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STRETCH FACTOR IN A PLANAR POISSON–DELAUNAY TRIANGULATION WITH A LARGE INTENSITY

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Abstract

Let $X := X_n \cup \{(0,0),(1,0)\}$, where X_n is a planar Poisson point process of intensity n. We provide a first nontrivial lower bound for the distance between the expected length of the shortest path between $(0,0)$ and $(1,0)$ in the Delaunay triangulation associated with X when the intensity of X_n goes to ∞. Simulations indicate that the correct value is about 1.04. We also prove that the expected length of the so-called upper path converges to $35/3\pi^2$, yielding an upper bound for the expected length of the smallest path.

Keywords: Delaunay triangulation; Poisson point process; stretch factor

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Secondary 84B41

1. Introduction

Let χ be a locally finite subset in $\mathbb{R}^2$, endowed with its Euclidean norm $\|\cdot\|$, such that each subset of size $n < 3$ is affinely independent and no four points lie on a circle. The Delaunay triangulation of χ is the unique triangulation with vertices in χ such that the circumdisk of each triangle contains no point of χ in its interior. The set of its edges is denoted by $\mathrm{Del}(\chi)$ and the graph $(\chi, \mathrm{Del}(\chi))$ is the so-called Delaunay graph associated with χ [20, p. 478]. Delaunay triangulations are a very popular structure in computational geometry [1] and are extensively used in many areas such as surface reconstruction [6] or mesh generation [8].

In this paper we investigate several paths in $\mathrm{Del}(\chi)$. By a path $P = P(s,t)$ between two points $s, t \in \chi$, we mean a sequence of segments $[Z_0, Z_1], [Z_1, Z_2], \ldots, [Z_{k-1}, Z_k]$, such that $Z_0 = s$, $Z_k = t$. In particular, we say that P is a path in $\mathrm{Del}(\chi)$ if it is a path such that each segment $[Z_i, Z_{i+1}]$ is an edge in $\mathrm{Del}(\chi)$. The investigation of paths is related to walking strategies which are commonly used to find the triangle containing a query point in a planar triangulation [13] or routeing in geometric networks [4]. One of the classical works concerns the so-called straight walk which deals with the set of triangles cut by the line segment $[s,t]$. In this context, Devroye *et al.* [14] considered n points evenly distributed in a convex domain and proved that the expected number of Delaunay edges crossed by a fixed line segment is $O(\sqrt{n})$. This result was improved upon by Bose and Devroye [3] who showed that, for n points also evenly distributed in a convex domain, the expectation of the maximal number of intersections between a line and edges of the triangulation is $\Theta(\sqrt{n})$.

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Another classical problem dealing with paths in triangulations is the investigation of the stretch factor associated with two nodes $s, t \in \chi$ in $\mathrm{Del}(\chi)$. This quantity is defined as $\ell(S_{s,t}(\chi))/\|s-t\|$, where $\ell(S_{s,t}(\chi))$ is the length of the shortest path between the source point s and the target point t. Many upper bounds were established for the stretch factor in the context of finite sets χ; see, e.g. [15] and [18]. The best upper bound established until now for deterministic finite sets χ is due to Xia [22] who proved that the stretch factor is lower than 1.998. For the lower bound, Xia and Zhang [23] found a configuration of points χ such that the stretch factor is greater than 1.5932.

In this paper we focus on a probabilistic version of the problem by taking a slight modification of the underlying point process. More precisely, we consider a homogeneous Poisson point process X_n of intensity n in $\mathbb{R}^2$. For such an infinite point process, studying the maximum of the stretch factor over any points $s, t \in X_n$ has no real sense. Actually, a configuration which is close to the one considered by Xia and Zhang [23], with a stretch factor close to 1.5932, occurs almost surely somewhere in the plane. Thus, we are interested in the case where s and t are fixed and added to the Poisson point process.

Another path in the Poisson–Delaunay triangulation, called the Voronoi path, consisting of the set of nuclei of the Voronoi cells that are crossed by a line, was investigated by Baccelli *et al.* [2]. They proved that this Voronoi path has expected length $4/\pi \simeq 1.27$, giving an upper bound for the expected stretch factor. This result has been improved by introducing shortcuts [12] and generalizing to higher dimensions [10].

1.1. Contributions

Let $X := X_n \cup \{s, t\}$, where X_n is a Poisson point process of intensity n,

$$s = (0, 0) \quad \text{and} \quad t = (1, 0).$$

The main focus of our paper is to provide bounds for the expectation of the stretch factor between s and t in the Delaunay triangulation $\mathrm{Del}(X)$. The difficulty in obtaining a lower bound for $\mathbb{E}[\ell(S_{s,t}(X))]$ comes from the fact that we do not know where the shortest path $S_{s,t}(X)$ is. Our first main result deals with the tail of the distribution of $\ell(S_{s,t}(X))$.

Theorem 1.1. *There exists $\delta \geq 7 \times 10^{-9}$ such that*

$$\mathbb{P}[\ell(S_{s,t}(X)) \leq 1 + \delta] = O(n^{-1/2}).$$

As a consequence of Theorem 1.1 and Lemma 2.2 (which proves the existence of the limit), we easily deduce the following result.

Corollary 1.1. *There exists $\delta \geq 7 \times 10^{-9}$ such that*

$$\lim_{n\to\infty} \mathbb{E}[\ell(S_{s,t}(X))] \geq 1 + \delta.$$

To the best of the authors' knowledge, the results of this paper provide the first nontrivial lower bound (i.e. strictly greater than 1) for the stretch factor when the intensity of the underlying Poisson point process goes to ∞. However, our lower bound is far from optimal since simulations suggest that $\lim_{n\to\infty} \mathbb{E}[\ell(S_{s,t}(X))] \simeq 1.04$. We note that our result is closely related to a theorem recently proved by Hirsch *et al.* [17, Theorem 26]. Indeed, they showed that $\inf_{n\geq 1} \mathbb{E}[\ell(S_{s,t}(X))] > 1$. However, their technique cannot provide an explicit lower bound for the stretch factor. In the following proposition, we also state an upper bound.

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Theorem 1.2. *With the above notation, we have*

$$\lim_{n\to\infty} \mathbb{E}[\ell(S_{s,t}(X))] \le \frac{35}{3\pi^2} \simeq 1.182.$$

The upper bound considered above is established by bounding the length of a particular path in the Delaunay triangulation. In particular, our theorem provides a more precise upper bound for the expectation of the stretch factor than the one obtained by using the Voronoi path, in which case it was proved that the expected stretch factor is lower than $4/\pi \simeq 1.27$.

2. Preliminaries

Notation. Let $s = (0, 0)$ and $t = (k, 0)$ for some integer k. In this section, as in the previous one and the next one, we let $k = 1$. In Section 4, the integer k is arbitrary. Let X_n be a homogeneous Poisson point process of intensity n in $\mathbb{R}^2$ and $X = X_n \cup \{s, t\}$. We denote by $\mathrm{Del}(X)$ the Delaunay triangulation associated with X. The following notation will be used throughout the paper.

- For any point $p \in \mathbb{R}^2$, we write $p = (x_p, y_p)$.
- For any segment $e \subset \mathbb{R}^2$, we denote by $h(e) \ge 0$ and $|\widehat{e}| \in [0, \pi/2]$ the length of the horizontal projection of e and the absolute value of the angle with the x-axis, respectively.
- For any points $p_1, \dots, p_k \in \mathbb{R}^2$, we write $p_{1:k} := (p_1, \dots, p_k)$. When $p_1, \dots, p_k \in \mathbb{R}^2$ are pairwise distinct, we write the k-tuple of points as $p_{1:k}^{\neq}$. Such notation will be used in the summation index. Moreover, with a slight abuse of notation, we write $\{p_{1:k}^{\neq}\} := \{p_1, \dots, p_k\}$.
- For each 3-tuple of points $p_{1:3}^{\neq} \in (\mathbb{R}^2)^3$ which do not belong to the same line, we denote by $\Delta(p_{1:3})$ and $B(p_{1:3})$ the triangle spanned by $p_{1:3}$ and the (open) circumdisk associated with $p_{1:3}$, respectively. We also denote by $R(p_{1:3})$ the radius of $B(p_{1:3})$.
- For each $z \in \mathbb{R}^2$ and $r \ge 0$, let $B(z, r)$ be the (open) disk centered at z with radius r.
- Let $\mathbb{S} \subset \mathbb{R}^2$ be the unit circle and let σ be the uniform distribution on $\mathbb{S}$ such that $\sigma(\mathbb{S}) = 2\pi$.
- For any Borel subset $B \subset \mathbb{R}^2$, let $\mathcal{A}(B)$ be the area of B. In particular, for each $u_{1:3} \in \mathbb{S}^3$, we have

$$\mathcal{A}(\Delta(u_{1:3})) = \frac{1}{2}\left|\det \begin{pmatrix} 1 & 1 & 1 \\ \cos\beta_1 & \cos\beta_2 & \cos\beta_3 \\ \sin\beta_1 & \sin\beta_2 & \sin\beta_3 \end{pmatrix}\right|,$$

  where $\beta_i \in [0, 2\pi)$ is the angle between u_i and $(1, 0)$, with $1 \le i \le 3$.

Outline. In Section 2 we begin with some preliminaries. In Section 3 we provide estimates for the length of a particular path. These estimates directly imply Theorem 1.2. Section 4 constitutes the main part of our paper and deals with the lower bound for the shortest path. Our main idea is to discretize the plane into pixels and to consider the so-called lattice animals. We derive Theorem 1.1 by investigating the size of these lattice animals and by adapting tools of percolation theory. A table of integrals which are used throughout the paper is provided in Appendix A.

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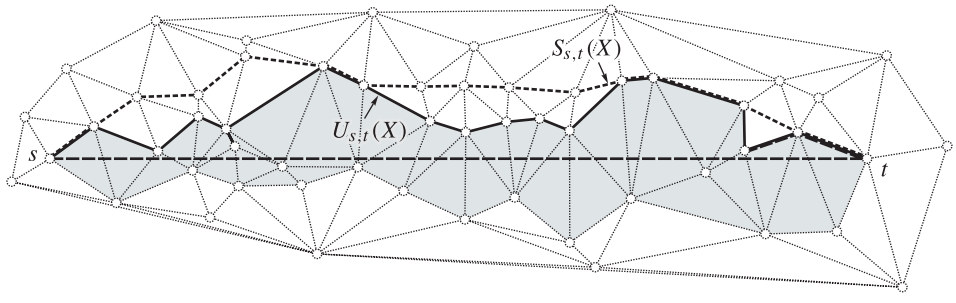


FIGURE 1: The paths $S_{s,t}(X)$ (*dotted*) and $U_{s,t}(X)$ (*solid*).

2.1. Paths

Given $s, t \in \mathbb{R} \times \{0\}$, we denote by $\mathcal{P}_{s,t}(X)$ the family of paths in $\mathrm{Del}(X)$ between the points s and t. For each path $P \in \mathcal{P}_{s,t}(X)$, we denote by $\ell(P)$ the Euclidean length of P and $\mathrm{card}(P)$ its number of edges. We introduce two types of paths in $\mathcal{P}_{s,t}(X)$, namely $S_{s,t}(X)$ and $U_{s,t}(X)$ as follows.

Shortest path $S_{s,t}(X)$: this path minimizes the length between s and t in the Delaunay triangulation $\mathrm{Del}(X)$. Note that such a path is almost surely (a.s.) unique.

Upper path $U_{s,t}(X)$: this path is defined as the sequence of all edges in $\mathbb{R} \times \mathbb{R}_+$ which belong to Delaunay triangles that intersects $[s, t]$. Some of these edges may be traversed in both ways (e.g. this is the case for one of the edges incident to the fifth vertex of the solid path in Figure 1).

These two paths are depicted in Figure 1.

2.2. Sum of the lengths for a typical vertex

The following lemma will be used to derive Proposition 3.1.

