This is equivalent to starting the iteration $x_{n+1} = (-ax_n - b)^{1/5}$ at the value $x_0 = 1$, but the article [1] does not give conditions for the convergence of this process.

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99.14 A recurrence relation for the limit point of nested triangles

There are several unsolved problems in the literature related to the limit point of various nested triangles, see Croft, Falconer, and Guy [1, 2, 3]. In this note, a recursive sequence is found for the limit point of nested triangles obtained by angle bisectors. We consider only the special case of an isosceles triangle.

Let T_1 be the given isosceles triangle with vertices $A_1B_1C_1$ and let C_1 be the vertex angle. Orient T_1 as shown in Figure 1 with C_1 at the origin. Let z_1, $\bar{z}_1$ represent the points A_1, B_1, respectively. See Figure 1.

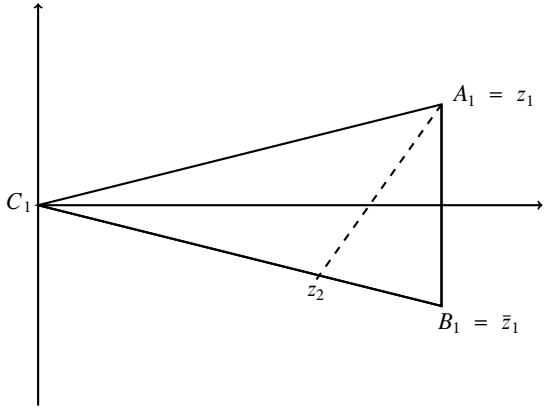


FIGURE 1: The initial triangle T_1

If z_2 is the point where the angle bisector of A_1 intersects side B_1C_1, then by the ratio theorem (or section formula) combined with the angle bisector theorem, z_2 may be expressed as

$$z_2 = \frac{|z_1|\,\bar{z}_1}{|z_1| + |2\ \mathrm{Im}\,(z_1)|}. \tag{1}$$

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Similarly, as in Figure 2, z_3 is found to be

$$z_3 = \frac{|\mathrm{Re}\,(z_1) - z_2|\,\bar{z}_2 + |2\ \mathrm{Im}\,(z_2)|\,\mathrm{Re}\,(z_1)}{|\mathrm{Re}\,(z_1) - z_2| + |2\ \mathrm{Im}\,(z_12)|}. \tag{2}$$

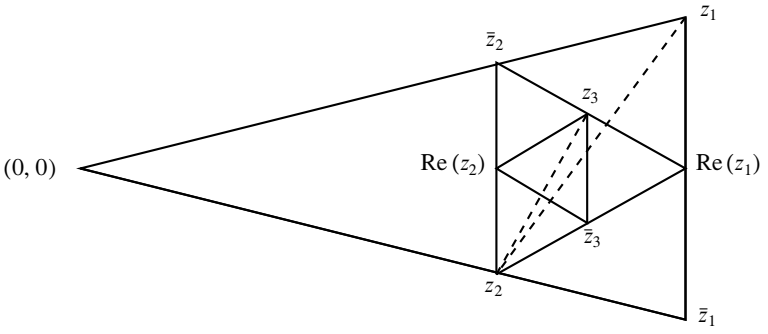


FIGURE 2: T_1 and the nested triangles T_2 and T_3.
(The dashed lines are the angle bisectors.)

For simplicity, we denote the point where the $(n-1)$th angle bisector (dashed) intersects the opposite side as z_n and its conjugate, reflected across the real axis, as $\bar{z}_n$. See Figure 3.

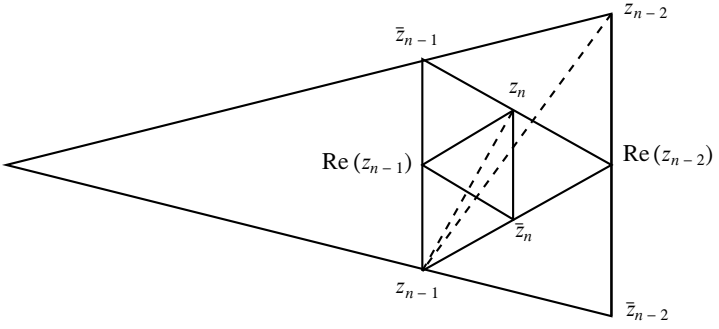


FIGURE 3: A sequence of nested triangles T_{n-2}, T_{n-1}, T_n and their angle bisectors.

In general, we have, for $n \geqslant 3$,

$$z_n = \frac{|\mathrm{Re}\,(z_{n-2}) - z_{n-1}|\,\bar{z}_{n-1} + |2\ \mathrm{Im}\,(z_{n-1})|\ \mathrm{Re}\,(z_{n-2})}{|\mathrm{Re}\,(z_{n-2}) - z_{n-1}| + |2\ \mathrm{Im}\,(z_{n-1})|}.$$

The real part of z_n is given by

$$x_n = \frac{x_{n-1}\sqrt{(x_{n-2} - x_{n-1})^2 + y_{n-1}^2} + 2\,|y_{n-1}|\,x_{n-2}}{\sqrt{(x_{n-2} - x_{n-1})^2 + y_{n-1}^2} + 2\,|y_{n-1}|}. \tag{3}$$

Similarly, the imaginary part of z_n is given by

$$y_n = \frac{-y_{n-1}\sqrt{(x_{n-2} - x_{n-1})^2 + y_{n-1}^2}}{\sqrt{(x_{n-2} - x_{n-1})^2 + y_{n-1}^2} + 2\,|y_{n-1}|}. \tag{4}$$

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Although the algebra is a bit tedious, we can eliminate y_n and y_{n-1} between (3) and (4) above and produce a recurrence relation for x_n in terms of only the real parts of z_{n-1} and z_{n-2}. This value of x_n will be the x-coordinate of the base angles of the $(n-1)$th nested triangle.

After much algebra, we arrive at the relationship

$$\left(\frac{x_n - x_{n+1}}{x_n - x_{n-2}}\right)^2 = \frac{4(x_{n+1} - x_{n-1})^2}{5(x_n - x_{n-2})^2 - (x_{n-1} - x_n)^2}.$$

Taking square roots (using the positive root) and re-indexing back to x_n, we get for $n > 3$

$$x_n = \frac{x_{n-1}\sqrt{5(x_{n-1} - x_{n-3})^2 - (x_{n-2} - x_{n-1})^2} + 2|x_{n-1} - x_{n-3}|x_{n-2}}{\sqrt{5(x_{n-1} - x_{n-3})^2 - (x_{n-2} - x_{n-1})^2} + 2|x_{n-1} - x_{n-3}|}. \quad (5)$$

Here x_1 is given and the values for x_2 and x_3 are easily found using equations (1) and (2) respectively.

It is shown in [3] that the *limiting shape* of nested triangles formed by angle bisectors is equilateral. Additionally, any set of nested equilateral triangles converge to a common point, namely the common centroid. It follows that the limiting value of the x-coordinate of the base angles of nested isosceles triangles formed by angle bisectors will serve as the *limit point* of the nested isosceles triangles.

As an illustration of the convergence of (5), consider the special case where the initial triangle T_1 is an equilateral triangle with $z_1 = \sqrt{3} + i$. We find $x_{40} \approx 1.1547005383784$. By elementary geometry, the x-coordinate of the centroid is found to be $x = \frac{2}{3}\sqrt{3} \approx 1.1547005383793$.

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