ON A CERTAIN MULTIVARIATE NON-NORMAL DISTRIBUTION

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Communicated by D. V. Lindley

Received 24 July 1959

ABSTRACT. Certain regression and estimation properties of a multivariate distribution closely related to the multinormal one have been presented.

1. The following distribution appears of interest as an alternative to the normal one in tests of significance in multivariate analysis. Consider the density function

$$f_{n,p}(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\Lambda}) = K\left\{1 + \frac{1}{n} Q(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\Lambda})\right\}^{-\frac{1}{2}(n+p)}, \tag{1}$$

where $$Q(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\Lambda}) = (\mathbf{x}-\boldsymbol{\theta})' \boldsymbol{\Lambda}(\mathbf{x}-\boldsymbol{\theta}) = \sum_{i,j} \lambda_{ij}(x_i-\theta_i)(x_j-\theta_j)$$

is a positive definite quadratic form in the random variables $\mathbf{x}' = (x_1, x_2, \ldots, x_p)$ with symmetric matrix $\boldsymbol{\Lambda} = \|\lambda_{ij}\|$ of rank equal to p, and $\boldsymbol{\theta}' = E(\mathbf{x}') = (\theta_1, \theta_2, \ldots, \theta_p)$. Since the integral of (1) over the whole space is equal to unity, the constant K is determined as

$$K = K(n,p) = \frac{|\boldsymbol{\Lambda}|^{\frac{1}{2}}}{(\sqrt{n\pi})^p} \frac{\Gamma\{\frac{1}{2}(n+p)\}}{\Gamma(\frac{1}{2}n)}. \tag{2}$$

In the special case $p = 1$, (1) coincides with Student's distribution (Fisher (2)) with n degrees of freedom. If $p = 2$, (1) is a slight modification of the bivariate surface of Karl Pearson (4). Finally, it will be noted that as n becomes infinite, $f_{n,p}$ tends uniformly to the multivariate normal density $N(\boldsymbol{\theta}, \boldsymbol{\Sigma})$ if we set $\boldsymbol{\Sigma} = \boldsymbol{\Lambda}^{-1}$. Dunnett and Sobel (1) have been concerned with a form of (1) in connexion with the distribution of a sequence of Student t-ratios based on a common mean square.

2. If we now assume $\boldsymbol{\theta} = \mathbf{0}$, we may write

$$f_{n,p}(\mathbf{x}, \mathbf{0}, \boldsymbol{\Lambda}) = K\left\{1 + \frac{1}{n} Q_0^* + \frac{1}{n} \lambda_{pp}(x_p - X_p^0)^2\right\}^{-\frac{1}{2}(n+p)} \tag{3}$$

in terms of $$\mathbf{x}^0 = (x_1^0, \ldots, x_{p-1}^0), \quad X_p^0 = -\frac{1}{\lambda_{pp}} \sum_{i=0}^{p-1} \lambda_{ip} x_i^0$$

and the matrix $\boldsymbol{\Lambda}^* = \|\lambda_{ij}^*\|$ if we set $\lambda_{ij}^* = \lambda_{ij} - (\lambda_{ip}\lambda_{jp})/\lambda_{pp}$ and $Q_0^* = \mathbf{x}^{0\prime} \boldsymbol{\Lambda}^* \mathbf{x}^0$, for $i, j = 1, \ldots, p-1$. It may then be verified that the regression of x_p on the fixed variates $x_1^0, \ldots, x_{p-1}^0$ $E(x_p|\mathbf{x}^0) = -X_p^0$. From the conditional distribution it follows that the conditional variance

$$\operatorname{var}(x_p|\mathbf{x}^0) = \frac{1}{\lambda_{pp}} \frac{n}{n+p-3} \left\{1 + \frac{1}{n} \sum_{i,j=1}^{p-1} \lambda_{ij}^* x_i^0 x_j^0\right\} \tag{4}$$

in comparison with the value $= 1/\lambda_{pp}$ for the corresponding multivariate normal distribution.

https://doi.org/10.1017/S030500410003543X Published online by Cambridge University Press

Since the quadratic form Q in (1) is positive definite of rank equal to p, we may introduce a linear transformation: $\mathbf{x} = \mathbf{A}\mathbf{y}$, $\mathbf{A} = \|a_{ij}\|$ such that $\mathbf{x}'\mathbf{\Lambda}\mathbf{x} = \mathbf{y}'\mathbf{y}$, or $\mathbf{A}'\mathbf{\Lambda}\mathbf{A} = \mathbf{I}$ is the unit $(p \times p)$ matrix. Thus, instead of (1) we may consider the density function

$$g_{n,p}(\mathbf{y}) = C(n,p)\{1 + (\mathbf{y}'\mathbf{y})/n\}^{-\frac{1}{2}(n+p)}, \tag{5}$$

where $C(n,p) = K(n,p)/|\mathbf{\Lambda}|^{\frac{1}{2}}$. The characteristic function of (5) is defined as

$$\begin{aligned} M(t_1, \ldots, t_p) &= E[\exp\{i(t_1 y_1 + \ldots + t_p y_p)\}] \\ &= \frac{1}{\sqrt{n\pi}} \frac{\Gamma\{\frac{1}{2}(n+1)\}}{\Gamma(\frac{1}{2}n)} \int_{-\infty}^{\infty} \frac{e^{i\tau y}\, dy}{(1 + y^2/n)^{\frac{1}{2}(n+1)}}, \end{aligned} \tag{6}$$

where: $\tau^2 = \sum_{i=1}^{p} t_i^2$ and (6) will be recognized as $M(\tau)$, the characteristic function of the corresponding univariate Student t-distribution with n degrees of freedom (Fisher and Healy (3)). These authors give an explicit polynomial representation for $M(\tau)$ if the number of degrees of freedom is odd.