Lemma 2.1. *Let X_n be a Poisson point process of intensity n and let L_s be the sum of the lengths of the edges with vertex $s = (0, 0)$ in $\mathrm{Del}(X_n \cup \{s\})$. Then $\mathbb{E}[L_s] = cn^{-1/2}$ for some constant c.*

In the sense of Palm theory, L_s is the sum of the lengths of the Delaunay edges starting from a typical vertex.

Proof of Lemma 1. Due to the scaling invariance property of a homogeneous Poisson point process, we can assume that $n = 1$. The fact that $\mathbb{E}[L_s]$ is finite is a consequence of the edge-star intensity relationship derived in [20, Chapter 10.1]. □

Note that the constant c can be made explicit by applying the Slivnyak–Mecke formula (see, e.g. [20, Theorem 3.3.5]) and an analogous version of the Blaschke–Petkantschin-type change of variables (see, e.g. [20, Theorem 7.3.1]) in which one of the vertices is held fixed.

2.3. Existence of the limit

Lemma 2.2. *The expected length of the shortest path $\mathbb{E}[\ell(S_{s,t}(X))]$ between s and t converges as the intensity of X_n goes to ∞.*

Proof. By rescaling, it is enough to prove that, for each $n > 0$, the function $k \mapsto \mathbb{E}[\ell(S_{s,(k,0)}(X))]$, with $k \in \mathbb{R}_+$, converges as k goes to ∞. In doing so, we introduce a

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slight modification of $S_{s,(k,0)}(X)$ as follows. Let $\Sigma_{s,(k,0)}(X_n)$ be the shortest path from the nearest neighbor of s in $\mathrm{Del}(X_n)$ to the nearest neighbor of $(k,0)$ in $\mathrm{Del}(X_n)$.

The function $k \mapsto \mathbb{E}[\ell(\Sigma_{s,(k,0)}(X_n))]$ is clearly subadditive. It follows from Fekete's subadditive lemma that $\lim_{k\to\infty} k^{-1}\,\mathbb{E}[\ell(\Sigma_{s,(k,0)}(X_n))]$ exists. Moreover, according to Lemma 2.1, we know that $\mathbb{E}[|\ell(S_{s,(k,0)}(X)) - \ell(\Sigma_{s,(k,0)}(X_n))|] \le 2cn^{-1/2}$. □

Remark 2.1. In the proof of the above result, the main idea was to apply Fekete's subadditive lemma. We introduced the auxiliary path $\Sigma_{s,(k,0)}(X_n)$ in such a way that $k \mapsto \mathbb{E}[\ell(\Sigma_{s,(k,0)}(X_n))]$ is subadditive. This new path was needed because it is not trivial to prove that $k \mapsto \mathbb{E}[\ell(S_{s,(k,0)}(X_n))]$ is also subadditive. Actually, conditioning on the fact that an intermediate point $(x,0)$, with $0 < x < k$, belongs to the shortest path can increase or decrease the length of the shortest path.

3. Length of the upper path

In this section we estimate the expectation and the variance of the length of the upper path $U_{s,t}(X)$. The following proposition deals with the expectation.

Proposition 3.1. *Let X_n be a Poisson point process of intensity n. Then*

$$\mathbb{E}[\ell(U_{s,t}(X))] = \frac{35}{3\pi^2} + O(n^{-1/2}) \simeq 1.182.$$

The above result provides an upper bound for the expectation of the length of the shortest path and implies directly Theorem 1.2.

Proof of Proposition 3.1. Let

$$L_{X_n} := \sum_{p_{1:3}^{\neq} \in X_n^3} \ell_{X_n}(p_{1:3}), \tag{3.1}$$

where $\ell_{X_n}(p_{1:3})$ is the half-length of an edge crossing the line segment $[s,t]$:

$$\ell_{X_n}(p_{1:3}) = \tfrac{1}{2}\mathbf{1}_{[\Delta(p_{1:3})\in \mathrm{Del}(X_n)]}\mathbf{1}_{[p_{1:3}\in E^+]}\|p_2 - p_1\|$$

and where E^+ is the set of triples of points $p_{1:3}$ such that the triangle spanned by $p_{1:3}$ intersects $[s,t]$ and such that the circumdisk associated with $p_{1:3}$ contains neither s nor t:

$$E^+ := \{p_{1:3}^{\neq} \in (\mathbb{R}^2)^3 \colon \Delta(p_{1:3}) \cap [s,t] \neq \varnothing,\, B(p_{1:3}) \cap \{s,t\} = \varnothing,\, y_{p_1}, y_{p_2} > 0,\, y_{p_3} < 0\}.$$

In the expression of L_{X_n}, we have considered the lengths of the edges in $U_{s,t}(X)$, except the ones which contain the points s and t. By Lemma 2.1, the expected lengths of these two edges is $O(n^{-1/2})$. Hence, it is enough to show that $\mathbb{E}[L_{X_n}] = 35/3\pi^2 + O(n^{-1/2})$. In doing so, we apply the Slivnyak–Mecke formula (see, e.g. [20, Theorem 3.3.5]). This yields

$$\mathbb{E}[L_{X_n}] = \frac{1}{2}n^3 \int_{\mathbb{R}^6} \mathbb{P}[\Delta(p_{1:3}) \cap X_n = \varnothing]\mathbf{1}_{[p_{1:3}\in E^+]}\|p_2 - p_1\|\,\mathrm{d}p_{1:3}.$$

It follows from the Blaschke–Petkantschin-type change of variables (see, e.g. [20, Theorem 7.3.1]) that

$$\mathbb{E}[L_{X_n}] = \frac{1}{2}n^3 \int_{\mathbb{R}_+}\int_{\mathbb{R}^2}\int_{\mathbb{S}^3} \mathrm{e}^{-n\pi r^2}\mathbf{1}_{[z+ru_{1:3}\in E^+]} r\|u_2 - u_1\| r^3 2\,\mathcal{A}(\Delta(u_{1:3}))\sigma^3(\mathrm{d}u_{1:3}) \times \mathrm{d}y_z\,\mathrm{d}x_z\,\mathrm{d}r, \tag{3.2}$$

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where $z = (x_z, y_z)$. With a slight abuse of notation, we have written $z + ru_{1:3}$ to denote the 3-tuple $(z + ru_1, z + ru_2, z + ru_3)$. Note that the integrand in the above equation is not 0, only if $z \in [0, 1] \times [-r, r]$. To deal with $\mathbf{1}_{[z+ru_{1:3} \in E^+]}$, we introduce an event E'^+ defined as a slight modification of the event E^+. For the event E'^+, we no longer require that the triangle $\Delta(p_{1:3})$ intersects the segment $[s, t]$ but we consider the configuration in which the triangle intersects the supporting line:

$$E'^+ := \{p_{1:3} \in (\mathbb{R}^2)^3 \colon y_{p_1}, y_{p_2} > 0, y_{p_3} < 0\}.$$

In particular, we have $E^+ \subset E'^+$. On the contrary, if $z + ru_{1:3} \in E'^+$ and if $z \in ([0, 1] \times [-r, r]) \setminus B^{\cup}(s, t, r)$, we have $z + ru_{1:3} \in E^+$, where

$$B^{\cup}(s, t, r) := (B(s, r) \cap \{z \colon x_z \geq 0\}) \cup (B(t, r) \cap \{z \colon x_z \leq 1\}).$$

We show below that the right-hand side of (3.2) is $O(n^{-1/2})$ if the integration over z is restricted to $B^{\cup}(s, t, r)$. Indeed,

$$\begin{aligned}
&n^3 \int_0^\infty \int_{B^{\cup}(s,t,r)} \int_{\mathbb{S}^3} \mathrm{e}^{-n\pi r^2} r^4 2\, \mathcal{A}(\Delta(u_{1:3}))\sigma^3(\mathrm{d}u_{1:3})\, \mathrm{d}y_z\, \mathrm{d}x_z\, \mathrm{d}r \\
&\quad \leq n^3 \int_0^\infty \pi r^2 \int_{\mathbb{S}^3} \mathrm{e}^{-n\pi r^2} r^4 2\, \mathcal{A}(\Delta(u_{1:3}))\sigma^3(\mathrm{d}u_{1:3})\, \mathrm{d}y_z\, \mathrm{d}x_z\, \mathrm{d}r \\
&\quad \leq \pi n^3 \times \frac{15}{16\pi^3 n^{7/2}} \times 24\pi^2 \\
&\quad = \frac{45}{2\sqrt{n}} \\
&\quad = 22.5\, n^{-1/2},
\end{aligned}$$

where the integrals over r and $u_{1:3}$ are given by (A.1) and (A.2) in Appendix A. Replacing E^+ by E'^+ in (3.2), it follows that

$$\begin{aligned}
\mathbb{E}[L_{X_n}] = \frac{n^3}{2} \int_0^\infty \int_0^1 \int_{-r}^r \int_{\mathbb{S}^3} \mathrm{e}^{-n\pi r^2} \mathbf{1}_{[z+ru_{1:3} \in E'^+]} \|u_2 - u_1\| r^4 2\, \mathcal{A}(\Delta(u_{1:3}))\sigma^3(\mathrm{d}u_{1:3}) \\
\times\, \mathrm{d}y_z\, \mathrm{d}x_z\, \mathrm{d}r + O(n^{-1/2}).
\end{aligned}$$

Let $r \geq 0$ be fixed. By taking the change of variables $h = y_z/r$ (see Figure 2), we obtain

$$\begin{aligned}
&\int_{-r}^r \int_{\mathbb{S}^3} \mathbf{1}_{[z+ru_{1:3} \in E'^+]} \|u_2 - u_1\|\, \mathcal{A}(\Delta(u_{1:3}))\sigma^3(\mathrm{d}u_{1:3})\, \mathrm{d}y_z \\
&\quad = r \int_{-1}^1 \int_{\pi+\arcsin h}^{2\pi-\arcsin h} \int_{-\arcsin h}^{\pi+\arcsin h} \int_{-\arcsin h}^{\beta_2} \det \begin{pmatrix} 1 & 1 & 1 \\ \cos\beta_1 & \cos\beta_2 & \cos\beta_3 \\ \sin\beta_1 & \sin\beta_2 & \sin\beta_3 \end{pmatrix} \\
&\qquad\qquad \times 2 \sin\frac{\beta_2 - \beta_1}{2}\, \mathrm{d}\beta_1\, \mathrm{d}\beta_2\, \mathrm{d}\beta_3\, \mathrm{d}h \\
&\quad = \frac{35\pi}{3} r,
\end{aligned}$$

where the last line is due to (A.3). It follows that

$$\mathbb{E}[L_{X_n}] = n^3 \frac{35\pi}{3} \int_0^\infty \int_0^1 \mathrm{e}^{-n\pi r^2} r^5\, \mathrm{d}x_z\, \mathrm{d}r + O(n^{-1/2}) = \frac{35}{3\pi^2} + O(n^{-1/2}). \qquad \square$$

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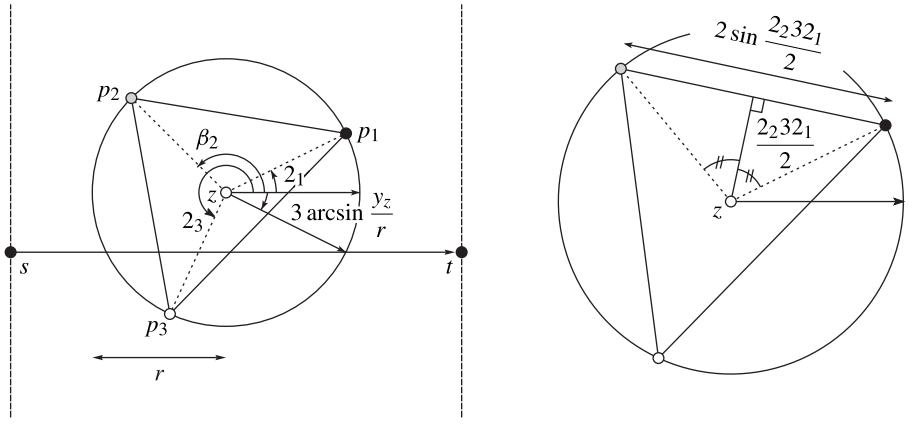


FIGURE 2: Notation for the proof of Proposition 3.1.

