 1972), Lecture Notes in Mathematics, 350, Springer, Berlin, 1973, pp. 191–268.
[9] H. P. F. Swinnerton-Dyer, *On l-adic representations and congruences for coefficients of modular forms*. In: W. Kuijk and J.-P. Serre (eds.), Modular functions of one variable III, proceedings international Summer School, University of Antwerp, 1972, Lecture Notes in Mathematics, 350, 1973, pp. 1–55.

University of Illinois, USA
e-mail: sahlgren@illinois.edu

Chalmers tekniska högskola och Göteborgs Universitet, Sweden
e-mail: martin@raum-brothers.eu

University of North Texas, USA
e-mail: richter@unt.edu

https://doi.org/10.4153/S0008414X2610220X Published online by Cambridge University Press