The (even) moments of (5) are

$$\begin{aligned} \mu_{2r_1, \ldots, 2r_p} &= \int_{-\infty}^{\infty} y_1^{2r_1 - 1} \ldots y_p^{2r_p - 1} g_{n,p}(\mathbf{y})\, d(\mathbf{y}^2) \\ &= \frac{n^{R_p}\, \Gamma(\frac{1}{2}n - R_p)}{(\sqrt{\pi})^p\, \Gamma(\frac{1}{2}n)} \prod_{i=1}^{p} \Gamma(r_i + \tfrac{1}{2}) \end{aligned} \tag{7}$$

provided $2R_p = 2\left(\sum_{i=1}^{p} r_i\right) < n$. Incidentally, since the density function $f_{n\,p}(\mathbf{x}, \mathbf{0}, \mathbf{\Lambda})$ is constant on the contours of a hyperellipsoid: $\chi^2 = \mathbf{x}'\mathbf{\Lambda}\mathbf{x} = \mathbf{y}'\mathbf{y} = $ constant, a transformation of the distribution to generalized spherical coordinates with $(p-1)$ angles $\phi_1, \ldots, \phi_{p-1}$ and radius vector equal to χ shows that the distribution of χ is proportional to

$$\frac{\chi^{p-1}}{(1 + \chi^2)^{\frac{1}{2}(n+p)}}\, d\chi \tag{8}$$

and so $\chi^2/(1+\chi^2)$ has the $B(\frac{1}{2}p, \frac{1}{2}n)$ distribution, a result which may also be obtained from the characteristic function of χ^2.

3. Regarding the matrix $\mathbf{\Lambda} = \|\lambda_{ij}\|$, we first define $\operatorname{cov}(x_i, x_j | n, p) = \sigma_{ij}(n,p)$, and then differentiate with respect to λ_{ij} under the sign of the integral of $f_{n,p}$ over the whole space, so that for $i, j = 1, \ldots, p$

$$\frac{\partial \log K(n,p)}{\partial \lambda_{ij}} = \lambda^{ij} = \frac{n}{n+2}\, \sigma_{ij}(n+2, p). \tag{9}$$

Concerning the estimation of $\boldsymbol{\theta}$, $\mathbf{\Lambda}$ on the basis of s independent sample vectors $\{\mathbf{x}_\alpha\}$, $(\alpha = 1, \ldots, s)$, the likelihood function is

$$\begin{aligned} L &= \log \prod_{\alpha=1}^{s} f_{n,p}(\mathbf{x}_\alpha, \boldsymbol{\theta}, \mathbf{\Lambda}) \\ &= \text{constant} + \tfrac{1}{2}s \log |\mathbf{\Lambda}| - \tfrac{1}{2}(n+p) \sum_{\alpha=1}^{s} \log\left\{1 + \frac{1}{n}(\mathbf{x}_\alpha - \boldsymbol{\theta})'\, \mathbf{\Lambda}(\mathbf{x}_\alpha - \boldsymbol{\theta})\right\} \end{aligned} \tag{10}$$

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so that the maximum likelihood equations are

$$\left.\begin{aligned}\frac{\partial L}{\partial \theta_i} &= \frac{(n+p)}{n}\sum_{\alpha=1}^{s}\left(\sum_{j=1}^{p}\hat{\lambda}_{ij}(x_{j\alpha}-\hat{\theta}_j)\right)\left(1+\frac{1}{n}\sum_{i,j=1}^{p}\hat{\lambda}_{ij}(x_{i\alpha}-\hat{\theta}_i)(x_{j\alpha}-\hat{\theta}_j)\right)^{-1}=0,\\ \frac{\partial L}{\partial \lambda_{ij}} &= s\hat{\lambda}^{ij}-\frac{(n+p)}{n}\sum_{\alpha=1}^{s}(x_{i\alpha}-\hat{\theta}_i)(x_{j\alpha}-\hat{\theta}_j)\left(1+\frac{1}{n}\sum_{i,j=1}^{p}\hat{\lambda}_{ij}(x_{i\alpha}-\hat{\theta}_i)(x_{j\alpha}-\hat{\theta}_j)\right)^{-1}=0\end{aligned}\right\} \tag{11}$$

for $i, j = 1, \ldots, p$. The solutions of equations (11) are quite involved generally, and resort must be had to iterative methods for their solutions.

The asymptotic variances and covariances of the maximum likelihood solutions may be obtained directly, however. If for simplicity we assume $\mathbf{\Lambda}$ known, then $\operatorname{cov}(\hat{\theta}_i, \hat{\theta}_j) = v_{ij}/s$, where

$$v^{ij} = -E\left(\frac{\partial^2 \log f_{n,p}}{\partial\theta_i \partial\theta_j}\right) = \frac{n+p}{n+p+2}\lambda_{ij}, \tag{12}$$

where $\|v^{ij}\| = \mathbf{V}^{-1} = \|v_{ij}\|^{-1}$.

It is intended to publish separately some further results on the sampling distribution of the variances and covariances, the generalized T^2 and variance statistics from the multivariate density (1).

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RECURRENT RADIATION IN GENERAL RELATIVITY

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Received 16 June 1960

The purpose of this note is to describe the relation between the recurrent spaces of Ruse (1) and Walker (2), and plane waves in general relativity. A space is recurrent if its Riemann tensor satisfies the equation*

$$R_{ijkl;m} = \lambda_m R_{ijkl}, \tag{1}$$

where the vector field λ_m is non-zero (except perhaps at isolated points). At any point such a Riemann tensor is covariantly constant along all directions orthogonal to λ_m,

* A different discussion of (1) has been given by R. Penrose (3).

https://doi.org/10.1017/S030500410003543X Published online by Cambridge University Press