In the following proposition we deal with the variance of the length of the upper path.

Proposition 3.2. *Let X_n be a Poisson point process of intensity n. Let $\ell(U_{s,t}(X))$ be the length of the upper path $U_{s,t}(X)$ in* Del(X) *from s to t. Then*

$$\mathbb{V}[\ell(U_{s,t}(X))] = O(n^{-1/2}).$$

Proof. In the same spirit as in the proof of Proposition 3.1, it is enough to show that $\mathbb{V}[L_{X_n}] = O(n^{-1/2})$. The main idea is to apply a theorem due to Thäle and Yukich on the variance of linear statistics (see Theorem 2.2 of [21]). We first recall their framework, by introducing several definitions and notation.

We say that $A \subset \mathbb{R}^2$ is an admissible set if it is a compact set which is gentle (see [21, p. 2379] for the few restrictive conditions), regular closed (i.e. such that A equals the closure of its interior), and satisfies $H^2(\partial^2 A) = 0$, where H^2 is the two-dimensional Hausdorff measure. Let $\xi(p, \chi, \partial A)$ be a translation rotation-invariant function defined on triples $(p, \chi, \partial A)$, where $\chi \subset \mathbb{R}^2$ is locally finite, $p \in \chi$, and where $A \subset \mathbb{R}^2$ is an admissible set. The function $\xi(p, \chi, \partial A)$ is referred to as the score function. For shorthand, we write $\xi(p, \chi \cup \{p\}, \partial A)$ as $\xi(p, \chi, \partial A)$ if $p \notin \chi$. Now we state several properties of a score function.

- We say that ξ is *homogeneously stabilizing* if, for all $\tau \in (0, \infty)$ and all line L, there is $R := R^{\xi}(X_\tau, L) \in (0, \infty)$ a.s. (a radius of stabilization), where X_τ is a homogeneous Poisson point process of intensity τ in $\mathbb{R}^2$, such that

$$\xi(o, X_\tau \cap B(o, R), L) = \xi(o, (X_\tau \cap B(o, R)) \cup \mathcal{A}, L) \tag{3.3}$$

  for all locally finite $\mathcal{A} \subset (B(o, R))^c$.

- We say that ξ is *exponentially stabilizing* with respect to A if for all $p \in \mathbb{R}^2$ there is a radius of stabilization $R := R^{\xi}(p, P_n) \in (0, \infty)$ a.s. such that

$$\begin{aligned}&\xi(n^{1/2}p, (n^{1/2}P_n) \cap B(p, R), n^{1/2}\partial A)\\ &\quad = \xi(n^{1/2}p, ((n^{1/2}P_n) \cap B(p, R)) \cup (n^{1/2}\mathcal{A}), n^{1/2}\partial A)\end{aligned}$$

  for all locally finite $\mathcal{A} \subset (B(p, n^{-1/2}R))^c$ and where the tail probability $\tau(t) := \sup_{n>0,\, p\in\mathbb{R}^2} \mathbb{P}[R^{\xi}(p, P_n) > t]$ satisfies $\limsup_{t\to\infty} t^{-1} \log \tau(t) < 0$.

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- The score function ξ satisfies the *m-moment condition* with respect to A if there exists $m > 2$ and there is a bounded integrable function $G^{\xi,m}: \mathbb{R} \to \mathbb{R}_+$ with $\int_{\mathbb{R}} r(G^{\xi,m}(r))^{1/m}\,\mathrm{d}r < \infty$ such that, for all $r \in \mathbb{R}$,

$$\sup_{z\in\mathbb{R}^2\cup\varnothing}\ \sup_{p\in\mathcal{R}}\ \sup_{n>0} \mathbb{E}[|\xi(n^{1/2}p + ru_p, n^{1/2}(P_n \cup \{z\}), n^{1/2}\partial A)|^m] \le G^{\xi,m}(|r|),$$

  where $\mathcal{R}$ is the boundary of $[0,1]^2$ without its extremal points, and where u_p is the unit outward-pointing normal vector to $\partial[0,1]^2$ at p.

- The score is *well-approximated by P_n input on half-spaces* if, for all admissible set A, almost all $x \in \partial A$, and all $w \in \mathbb{R}^2$, we have

$$\lim_{n\to\infty} \mathbb{E}[|\xi(w, n^{1/2}(P_n - y), n^{1/2}(\partial A - y)) - \xi(w, n^{1/2}(P_n - y), L_y)|],$$

  where L_y is the line tangent to o at $\partial(A - y)$, with $y \in \partial A$.

We rewrite below a weak version of Theorem 2.2 of [21] in our context.

Theorem 3.1. (Thäle and Yukich [21, Theorem 2.2].) *Let P_n be a Poisson point process of intensity n in $[-2,2]^2$. Let $\xi(p,\chi,A)$ be a score function which is homogeneously stabilizing, exponentially stabilizing, satisfies the moment condition for some $m > 2$, and is well-approximated by P_n on half-spaces. Then there exists a constant $\sigma^2 \in [0,\infty)$ such that*

$$\lim_{n\to\infty} n^{-1/2}\mathbb{V}\left[\sum_{x\in P_n} \xi(n^{1/2}x, n^{1/2}P_n, n^{1/2}\partial[0,1]^2)\right] = \sigma^2.$$

In Theorem 2.2 of [21], the authors considered a Poisson point process $\mathcal{P}_\lambda := \mathcal{P}_{\lambda\kappa}$ of intensity $\lambda\kappa$ in $[0,1]^d$, $d \ge 2$, with κ a bounded density on $[0,1]^d$, a fixed admissible set $A_0 \subset [0,1]^d$, and a score function ξ defined on triples (x,χ,A). By scaling and by translation invariance, we consider a homogeneous Poisson point process P_n of intensity n in $[-2,2]^2$ by letting $A_0 := [0,1] \times [0,-1]$.

Returning to the proof of Proposition 3.2, we show how the above result implies that $\mathbb{V}[L_{X_n}] = O(n^{-1/2})$. Let L_{P_n} be defined as L_{X_n} by replacing X_n by P_n in (3.1). It is enough to prove that $\mathbb{V}[L_{P_n}] = O(n^{-1/2})$. In doing so, let $\xi(p,\chi,\partial A)$ be the translation rotation-invariant score function defined as follows. If $p \in \chi \cap A^c$ then

$$\xi(p,\chi,\partial A) = \frac{1}{2}\sum_{\{p_2,p_3\}\subset P_n} \mathbf{1}_{[\Delta(p,p_2,p_3)\in \mathrm{Del}(P_n)]}\mathbf{1}_{[p_2\notin A,\, p_3\in A]}\|p_2 - p\|, \tag{3.4}$$

where $\chi \subset \mathbb{R}^2$ is a locally finite set and where A is an admissible set. Otherwise, we let $\xi(p,\chi,\partial A) = 0$. In view of limits, such as (3.3), we need to define the score on the line $\mathbb{R}\times\{0\}$. We thus set $\xi(p,\chi,\mathbb{R}\times\{0\})$ as in (3.4) by replacing the admissible set A by $\mathbb{R}\times\mathbb{R}_-$ and its boundary ∂A by $\mathbb{R}\times\{0\}$.

Let $S_n := \sum_{p\in P_n} \xi(p, P_n, \partial A_0)$, where we recall that $A_0 = [0,1]\times[-1,0]$. Hence, S_n is the length of the (closed) path in $\mathrm{Del}(P_n) \cap ([0,1]^2)^c$ associated with the Delaunay triangles intersecting the boundary of A_0. We show below that the score $\chi(p,\chi,\partial A)$ satisfies the four assumptions of Theorem 3.1. This will be sketched since many similar examples derived from Theorem 3.1 can be found in the literature; see, e.g. Theorems 1.1–1.5 of [21] (see also four applications in Section 2 of [24] which are derived from a result which is very similar to Theorem 3.1).

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For the first assumption, let L be a line and recall that X_τ is a homogeneous Poisson point process of intensity τ. Equation (3.3) is satisfied by letting $R^\xi(X_\tau, L)$ be twice the radius of the smallest disk containing the Voronoi cell with nucleus o in the Voronoi tessellation associated with the point process $X_\tau \cup \{0\}$. Indeed, adding a point outside $B(o, R^\xi(X_\tau, L))$ does not modify the Delaunay edges with vertex o. This proves that ξ is homogeneously stabilizing. In the same spirit, we can show that ξ is exponentially stabilizing since the tail of the distribution of the circumradius of the typical Voronoi cell converges to 0 at a rate at least exponential (see, e.g. Equation (6) of [5]). The fact that ξ is well approximated by P_n input on half-spaces is trivial since $A_0 = [0, 1] \times [0, -1]$. It remains to prove that ξ satisfies the m-moment condition. We use similar arguments as the ones appearing in the proof of Theorem 2.1 of [24].

Let $z \in \mathbb{R}^2$ be fixed and let $p \in \mathcal{R}$. We denote by $L_{z,p,n}$ the sum of the lengths of the Delaunay edges with vertex p in $\mathrm{Del}(n^{1/2}(P_n \cup \{z\}))$. Moreover, we also let $C_{n^{1/2}(P_n\cup\{z\})}(n^{1/2}p + ru_p)$ as the Voronoi cell with nucleus $n^{1/2}p + ru_p$ in the Voronoi tesselation associated with the point process $n^{1/2}(P_n \cup \{z\})$. Assume that the score $\xi(n^{1/2}p + ru_p, n^{1/2}(P_n \cup \{z\}), n^{1/2}\partial A_0)$ is nonzero. In this case, there exists a Delaunay edge incident to $n^{1/2}p + ru_p \in (n^{1/2}A_0)^c$ (with length at least r) such that the endpoint belongs to $n^{1/2}A_0$. Hence, $2C_{n^{1/2}(P_n\cup\{z\})}(n^{1/2}p + ru_p)$ intersects $n^{1/2}A_0$. It follows that

$$\xi(n^{1/2}p + ru_p, n^{1/2}(P_n \cup \{z\}), n^{1/2}\partial A_0) \le L_{z,p,n}\mathbf{1}_{[2\cdot C_{n^{1/2}(P_n\cup\{z\})}(n^{1/2}p+ru_p)\cap n^{1/2}\partial A_0 \neq \varnothing]}.$$

In the same spirit as Lemma 2.2, we can prove that $L_{z,p,n}$ has finite moments of all orders, uniformly in p and z. Moreover, it may be seen that the probability of the event

$$\{C_{n^{1/2}(P_n\cup\{z\})}(n^{1/2}p + ru_p) \cap n^{1/2}\partial A_0 \neq \varnothing\}$$

decays exponentially fast in r, uniformly in p and z. The Cauchy–Schwarz inequality ensures that the m-moment condition is satisfied, completing the proof. □

As a corollary, we obtain an estimate of the tail of the length of the upper path.

Corollary 3.1. *With the same notation as above, we have*

$$\mathbb{P}[\ell(U_{s,t}(X)) > 1.2] = O(n^{-1/2}).$$

Proof. From Chebyshev's inequality, it follows that

$$\mathbb{P}[\ell(U_{s,t}(X)) > 1.2] \le \frac{\mathbb{V}[\ell(U_{s,t}(X))]}{1.2 - \mathbb{E}[\ell(U_{s,t}(X))]}.$$

This concludes the proof according to Proposition 3.1 and Proposition 3.2. □

4. Lower bound on the shortest path

In this section we prove Theorem 1.1. By scaling invariance, our problem is of the same type as if the intensity n is constant and $s = (0, 0)$, $t = (k, 0)$, where $k \in \mathbb{N}^*$ goes to ∞. The main idea is to discretize the plane $\mathbb{R}^2$ into squares (called pixels) and to define a lattice animal. For each pixel, we introduce a horizontality property. This property holds if there exists at least one path, through the pixel, which is almost horizontal (in a sense which will be specified). Then we prove that there exists a nonnegligible proportion of pixels intersected by the smallest path for which the horizontality property does not hold. This will provide a lower bound for the length of the smallest path.

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We begin our proof by introducing formally the notions of *pixels*, *animal lattices*, and *horizontality property*. Then we establish auxiliary results which will be the key ingredients to derive Theorem 1.1. These auxiliary results will be proved at the end of this section.

4.1. Preliminaries

4.1.1. *Pixels and lattice animals.* Recall that $s = (0, 0)$ and $t = (k, 0)$ for some integer k. We discretize $\mathbb{R}^2$ into pixels as follows. Let $\boldsymbol{G} = (\mathbb{Z}^2, E)$ be the graph with set of edges satisfying $(v, w) \in E$ if and only if $||v - w|| = 1$, where $v, w \in \mathbb{Z}^2$. In digital geometry, the graph is known as a *4-connected neighborhood* and each vertex of $\boldsymbol{G}$ is called a *pixel*. Moreover, for each $v \in \mathbb{Z}^2$, we consider different scaled versions of squares:

$$C(v) := v \oplus \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]^2, \qquad C^{\varepsilon}(v) := v \oplus \left[-\tfrac{1}{2} - \varepsilon, \tfrac{1}{2} + \varepsilon\right]^2, \qquad C_{\lambda}(v) := v \oplus \left[-\tfrac{1}{2}\lambda, \tfrac{1}{2}\lambda\right]^2,$$

where '$\oplus$' denotes the Minkowski sum, $\lambda \in \mathbb{N}^*$, and $\varepsilon \in \mathbb{R}_+^*$. With a slight abuse of notation, we also say that $C(v)$ is a pixel.

We define scaled and translated versions of the grid $\boldsymbol{G}$ as follows. For $\lambda \in \mathbb{N}^*$ and $\tau \in \mathbb{Z}^2$, we denote by $\lambda\boldsymbol{G}$ the grid of points in $(\lambda\mathbb{Z})^2$ with edges of length λ and $\tau + \lambda\boldsymbol{G}$ its translation by τ. We also split $\boldsymbol{G}$ into four subgrids as follows: each subgrid is referred to as a color $c \in$ colors, where colors := {green, pink, blue, yellow}. Each subgrid with color c is denoted by

$$\boldsymbol{G}_{(c)} = 2\boldsymbol{G} + O_{(c)},$$

where

$$O_{(\text{green})} = (0, 0), \qquad O_{(\text{pink})} = (1, 0), \qquad O_{(\text{blue})} = (0, 1), \qquad O_{(\text{yellow})} = (1, 1).$$

Note that when v and w are two pixels with the same color, the interiors of the squares $C_2(v)$ and $C_2(w)$ are the same. Such a property will be useful to ensure the independence of a suitable family of random variables.

We introduce below the notion of a so-called animal [9]. Given a graph $G' := (V', E')$, a *lattice animal* is a collection of vertices $\boldsymbol{A} \subset V'$ such that, for every pair of distinct vertices $v, w \in \boldsymbol{A}$, there is a path in G' connecting v, w visiting only vertices in $\boldsymbol{A}$. With each path P in the Delaunay triangulation, we associate the so-called lattice animal of P in $\boldsymbol{G}$ (see Figure 3):

$$\boldsymbol{A}(P) = \{v \text{ vertex of } \boldsymbol{G} \colon C(v) \cap P \neq \varnothing\}.$$

In the same spirit as above, for each $c \in$ colors, we let

$$\boldsymbol{A}_{(c)}(P) = \{v \text{ vertex of } \boldsymbol{G}_{(c)} \colon C_2(v) \cap P \neq \varnothing\}.$$

We also consider animals at different scales (see Figure 4):

$$\boldsymbol{A}_{(c),\lambda}(P) = \{v \text{ vertex of } \lambda\boldsymbol{G} + O_{(c)} \colon C_{\lambda}(v) \cap P \neq \varnothing\}.$$

4.1.2. *Properties on pixels.* For any pixel $v \in \mathbb{Z}^2$, we consider three events, namely, $\mathcal{I}_{\varepsilon}(v)$, $\mathcal{H}_{\rho}(v)$, and $\mathcal{H}'_{\varepsilon,\alpha,\kappa}(v)$. These events depend on four parameters $\varepsilon > 0$, $\rho > 0$, $\alpha \in [0, \pi/2]$, and $\kappa > 1$ described below.

- *Independence property* $\mathcal{I}_{\varepsilon}(v)$. This event holds if, for any Delaunay triangle in $\mathrm{Del}(X)$ intersecting $C^{\varepsilon}(v)$, the circumdisk of the triangle is included in $C_2(v)$ and $\|s - v\| \geq 2$ and $\|t - v\| \geq 2$ (to avoid pixels that are neighbors of s or t). Note that this event is $\sigma(X_n \cap C_2(v))$-measurable.

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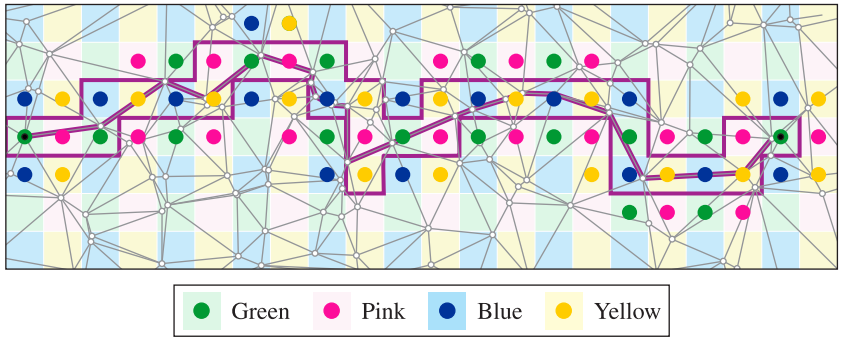


FIGURE 3: Lattice animals. The pixels $C(v)$ for $v \in \boldsymbol{A}(P)$ are shown as squares and the path P is the solid thick line. The $\boldsymbol{A}_{(c)}(P)$ for each $c \in$ colors are shown as circles.

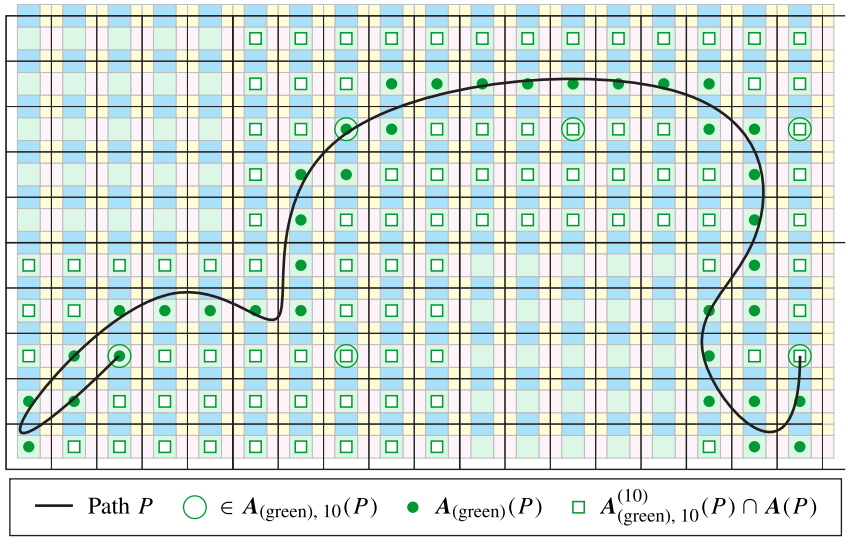


FIGURE 4: The set $\boldsymbol{A}^{(\lambda)}_{(\text{green}),\lambda}(P)$ for a path P.

- *Strong horizontality property* $\mathcal{H}_\rho(v)$. This event holds if there exists a path along Delaunay edges in $\mathrm{Del}(X)$, between $x_v - \frac{1}{2}$ and $x_v + \frac{1}{2}$, intersecting $C(v)$ and with length smaller than $1 + \rho$. A path satisfying this property is denoted by $\mathcal{PH}_\rho(v)$. The first and last edges of such a path are clipped by the vertical lines $x = x_v - \frac{1}{2}$ and $x = x_v + \frac{1}{2}$, respectively.

- *Weak horizontality property* $\mathcal{H}'_{\varepsilon,\alpha,\kappa}(v)$. This event holds if the total length $L_{\varepsilon,\alpha,\kappa}(v)$ of the horizontal projection of all edges in $\mathrm{Del}(X)$ intersecting $C^\varepsilon(v)$ and having an angle with respect to the x-axis smaller than α, is greater than $1/\kappa$, i.e.

$$L_{\varepsilon,\alpha,\kappa}(v) = \sum_{e,\, e\cap C^\varepsilon(v)\neq\varnothing,\, |\widehat{e}|\leq\alpha} h(e) \geq \frac{1}{\kappa}.$$

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As we will see in Lemma 4.3, under a suitable choice of $\varepsilon, \alpha, \kappa$, the strong horizontality property implies the weak horizontality property.

4.2. Auxiliary results

In this section we establish auxiliary results which will be used to derive Theorem 1.1. The first result provides a lower bound for the length of any path with respect to the number of pixels with a strong horizontality property.

Proposition 4.1. *Let $P \in \mathcal{P}_{s,t}(X)$ and $\rho > 0$, $\varepsilon > 0$. Then*

$$\ell(P) \geq k + \rho\Big(k - 4 \max_{c \in \text{colors}} \#_{\mathcal{H}}(\boldsymbol{A}_{(c)}(P))\Big),$$

where $\#_{\mathcal{H}}(\boldsymbol{A}_{(c)}(P))$ is the number of pixels v in $\boldsymbol{A}_{(c)}(P)$ such that $\mathcal{H}_\rho(v) \vee \neg \mathcal{I}_\varepsilon(v)$.

As a second auxiliary result, we prove that the number of pixels in $\boldsymbol{A}(S_{s,t}(X))$ which satisfy a property $\mathcal{Y}$ is not large with high probability. In the sequel, we will use the following notation:

$$\mathbb{Z}^{2,(c)}_{s,t} := \mathbb{Z}^2_{s,t} \cap \{v \in \mathbb{Z}^2 \text{ with color } c\},$$

where

$$\mathbb{Z}^2_{s,t} := \mathbb{Z}^2 \setminus \{v \in \mathbb{Z}^2 \colon \|v - s\| \leq 2 \text{ or } \|v - t\| \leq 2\}.$$

Proposition 4.2. *Let $p \in (0, 0.01]$ and let $\mathcal{Y} := (Y_v)_{v \in \mathbb{Z}^2}$ be a family of events such that, for any color c, the events $(Y_v)_{v \in \mathbb{Z}^{2,(c)}_{s,t}}$ are independent and $p = \mathbb{P}[Y_v]$ for each $v \in \mathbb{Z}^{2,(c)}_{s,t}$. For any $\boldsymbol{A} \subset \mathbb{Z}^2$, we denote by $\#_{\mathcal{Y}}(\boldsymbol{A}) = \sum_{v \in \boldsymbol{A}} \mathbf{1}_{[Y_v]}$ the number of pixels v in $\boldsymbol{A}$ such that the event Y_v holds. Then, for any color c, we have*

$$\mathbb{P}[\#_{\mathcal{Y}}(\boldsymbol{A}_{(c)}(S_{s,t}(X))) \geq k\sqrt{p}] = O(k^{-1/2}).$$

This result will be applied to the case where $\mathcal{Y}$ is the strong horizontality property. To apply such a result, we have to estimate the probability of the event $\mathcal{H}_\rho(v) \vee \neg \mathcal{I}_{\varepsilon_\rho}(v)$ for a suitable choice of ε_ρ. In the following proposition, we show that the probability that a pixel has a (strong) horizontality property is small.

Proposition 4.3. *Let $v \in \mathbb{Z}^2_{s,t}$ and $0 < \rho < 4 \times 10^{-6}$, and let $\varepsilon_\rho := \sqrt{\rho}\sqrt{2+\rho}$. Then*

$$\mathbb{P}[\mathcal{H}_\rho(v) \vee \neg \mathcal{I}_{\varepsilon_\rho}(v)] \leq P(\rho, n),$$

where

$$P(\rho, n) := 95n^3\, \mathrm{e}^{-0.194n} + (19n^2 + 13n + 4)\, \mathrm{e}^{-n\pi} + 31.76\left(\frac{3}{4} + \frac{\sqrt{\rho}\sqrt{\rho+2}}{2}\right)^2 \sqrt{\rho n}.$$

The above result constitutes one of the main difficulties in proving Theorem 1.1.

4.3. Proof of Theorem 1.1

Let ρ and n be such that $p \leq 0.01$. According to Proposition 4.1, we have

$$\begin{aligned}\mathbb{P}[\ell(S_{s,t}(X)) \leq k + \rho(k - 4k\sqrt{p})] &\leq \mathbb{P}\Big[\max_{c \in \text{colors}} \#_{\mathcal{H}}(\boldsymbol{A}_{(c)}(S_{s,t}(X))) \geq k\sqrt{p}\Big] \\ &\leq \sum_{c \in \text{colors}} \mathbb{P}[\#_{\mathcal{H}}(\boldsymbol{A}_{(c)}(S_{s,t}(X))) \geq k\sqrt{p}].\end{aligned}$$

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To bound the right-hand side, we apply Proposition 4.2 to the family of events $\mathcal{Y} := (Y_v)_{v\in\mathbb{Z}^2}$, where $Y_v := \mathcal{H}_\rho(v) \vee \neg\mathcal{I}_{\varepsilon_\rho}(v)$. Note that for any color c, the events $(Y_v)_{v\in\mathbb{Z}^{2,(c)}_{s,t}}$ are independent: this comes from the fact that the event $\mathcal{H}_\rho(v) \vee \neg\mathcal{I}_{\varepsilon_\rho}(v)$ is $\sigma(X_n \cap C_2(v))$-measurable and the fact that the interiors of $C_2(v)$ and $C_2(w)$ are disjoint when $v \neq w \in \mathbb{Z}^{2,(c)}_{s,t}$. Moreover, for any $v \in \mathbb{Z}^{2,(c)}_{s,t}$, the probability $p := \mathbb{P}[\mathcal{H}_\rho(v) \vee \neg\mathcal{I}_{\varepsilon_\rho}(v)]$ does not depend on v. Taking $\rho \le 4 \times 10^{-6}$, we deduce from Proposition 4.2 and Proposition 4.3 that

$$\mathbb{P}[\ell(S_{s,t}(X)) \le k + \rho(1 - 4\sqrt{P(\rho, n)})k] = O(k^{-1/2}).$$

To obtain the best lower bound in Theorem 1.1, we have to maximize $\rho(1 - 4\sqrt{P(\rho, n)})$. The values $\rho = 3.4 \times 10^{-8}$ and $n = 136$ yield a good estimate for this bound. Then we have $\rho < 4 \times 10^{-6}$ and $p \le P(\rho, n) \simeq 0.04 < 0.01$, so that the assumptions of Propositions 4.2 and 4.3 are satisfied. This concludes the proof of Theorem 1.1 since

$$(1 - 4\sqrt{P(\rho, n)}) \ge 7 \times 10^{-9}. \qquad \square$$

4.4. Proofs of the auxiliary results

Proof of Proposition 4.1. Splitting the path P into vertical columns, we have $\ell(P) = \sum_{i\in\mathbb{Z}} \ell(P \cap \mathrm{col}[i])$, where $\mathrm{col}[i] = [i - \frac{1}{2}, i + \frac{1}{2}] \times \mathbb{R}$ is the ith column for each $i \in \mathbb{Z}$. For $i \in \{1, \dots, k-1\}$, let $v[i]$ be the lowest pixel of $\boldsymbol{A}(P) \cap \mathrm{col}[i]$ such that there is a connected component of $P \cap \mathrm{col}[i]$ intersecting $C(v[i])$ and the left and right side of $\mathrm{col}[i]$. Note that such a pixel exists for each column $1 \le i \le k-1$. On the event $\neg\mathcal{H}_\rho(v[i]) \wedge \mathcal{I}_\varepsilon(v[i])$, we use the fact that $\ell(P \cap \mathrm{col}[i]) \ge 1 + \rho$. On the complement of this event, we use the trivial inequality $\ell(P \cap \mathrm{col}[i]) \ge 1$. Denoting by $N := \sum_{i=1}^{k-1} \mathbf{1}_{[\mathcal{H}_\rho(v[i]) \vee \neg\mathcal{I}_\varepsilon(v[i]))]}$ the number of horizontal pixels on the path, it follows that

$$\begin{aligned}
\ell(P) &\ge \sum_{i=1}^{k-1} \mathbf{1}_{[\mathcal{H}_\rho(v[i]) \vee \neg\mathcal{I}_\varepsilon(v[i]))]} + \sum_{i=1}^{k-1} (1+\rho)\mathbf{1}_{[\neg\mathcal{H}_\rho(v[i]) \wedge \mathcal{I}_\varepsilon(v[i]))]} \\
&\ge N + (1+\rho)(k - N) \\
&= k + \rho(k - N).
\end{aligned}$$

Then we conclude the proof by observing that

$$N \le \sum_{v\in\boldsymbol{A}(P)} \mathbf{1}_{[\mathcal{H}_\rho(v) \vee \neg\mathcal{I}_\varepsilon(v)]} = \sum_{c\in\mathrm{colors}} \#_{\mathcal{H}}(\boldsymbol{A}_{(c)}(P)) \le 4 \max_{c\in\mathrm{colors}} \#_{\mathcal{H}}(\boldsymbol{A}_{(c)}(P)). \qquad \square$$

In order to prove Proposition 4.2, we establish a result which provides an upper bound for the size of the lattice animal of $S_{s,t}(X)$ with high probability.

Lemma 4.1. *Let $S_{s,t}(X) \in \mathcal{P}_{s,t}(X)$ with $\|s - t\| = k$. For any $\lambda \ge 2$, $c \in$ colors, and $k > 0$, let $\mathcal{E}(c, k, \lambda)$ be the event*

$$\mathcal{E}(c, k, \lambda) := \left\{ \mathrm{card}(\boldsymbol{A}_{(c),\lambda}(S_{s,t}(X))) < \frac{2.55\,k}{\lambda} + 1 \right\}. \tag{4.1}$$

Then $\mathbb{P}[\mathcal{E}(c, k, \lambda)] = 1 - O(k^{-1/2})$.

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Proof. According to a result of Gerard *et al.* [16], we know that

$$\operatorname{card}(\boldsymbol{A}(P)) \leq \frac{3\sqrt{2}}{2}\ell(P) + 1$$

for any $s, t \in \mathbb{R}^2$ and for any path P between s and t, not necessarily in $\mathrm{Del}(X)$ (a more concise proof of the above inequality can be found in [7]). As a 'colored version', we obtain, for each $c \in$ colors,

$$\operatorname{card}(\boldsymbol{A}_{(c),\lambda}(S_{s,t}(X))) \leq \frac{3\sqrt{2}}{2}\,\frac{\ell(S_{s,t}(X))}{\lambda} + 1.$$

Lemma 4.1 is a direct consequence of the above inequality and the fact that $\ell(S_{s,t}(X)) < 1.2k$ with probability $1 - O(k^{-1/2})$ according to Corollary 3.1. The term $2.55\,k/\lambda + 1$ comes from the fact that $1.2 \cdot 3\sqrt{2}/2 < 2.55$. □

Proof of Proposition 4.2. Our proof relies on an adaptation of Devillers and Hemsley [11, Lemma 7]. The main idea is to discretize the shortest path at different scales and to use standard ideas of (site) percolation theory. Let $\lambda \in 4\mathbb{N} + 2$. This restriction on λ ensures the following assertion: for each $v \in \boldsymbol{G}_{(c)}$, for each $w \in \lambda\boldsymbol{G} + O_{(c)}$, if the Lebesgue measure of the intersection of the squares $C_2(v)$ and $C_\lambda(w)$ differs from 0, then $C_2(v)$ is included in $C_\lambda(w)$. Taking $\mathcal{E}(c, k, \lambda)$ as in (4.1), we obtain, for each $x > 0$,

$$\begin{aligned}\mathbb{P}[\#_{\mathcal{Y}}(\boldsymbol{A}_{(c)}(S_{s,t}(X))) \geq xk\sqrt{p}] &= \mathbb{P}[\{\#_{\mathcal{Y}}(\boldsymbol{A}_{(c)}(S_{s,t}(X))) \geq xk\sqrt{p}\} \cap \mathcal{E}(c, k, \lambda)] \\ &\quad + \mathbb{P}[\{\#_{\mathcal{Y}}(\boldsymbol{A}_{(c)}(S_{s,t}(X))) \geq xk\sqrt{p}\} \cap (\neg\mathcal{E}(c, \lambda, k))].\end{aligned}$$

From Lemma 4.1, the second term is equal to $O(k^{-1/2})$.

The animal $\boldsymbol{A}_{(c),\lambda}(S_{s,t}(X))$ can be viewed as a sequence of connected squares of size λ, starting at the square containing s. On the event $\mathcal{E}(c, k, \lambda)$, the animal $\boldsymbol{A}_{(c),\lambda}(S_{s,t}(X))$ belongs to the family $\mathcal{A}_{(c),\lambda}(k)$ of animals $\boldsymbol{A} \subset \lambda\boldsymbol{G} + O_{(c)}$ such that $O_{(c)} \in \boldsymbol{A}$ and $\operatorname{card}(\boldsymbol{A}) \leq \lfloor 2.55k/\lambda + 1 \rfloor$. Each animal $\boldsymbol{A} \in \mathcal{A}_{(c),\lambda}(k)$ can be encoded as a word on four letters {S, E, N, W} (standing for south, east, north, and west) of length $\operatorname{card}(\boldsymbol{A})$. Hence, $\operatorname{card}(\mathcal{A}_{(c),\lambda}(k)) \leq 4^{\lfloor 2.55\,k/\lambda+1 \rfloor} \leq 4^{2.55\,k/\lambda+1}$. Moreover, since $\lambda \in 4\mathbb{N} + 2$, we have

$$\boldsymbol{A}_{(c)}(S_{s,t}(X)) \subset \boldsymbol{A}^{(\lambda)}_{(c),\lambda}(S_{s,t}(X)),$$

where, for an animal $\boldsymbol{A} \subset \lambda\boldsymbol{G} + O_{(c)}$, we let $\boldsymbol{A}^{(\lambda)} := \{v \in \boldsymbol{G}_{(c)} : \text{there exists } w \in \boldsymbol{A},\ v \in C_\lambda(w)\}$ (see Figure 4).

This implies that

$$\begin{aligned}&\mathbb{P}[\{\#_{\mathcal{Y}}(\boldsymbol{A}_{(c)}(S_{s,t}(X))) \geq k\sqrt{p}\} \cap \mathcal{E}(c, k, \lambda)] \\ &\quad\leq \mathbb{P}[\{\#_{\mathcal{Y}}(\boldsymbol{A}^{(\lambda)}_{(c),\lambda}(S_{s,t}(X))) \geq k\sqrt{p}\} \cap \mathcal{E}(c, k, \lambda)] \\ &\quad\leq \mathbb{P}\Bigg[\bigcup_{\boldsymbol{A}\in\mathcal{A}_{(c),\lambda}(k)} \{\#_{\mathcal{Y}}(\boldsymbol{A}^{(\lambda)}) \geq k\sqrt{p}\}\Bigg] \\ &\quad\leq \sum_{\boldsymbol{A}\in\mathcal{A}_{(c),\lambda}(k)} \mathbb{P}[\#_{\mathcal{Y}}(\boldsymbol{A}^{(\lambda)}) \geq k\sqrt{p}] \\ &\quad\leq \sum_{\boldsymbol{A}\in\mathcal{A}_{(c),\lambda}(k)} \mathbb{P}\Bigg[\sum_{v\in\boldsymbol{A}^{(\lambda)}\cap\mathbb{Z}^{2,(c)}_{s,t}} \mathbf{1}_{[Y_v]} \geq k\sqrt{p} - 10\Bigg]. \qquad (4.2)\end{aligned}$$

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The last inequality comes from the fact that $\sum_{v\in \boldsymbol{A}^{(\lambda)}\cap\mathbb{Z}^{2,(c)}_{s,t}} \mathbf{1}_{[Y_v]} \geq \#\mathcal{Y}(\boldsymbol{A}^{(\lambda)}) - 10$ since the number of pixels in $\mathbb{Z}^2$ with color c, which are not in $\mathbb{Z}^{2,(c)}_{s,t}$, is smaller than 10. From the assumption of Proposition 4.2, we know that, for each $\boldsymbol{A} \in \mathcal{A}_{(c),\lambda}(k)$, the random variable $\sum_{v\in \boldsymbol{A}^{(\lambda)}\cap\mathbb{Z}^{2,(c)}_{s,t}} \mathbf{1}_{[Y_v]}$ is a binomial distribution with parameters (N, p), where $N := \operatorname{card}(\boldsymbol{A}^{(\lambda)} \cap \mathbb{Z}^{2,(c)}_{s,t}) \leq 0.64\lambda k$ since $\operatorname{card}(\boldsymbol{A}^{(\lambda)}) \leq (2.55k/\lambda)(\lambda^2/4) + 1$, $s \in \boldsymbol{A}^{(\lambda)} \setminus \mathbb{Z}^{2,(c)}_{s,t}$, and $2.55/4 \leq 0.64$. Assume that λ is such that

$$k\sqrt{p} - 10 \geq 0.65\lambda kp. \tag{4.3}$$

Since $k\sqrt{p} - 10 > Np$, it follows from a concentration inequality for a binomial random variable (see, e.g. [19, Lemma 1.1]) that, for all $\boldsymbol{A} \in \mathcal{A}_{(c),\lambda}(k)$,

$$\mathbb{P}\left[\sum_{v\in \boldsymbol{A}^{(\lambda)}\cap\mathbb{Z}^{2,(c)}_{s,t}} \mathbf{1}_{[Y_v]} \geq k\sqrt{p} - 10\right] \leq \exp\left(-NpH\left(\frac{k\sqrt{p} - 10}{Np}\right)\right),$$

where $H\colon (0,\infty) \to [0,\infty)$ is defined by $H(a) = 1 - a + a\log a$ for each $a > 0$. Since, for each $z > 0$, the function $a \mapsto aH(z/a)$ is decreasing on $[z,\infty)$ and since $Np < 0.64\lambda kp$, we have

$$\mathbb{P}\left[\sum_{v\in \boldsymbol{A}^{(\lambda)}\cap\mathbb{Z}^{2,(c)}_{s,t}} \mathbf{1}_{[Y_v]} \geq k\sqrt{p} - 10\right] \leq \exp\left(-0.64\lambda kpH\left(\frac{k\sqrt{p} - 10}{0.64\lambda kp}\right)\right).$$

Since H is increasing on $[1,\infty)$, from (4.3), it follows that

$$\mathbb{P}\left[\sum_{v\in \boldsymbol{A}^{(\lambda)}\cap\mathbb{Z}^{2,(c)}_{s,t}} \mathbf{1}_{[Y_v]} \geq k\sqrt{p} - 10\right] \leq \exp\left(-0.64\lambda kpH\left(\frac{0.65}{0.64}\right)\right).$$

The right-hand side is negligible compared to $k^{-1/2}$ since $\lambda p > 0$ and $H(0.65/0.64) > 0$. It remains to choose λ in such a way that (4.3) holds. To do so, we take $\lambda \in [0, 1.5/\sqrt{p}] \cap (4\mathbb{N} + 2)$. The fact that such a λ exists is ensured by the condition $p < 0.01$. □

4.5. Proof of Proposition 4.3

To pave the way, we proceed into two steps. In Section 4.5.1 we choose parameters ε and α in such a way that the strong horizontality property is stronger than the weak horizontality property. In Sections 4.5.2 and 4.5.3 we provide bounds for the probability that a pixel has a weak horizontality property or an independence property.

4.5.1. *Strong versus weak horizontality.*

Lemma 4.2. *Let $v \in \mathbb{Z}^2$, $\rho > 0$, and $\varepsilon_\rho := \sqrt{\rho}\sqrt{2+\rho}$. If the property $\mathcal{H}_\rho(v)$ holds then $\mathcal{PH}_\rho(v) \subset C^{\varepsilon_\rho}(v)$.*

Proof. Assume that $v = 0$ without loss of generality. Up to a vertical translation, the shortest path between the lines $x = -\frac{1}{2}$ and $x = \frac{1}{2}$ crossing $C(0)$ and intersecting the complement of $C^{\varepsilon_\rho}(0)$ is the segment from $(-\frac{1}{2}, \frac{1}{2})$ to $(\frac{1}{2}, \frac{1}{2} + \varepsilon_\rho)$. Since the length of this segment is $\sqrt{1+\varepsilon_\rho^2} = 1 + \rho$, we necessarily have $\mathcal{PH}_\rho(v) \subset C^{\varepsilon_\rho}(v)$. □

Lemma 4.3. *Let $v \in \mathbb{Z}^2$, $\rho > 0$, and $0 < \kappa < 1$ such that $(\kappa/(\kappa-1))\rho < \pi^2/8$ and let $\alpha_{\rho,\kappa} := \sqrt{2(\kappa/(\kappa-1))\rho}$. If the property $\mathcal{H}_\rho(v)$ holds then the same is true for $\mathcal{H}'_{\varepsilon_\rho,\alpha_{\rho,\kappa},\kappa}(v)$.*

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Proof. We proceed with a proof by contradiction, assuming $\mathcal{H}_\rho(v)$ and $\neg\mathcal{H}'_{\varepsilon_\rho,\alpha_{\rho,\kappa},\kappa}(v)$. The main idea is to split the edges e in $\mathcal{PH}_\rho(v)$ with respect to their angles $|\widehat{e}|$. Indeed,

$$1+\rho \geq \ell(\mathcal{PH}_\rho(v)) = \sum_{e\in\mathcal{PH}_\rho(v),\,|\widehat{e}|\leq\alpha_{\rho,\kappa}} \ell(e) + \sum_{e\in\mathcal{PH}_\rho(v),\,|\widehat{e}|>\alpha_{\rho,\kappa}} \ell(e),$$

where the inequality comes from the property $\mathcal{H}_\rho(v)$. For each $e\in\mathcal{PH}_\rho(v)$, we use the trivial inequality $l(e)\geq h(e)$ when $|\widehat{e}|\leq\alpha_{\rho,\kappa}$. If $|\widehat{e}|>\alpha_{\rho,\kappa}$, we note that, for $(\kappa/(\kappa-1))\rho<\pi^2/8$,

$$\ell(e) > \frac{h(e)}{\cos\alpha_{\rho,\kappa}} \geq \left(1+\frac{\alpha_{\rho,\kappa}^2}{2}\right)h(e) = \left(1+\frac{\kappa}{\kappa-1}\rho\right)h(e),$$

where the second inequality comes from the fact that $1/\cos\alpha\geq(1+\alpha^2/2)$ for any $\alpha\in[0,\pi/2)$.

It follows that

$$\begin{aligned}1+\rho &> \left(1+\frac{\kappa}{\kappa-1}\rho-\frac{\kappa}{\kappa-1}\rho\right)\sum_{e\in\mathcal{PH}_\rho(v),|\widehat{e}|\leq\alpha_{\rho,\kappa}} h(e)\\ &\quad+\left(1+\frac{\kappa}{\kappa-1}\rho\right)\sum_{e\in\mathcal{PH}_\rho(v),|\widehat{e}|>\alpha_{\rho,\kappa}} h(e)\\ &= \left(1+\frac{\kappa}{\kappa-1}\rho\right)\sum_{e\in\mathcal{PH}_\rho(v)} h(e) - \frac{\kappa}{\kappa-1}\rho\sum_{e\in\mathcal{PH}_\rho(v),|\widehat{e}|\leq\alpha_{\rho,\kappa}} h(e).\end{aligned}$$

By assumption, the property $\mathcal{H}'_{\varepsilon_\rho,\alpha_{\rho,\kappa},\kappa}(v)$ does not hold. Then, from Lemma 4.2, we deduce that

$$1+\rho > \left(1+\frac{\kappa}{\kappa-1}\rho\right) - \frac{\kappa}{\kappa-1}\rho L_{\varepsilon_\rho,\alpha_{\rho,\kappa},\kappa}(v) > 1+\frac{\kappa}{\kappa-1}\rho-\frac{\kappa}{\kappa-1}\rho\frac{1}{\kappa} = 1+\rho,$$

obtaining a contradiction. □

4.5.2. *Pixel probabilities: probability for the independence property.* First, we provide an upper bound for the probability that a pixel does not have the independence property.

Lemma 4.4. *Let $v\in\mathbb{Z}^2_{s,t}$ and let $\varepsilon<\frac{1}{700}$ be fixed. Then*

$$\mathbb{P}[\neg\mathcal{I}_\varepsilon(v)] \leq 95n^3\,\mathrm{e}^{-0.194n} + (19n^2+13n+4)\,\mathrm{e}^{-n\pi}.$$

In the above lemma, we have assumed that $\varepsilon<\frac{1}{700}$ to obtain an upper bound for $\mathbb{P}[\neg\mathcal{I}_\varepsilon(v)]$ which is independent of ε.

Proof of Lemma 4.4. Let $N_\varepsilon(v)$ be the number of Delaunay triangles in $\mathrm{Del}(X_n)$ such that the associated circumdisk intersects simultaneously $C^\varepsilon(v)$ and the complement of $C_2(v)$. If the event $\mathcal{I}_\varepsilon(v)$ does not hold then $N_\varepsilon(v)\geq 1$ (here we have used the fact that $\mathcal{I}_\varepsilon(v)$ is $\sigma(X_n\cap C(v))$-measurable). From Markov's inequality, it follows that

$$\mathbb{P}[\neg\mathcal{I}_\varepsilon(v)] \leq \mathbb{P}[N_\varepsilon(v)\geq 1] \leq \mathbb{E}[N_\varepsilon(v)],$$

where

$$\mathbb{E}[N_\varepsilon(v)] = \frac{1}{3!}\mathbb{E}\Bigg[\sum_{p_{1:3}^{\neq}\in X_n^3} \mathbf{1}_{[\Delta(p_{1:3})\in\mathrm{Del}(X_n)]}\mathbf{1}_{[B(p_{1:3})\not\subset C_2(v);\,B(p_{1:3})\cap C^\varepsilon(v)\neq\varnothing]}\Bigg].$$

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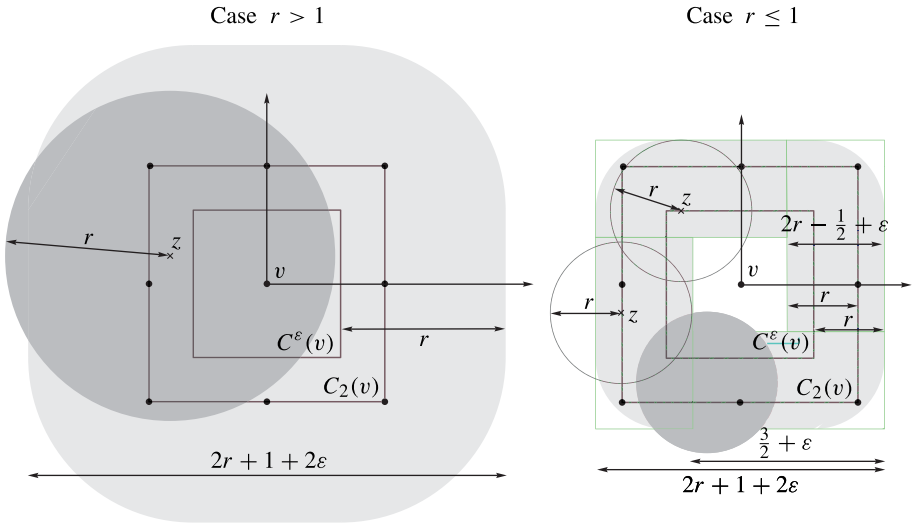


FIGURE 5: The squares $C^\varepsilon(v)$, $C_2(v)$ (*solid outline*), the set $C^\varepsilon(v) \oplus B(0,r)$ (*shaded light*) and the disk $B(z,r)$ (*shaded dark*).

From the Slivnyak–Mecke and the Blaschke–Petkantschin formulas, it follows that

$$\begin{aligned}
\mathbb{E}[N_\varepsilon(v)] &= \frac{n^3}{3!}\int_{(\mathbb{R}^2)^3} \mathbb{P}[B(z,r)\cap X_n = \varnothing]\mathbf{1}_{[B(z,r)\not\subset C_2(v);\ B(z,r)\cap C^\varepsilon(v)\neq\varnothing]}\,\mathrm{d}p_{1:3} \\
&= \frac{n^3}{6}\int_{\mathbb{R}_+}\int_{\mathbb{R}^2}\int_{\mathbb{S}^3} \mathrm{e}^{-n\,\mathcal{A}(B(z,r))}\mathbf{1}_{[B(z,r)\not\subset C_2(v);\ B(z,r)\cap C^\varepsilon(v)\neq\varnothing]} \\
&\qquad\qquad \times r^3 2\,\mathcal{A}(\Delta(u_{1:3}))\sigma^3(\mathrm{d}u_{1:3})\,\mathrm{d}z\ \mathrm{d}r \\
&= \frac{n^3}{6}24\pi^2\int_{\mathbb{R}_+}\int_{\mathbb{R}^2} \mathrm{e}^{-n\pi r^2}\mathbf{1}_{[B(z,r)\not\subset C_2(v);\ z\in C^\varepsilon(v)\oplus B(0,r)]}r^3\,\mathrm{d}z\ \mathrm{d}r, \qquad (4.4)
\end{aligned}$$

where the last line comes from (A.2) and the fact that $B(z,r)\cap C^\varepsilon(v) \neq \varnothing$ if and only if $z \in C^\varepsilon(v)\oplus B(0,r)$. To deal with $B(z,r) \not\subset C_2(v)$, we consider two cases as follows (see Figure 5).

Case 1. If $r \le 1$, we use the fact that

$$\begin{aligned}
&B(z,r) \not\subset C_2(v) \quad \textit{and} \quad z \in C^\varepsilon(v)\oplus B(0,r) \\
&\qquad\Longrightarrow \quad z \in \left[-r-\tfrac{1}{2}-\varepsilon, r+\tfrac{1}{2}+\varepsilon\right]^2 \setminus [r-1, 1-r]^2.
\end{aligned}$$

Note that we have to choose $r > (1-2\varepsilon)/4 > 0.249$ to ensure that the set of the right-hand side is not empty. This set has an area smaller than $4(2r+\varepsilon-\frac{1}{2})(\frac{3}{2}+\varepsilon) < 12.016r$.

Case 2. If $r > 1$, we use the trivial assertion

$$B(z,r) \not\subset C_2(v) \quad \textit{and} \quad z \in C^\varepsilon(v)\oplus B(0,r) \Rightarrow z \in \left[-r-\tfrac{1}{2}-\varepsilon, r+\tfrac{1}{2}+\varepsilon\right]^2.$$

The set of the right-hand side has an area which is lower than $(1+2\varepsilon+2r)^2 < 3.004r^2$.

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By integrating over z, from (4.4), it follows that

$$\begin{aligned}\mathbb{P}[\neg \mathcal{I}_\varepsilon(v)] &\le 4\pi^2 n^3 \left(\int_{0.249}^1 \mathrm{e}^{-n\pi r^2} 12.016\, r^4 \,\mathrm{d}r + \int_1^\infty \mathrm{e}^{-n\pi r^2} 3.004\, r^5 \,\mathrm{d}r \right) \\ &\le 4\pi^2 n^3 \left(\mathrm{e}^{-n\pi 0.249^2} \int_{0.249}^1 12.016\, r^4 \,\mathrm{d}r + \int_1^\infty \mathrm{e}^{-n\pi r^2} 3.004\, r^5 \,\mathrm{d}r \right) \\ &\le 95 n^3 \,\mathrm{e}^{-0.194n} + (19n^2 + 13n + 4)\,\mathrm{e}^{-n\pi}. \end{aligned}$$

□

4.5.3. *Pixel probabilities: probability for the weak horizontality property.* We provide an upper bound for the probability that a pixel has a weak horizontality property conditional on the fact that it has the independence property.

Lemma 4.5. *Let $v \in \mathbb{Z}^2_{s,t}$ and $\rho < 0.4$. Then there exists $\kappa > 0$ such that $(\kappa/(\kappa-1))\rho < \pi^2/8$ and*

$$\mathbb{P}[\mathcal{H}'_{\varepsilon_\rho,\alpha_{\rho,\kappa},\kappa}(v) \wedge \mathcal{I}_{\varepsilon_\rho}(v)] \le 31.8 \left(\frac{3}{4} + \frac{\sqrt{\rho}\sqrt{\rho+2}}{2} \right)^2 \sqrt{\rho} n,$$

where $\varepsilon_\rho := \sqrt{\rho}\sqrt{2+\rho}$ and $\alpha_{\rho,\kappa} := \sqrt{2(\kappa/(\kappa-1))\rho} \in [0, \pi/2]$ are the same as in Lemma 4.3.

Proof. According to Markov's inequality, we have

$$\begin{aligned}\mathbb{P}[\mathcal{H}'_{\varepsilon_\rho,\alpha_{\rho,\kappa},\kappa}(v) \wedge \mathcal{I}_{\varepsilon_\rho}(v)] &= \mathbb{P}\left[\left\{ L_{\varepsilon_\rho,\alpha_{\rho,\kappa},\kappa}(v) \ge \frac{1}{\kappa} \right\} \wedge \mathcal{I}_{\varepsilon_\rho}(v) \right] \\ &\le \kappa \mathbb{E}[L_{\varepsilon_\rho,\alpha_{\rho,\kappa},\kappa}(v) \mathbf{1}_{[\mathcal{I}_{\varepsilon_\rho}(v)]}]. \end{aligned} \tag{4.5}$$

Now, recalling that on the event $\mathcal{I}_{\varepsilon_\rho}(v)$, any triangle, and thus any edge, intersecting $C^{\varepsilon_\rho}(v)$ has its vertices inside $C_2(v)$, we have

$$\begin{aligned}\mathbb{E}[L_{\varepsilon_\rho,\alpha_{\rho,\kappa},\kappa}(v) \mathbf{1}_{[\mathcal{I}_{\varepsilon_\rho}(v)]}] \le \frac{1}{2} \mathbb{E}\Bigg[\sum_{p_1 \ne 3 \in X_n^3} & h(p_1 p_2) \mathbf{1}_{[p_{1:3} \in \mathrm{Del}(X_n)]} \mathbf{1}_{[|\widehat{p_1 p_2}| < \alpha_{\rho,\kappa}; x_{p_1} \le x_{p_2}]} \\ & \times \mathbf{1}_{[B(p_{1:3}) \subset C_2(v)]} \mathbf{1}_{[B(p_{1:3}) \cap C^{\varepsilon_\rho}(v) \ne \varnothing]} \Bigg], \end{aligned}$$

where the $\frac{1}{2}$ factor comes from the fact that each edge is counted twice in the sum (once for each incident triangle). We apply the Slivnyak–Mecke and the Blaschke–Petkantschin formulas. This yields

$$\begin{aligned}&\mathbb{E}[L_{\varepsilon_\rho,\alpha_{\rho,\kappa},\kappa}(v) \mathbf{1}_{[\mathcal{I}_{\varepsilon_\rho}(v)]}] \\ &\quad \le \frac{1}{2} n^3 \int_{C_2(v)} \int_0^1 \int_{\mathbb{S}^3} \mathrm{e}^{-n\,\mathcal{A}(B(z,r))} \mathbf{1}_{[|\widehat{u_1 u_2}| < \alpha_{\rho,\kappa}; x_{u_1} \le x_{u_2}]} \mathbf{1}_{[B(z,r) \subset C_2(v)]} \mathbf{1}_{[B(z,r) \cap C^{\varepsilon_\rho}(v) \ne \varnothing]} \\ &\qquad\qquad \times r h(u_1 u_2) r^3 2\,\mathcal{A}(\Delta(u_{1:3})) \sigma^3(\mathrm{d}u_{1:3})\,\mathrm{d}r\,\mathrm{d}z \end{aligned}$$

since $B(z,r) \subset C_2(v)$ implies that $z \in C_2(v)$ and $r \le 1$.

Hence, $\mathbb{E}[L_{\varepsilon_\rho,\alpha_{\rho,\kappa},\kappa}(v) \mathbf{1}_{[\mathcal{I}_{\varepsilon_\rho}(v)]}] \le a(n,\rho) b(\rho,\kappa)$, where

$$\begin{aligned} a(n,\rho) &:= \frac{1}{2} n^3 \int_{C_2(v)} \int_0^1 \mathrm{e}^{-n\pi r^2} r^4 \mathbf{1}_{[B(z,r) \subset C_2(v)]} \mathbf{1}_{[B(z,r) \cap C^{\varepsilon_\rho}(v) \ne \varnothing]} \,\mathrm{d}r\,\mathrm{d}z, \\ b(\rho,\kappa) &:= \int_{\mathbb{S}^3} \mathbf{1}_{[|\widehat{u_1 u_2}| < \alpha_{\rho,\kappa};\, x_{u_1} \le x_{u_2}]} 2\,\mathcal{A}(\Delta(u_{1:3})) h(u_1 u_2) \sigma^3(\mathrm{d}u_{1:3}). \end{aligned}$$

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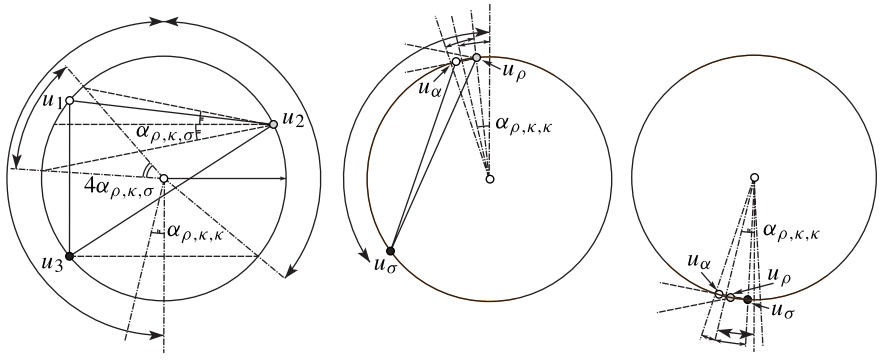


FIGURE 6: Domains of integration for $u(\beta_{1:3})$.

First, we provide an upper bound for $a(n,\rho)$. Since $B(z,r)\cap C^{\varepsilon_\rho}(v)\neq\varnothing$, we have $z\in v\oplus[-\frac{3}{4}-\frac{1}{2}\varepsilon_\rho,\frac{3}{4}+\frac{1}{2}\varepsilon_\rho]^2$. Dividing the square $v\oplus[-\frac{3}{4}-\frac{1}{2}\varepsilon_\rho,\frac{3}{4}+\frac{1}{2}\varepsilon_\rho]^2$ into four quadrants of equal size, it follows that

$$\begin{aligned} a(n,\rho) &\le \frac{1}{2}4n^3\int_0^{3/4+\varepsilon_\rho/2}\int_0^{3/4+\varepsilon_\rho/2}\int_0^{\infty} \mathrm{e}^{-n\pi r^2} r^4\,\mathrm{d}r\,\mathrm{d}y_z\,\mathrm{d}x_z \\ &= 2n^3\left(\frac{3}{4}+\frac{\varepsilon_\rho}{2}\right)^2\frac{3}{8\pi^2 n^{5/2}}. \end{aligned} \tag{4.6}$$

Second, to provide an upper bound for $b(\rho,\kappa)$, we write

$$u_{1:3} := u(\beta_{1:3}) := (u(\beta_1), u(\beta_2), u(\beta_3))$$

with $u(\beta_i)=(\cos\beta_i,\sin\beta_i)$. Up to the line symmetry with respect to the y-axis, we impose the condition that $\beta_3\in[\pi/2,3\pi/2]$. Up to the line symmetry with respect to the x-axis, we also impose the condition that $y_{u(\beta_2)}\ge y_{u_{\beta_3}}$, i.e. $\beta_2\in[\pi-\beta_3,\beta_3]$. Also, $u_1\in C(0,1)\cap\mathcal{C}(\beta_2)$, where $C(0,1)$ is the unit circle and where $\mathcal{C}(\beta_2)$ is the half-cone generated on the left of $u(\beta_2)$ by the x-axis, with vertex $u(\beta_2)$ and with angle $2\alpha_{\rho,\kappa}$, i.e.

$$\mathcal{C}(\beta_2) := u(\beta_2)+\{(r\cos\gamma, r\sin\gamma) : r\ge 0, \gamma\in[\pi-\alpha_{\rho,\kappa},\pi+\alpha_{\rho,\kappa}]\}.$$

We discuss four cases by splitting the domain of integration $[\pi-\beta_3,\beta_3]$ of β_2 as follows (see Figure 6).

1. If $\beta_2\in[\pi-\beta_3,\pi/2]$, we have $\beta_1\in[\pi-\beta_2-2\alpha_{\rho,\kappa},\pi-\beta_2+2\alpha_{\rho,\kappa}]$ (see Figure 6(a)).
2. If $\beta_2\in[\pi/2,\pi/2+\alpha_{\rho,\kappa}]$, the length of $C(0,1)\cap\mathcal{C}(\beta_2)$ is maximal when $\beta_2=\pi/2$, so that $\beta_1\in[\pi/2,\pi/2+2\alpha_{\rho,\kappa}]$. The area of the triangle $\Delta(u(\beta_{1:3}))$ is less than $2\alpha_{\rho,\kappa}$ and $h(u(\beta_1)u(\beta_2))$ is also less than $2\alpha_{\rho,\kappa}$ (see Figure 6(b)).
3. If $\beta_2\in[\pi/2+\alpha_{\rho,\kappa},3\pi/2-\alpha_{\rho,\kappa}]$, we have $C(0,1)\cap\mathcal{C}(\beta_2)=\varnothing$.
4. If $\beta_2\in[3\pi/2-\alpha_{\rho,\kappa},\beta_3]$, with $\beta_3>3\pi/2-\alpha_{\rho,\kappa}$, the length of $C(0,1)\cap\mathcal{C}(\beta_2)$ is maximal when $\beta_2=3\pi/2$, so that $\beta_1\in[3\pi/2-2\alpha_{\rho,\kappa},\pi/2]$. The area of the triangle $\Delta(u(\beta_{1:3}))$ is less than $2\alpha_{\rho,\kappa}$ and $h(u(\beta_1)u(\beta_2))$ is also less than $2\alpha_{\rho,\kappa}$ (see Figure 6(c)).

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With the integrals associated with cases 2 and 4 being bounded by $16\pi\alpha_{\rho,\kappa}^4$, it follows that

$$b(\rho,\kappa) \le 4\int_{\pi/2}^{3\pi/2}\int_{\pi-\beta_3}^{\pi/2}\int_{\pi-\beta_2-2\alpha_{\rho,\kappa}}^{\pi-\beta_2+2\alpha_{\rho,\kappa}} 2\,\mathcal{A}(\Delta(u(\beta_{1:3})))h(u(\beta_1)u(\beta_2))\,\mathrm{d}\beta_1\,\mathrm{d}\beta_2\,\mathrm{d}\beta_3 + 16\pi\alpha_{\rho,\kappa}^4.$$

Moreover, we know that

$$2\,\mathcal{A}(\Delta(u(\beta_{1:3}))) = \begin{pmatrix} 1 & 1 & 1 \\ \cos\beta_1 & \cos\beta_2 & \cos\beta_3 \\ \sin\beta_1 & \sin\beta_2 & \sin\beta_3 \end{pmatrix}, \qquad h(u(\beta_1)u(\beta_2)) = \cos\beta_1 - \cos\beta_2.$$

According to (A.4), this yields

$$b(\rho,\kappa) \le \tfrac{256}{9}\cos^3\alpha_{\rho,\kappa}\sin\alpha_{\rho,\kappa} + \tfrac{128}{3}\cos\alpha_{\rho,\kappa}\sin\alpha_{\rho,\kappa} + \tfrac{128}{3}\alpha_{\rho,\kappa} + 16\pi\alpha_{\rho,\kappa}^4 \le \tfrac{1024}{9}\alpha_{\rho,\kappa}.$$

This together with (4.5), (4.6), and the fact that $\mathbb{E}[L_{\varepsilon_\rho,\alpha_{\rho,\kappa},\kappa}(v)\mathbf{1}_{[\mathcal{I}_{\varepsilon_\rho}(v)]}] \le a(n,\rho)b(\rho,\kappa)$ with $\varepsilon_\rho = \sqrt{\rho}\sqrt{\rho+2}$ and $\alpha_{\rho,\kappa} = \sqrt{2(\kappa/(\kappa-1))\rho}$, implies that

$$\begin{aligned}\mathbb{P}[\mathcal{H}'_{\varepsilon_\rho,\alpha_{\rho,\kappa},\kappa}(v) \wedge \mathcal{I}_{\varepsilon_\rho}(v)] &\le \kappa 2n^3\left(\frac{3}{4} + \frac{\sqrt{\rho}\sqrt{\rho+2}}{2}\right)^2 \frac{3}{8\pi^2 n^{5/2}}\frac{1024}{9}\sqrt{2\frac{\kappa}{\kappa-1}\rho} \\ &= 2\frac{3}{8\pi^2}\frac{1024}{9}\sqrt{2}\left(\frac{3}{4} + \frac{\sqrt{\rho}\sqrt{\rho+2}}{2}\right)^2\sqrt{\frac{\kappa^3}{\kappa-1}}\sqrt{\rho n} \\ &\le 31.76\left(\frac{3}{4} + \frac{\sqrt{\rho}\sqrt{\rho+2}}{2}\right)^2\sqrt{\rho n}.\end{aligned}$$

In the last line, we have taken $\kappa = \frac{3}{2}$ since it is the value of κ which minimizes $\kappa^3/(\kappa-1)$. The condition $(\kappa/(\kappa-1))\rho < \pi^2/8$ is satisfied since $\rho < \pi^2/24 \simeq 0.4$ by assumption. □

We can now conclude the proof of Proposition 4.3. Indeed, such a result is a consequence of Lemmas 4.3–4.5 and the fact that

$$\mathbb{P}[\mathcal{H}_\rho(v) \vee \neg\mathcal{I}_{\varepsilon_\rho}(v)] \le \mathbb{P}[\neg\mathcal{I}_{\varepsilon_\rho}(v)] + \mathbb{P}[\mathcal{H}'_{\varepsilon_\rho,\alpha_{\rho,\kappa},\kappa}(v) \wedge \mathcal{I}_{\varepsilon_\rho}(v)].$$

The assumption $\rho < 4\times 10^{-6}$ ensures that $\varepsilon_\rho < \frac{1}{700}$ in Lemma 4.4 and $\rho < 0.4$ in Lemma 4.5, completing the proof. □

5. Concluding remarks

In this paper we have provided bounds for the expectation of the length of the shortest path. Experimental values for the expectation $\mathbb{E}[\ell(P)]$ and for the standard deviation $\sigma(\ell(P))$ of the lengths of the path $P = U_{s,t}(X)$, $S_{s,t}(X)$ are presented in Table 1. These simulations are written with CGAL (computational geometry algorithms library; see http://www.cgal.org) and are available in [7]. Our estimates are based on 100 trials of Poisson point processes with intensity $n = 10^7$. In particular, Proposition 3.1 is confirmed by our experiments. However, we note that the correct value for $\mathbb{E}[\ell(S_{s,t}(X))]$ is about 1.04.

Our lower bound for δ, appearing in Theorem 1.1, for the expected stretch factor in a Poisson–Delaunay triangulation is far from being tight since the experimental value is much

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TABLE 1: Experimental values for $U_{s,t}(X)$ and $S_{s,t}(X)$.

Path P	$\mathbb{E}[\ell(P)]$	$\sigma(\ell(P))$
$U_{s,t}(X)$	1.1826	0.0053
$S_{s,t}(X)$	1.0401	0.0004

larger. Although several constants in our proof can be improved slightly, this scheme for the proof can only yield lower bounds which are far from optimal. Indeed, the first point where our evaluation is quite crude is the approximation of the strong horizontality property by the weak horizontality property. Another point where we widely underevaluate δ comes from the fact that we use approximation by animals. Actually, a bad situation (when the shortest path is very short) corresponds to an animal with many pixels with a strong horizontality property. However, the converse is clearly not true. Thus, we believe that the proof of a tight constant necessitates other techniques.

Appendix A. Integrals

In this section we provide values for several integrals which are often used in the paper. These integrals can be computed by using tedious classical computations. These computations were carried out using MAPLE® (a MAPLE sheet is available in [7]). Thus,

$$\int_0^\infty \mathrm{e}^{-n\pi r^2} r^k \,\mathrm{d}r = \frac{\Gamma((k+1)/2)}{2(n\pi)^{(k+1)/2}}, \qquad k > -1, \tag{A.1}$$

$$\begin{aligned}\int_{[0,2\pi)^3} \left|\det \begin{pmatrix} 1 & 1 & 1 \\ \cos\beta_1 & \cos\beta_2 & \cos\beta_3 \\ \sin\beta_1 & \sin\beta_2 & \sin\beta_3 \end{pmatrix}\right| \mathrm{d}\beta_{1:3} \\ = 3! \int_0^{2\pi}\int_0^{\beta_3}\int_0^{\beta_2} \det \begin{pmatrix} 1 & 1 & 1 \\ \cos\beta_1 & \cos\beta_2 & \cos\beta_3 \\ \sin\beta_1 & \sin\beta_2 & \sin\beta_3 \end{pmatrix} \mathrm{d}\beta_1\,\mathrm{d}\beta_2\,\mathrm{d}\beta_3 \\ = 24\pi^2,\end{aligned} \tag{A.2}$$

$$\begin{aligned}\int_{-1}^1 \int_{\pi+\arcsin h}^{2\pi-\arcsin h} \int_{-\arcsin h}^{\pi+\arcsin h} \int_{-\arcsin h}^{\beta_2} \det \begin{pmatrix} 1 & 1 & 1 \\ \cos\beta_1 & \cos\beta_2 & \cos\beta_3 \\ \sin\beta_1 & \sin\beta_2 & \sin\beta_3 \end{pmatrix} \\ \times 2\sin\frac{\beta_2-\beta_1}{2}\,\mathrm{d}\beta_1\,\mathrm{d}\beta_2\,\mathrm{d}\beta_3\,\mathrm{d}h \\ = \tfrac{35}{3}\pi,\end{aligned} \tag{A.3}$$

$$\begin{aligned}\int_{\pi/2}^{3\pi/2} \int_{\pi-\beta_3}^{\pi/2} \int_{\pi-\beta_2-2\alpha_{\rho,\kappa}}^{\pi-\beta_2+2\alpha_{\rho,\kappa}} \det \begin{pmatrix} 1 & 1 & 1 \\ \cos\beta_1 & \cos\beta_2 & \cos\beta_3 \\ \sin\beta_1 & \sin\beta_2 & \sin\beta_3 \end{pmatrix} (\cos\beta_1 - \cos\beta_2)\,\mathrm{d}\beta_1\,\mathrm{d}\beta_2\,\mathrm{d}\beta_3 \\ = \tfrac{64}{9}\cos^3\alpha_{\rho,\kappa}\sin\alpha_{\rho,\kappa} + \tfrac{32}{3}\cos\alpha_{\rho,\kappa}\sin\alpha_{\rho,\kappa} + \tfrac{32}{3}\alpha_{\rho,\kappa}.\end{aligned} \tag{A.4}$$

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