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INCOHERENCE AND INCONSISTENCY

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Abstract. This paper scrutinizes the relationship between inconsistency and incoherence with a special focus on probabilistic measures of coherence. As is shown, while the majority of extant coherence measures face problems regarding the assessment of inconsistent sets of propositions, it is possible to adapt the measures in order to improve their performance. Furthermore, different intuitions regarding the degree of incoherence of inconsistent sets of propositions are surveyed and assessed with respect to extant measures. In this context, a refined approach to measuring coherence is introduced. As is argued, by means of this approach one can account for the diverging coherence intuitions regarding inconsistent sets independently of the discussion on the adequacy of different probabilistic explications of coherence. The last part of the paper is devoted to the question of whether there is a covariation between degrees of inconsistency and degrees of incoherence in the sense that the higher the degree of inconsistency of a set of propositions, the higher its degree of incoherence. Focusing on two straightforward measures of the degree of inconsistency, this latter question is answered in the negative.

§1. Introduction. A set of propositions is said to be coherent if the propositions hang together, agree or dovetail with each other. In this paper the relationship between consistency on the one hand and coherence on the other is examined. I take it that what is *at least* beyond doubt is that consistency is not sufficient for coherence. On the other hand, consistency might be a necessary condition for coherence in the sense that all inconsistent sets are also incoherent.[1] Nonetheless, even if we grant the necessity of consistency for coherence, it remains an open question whether all inconsistent sets are maximally incoherent or whether we should discriminate inconsistent sets of propositions with respect to their degree of incoherence.

In what follows I scrutinize various questions regarding the relationship between inconsistency and incoherence with a special focus on probabilistic measures of coherence (Shogenji, 1999; Glass, 2002; Olsson, 2002; Fitelson, 2003; Douven & Meijs, 2007). The paper is structured as follows. Section 2.1 is a brief summary of some of the extant probabilistic measures of coherence. In Section 2.2 these measures are assessed with respect to their ability to handle sets of *inconsistent* propositions regarding their degree of *incoherence*. As it turns out, the majority of coherence measures are unable to appropriately

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[1] A position along these lines seems to be endorsed by BonJour (1985, p. 95). However, even BonJour expresses some hesitance in this respect, noting that "some proponents of relevance logics may want to argue that in some cases a system of beliefs which was sufficiently rich and complex but which contained some trivial inconsistency might be preferable to a much less rich system which was totally consistent" (1985, endnote 7, p. 240). Here, we dispense with a detailed investigation of this issue.

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cope with inconsistent sets of propositions. Hence, Section 2.3 outlines a proposal for a quick fix based on slight modifications of the extant measures.

Section 3 is devoted to a comparison of two diverging positions concerning the degree of incoherence of inconsistent sets. I survey the arguments that speak in favor of a position that takes *all* inconsistent sets to be maximally incoherent *tout court* (3.1) and the more alleviated position that two inconsistent sets need not be assigned the same degree of incoherence (let alone the maximum degree of incoherence) (3.2). Summing up the pros and cons of each position, it seems that we are forced to make a decision between these two apparently incompatible but equally appealing options. However, it is argued in Section 3.3 that the different intuitions regarding the degree of incoherence of inconsistent sets of propositions are connected to different contexts of assessing coherence. In this regard, the paper surveys a recent approach to measuring coherence that seems to validate both intuitions.

Finally, Section 4 relates the discussion on degrees of incoherence to the one on *degrees of inconsistency* (see, e.g., Hunter & Konieczny, 2005). Focusing on two straightforward measures of inconsistency, it is shown that a higher degree of inconsistency does *not* inevitably entail a higher degree of incoherence.

The following formal preliminaries will be useful throughout the paper: Let $\mathcal{L}$ be a classical propositional language with a finite set of atomic formulae $\{p_1, \ldots, p_n\}$ closed under logical connectives $\{\neg, \wedge, \vee\}$. Arbitrary propositions of $\mathcal{L}$ will be denoted by Greek letter φ, ψ, χ and $\varphi_1, \ldots, \varphi_n$. Sets of formulae of $\mathcal{L}$ will be denoted by Δ, Γ, Σ and $\Delta_1, \ldots, \Delta_n$. Furthermore, "$\bot$" denotes an arbitrary contradiction made up of propositions in $\mathcal{L}$. For each set $\Delta \subseteq \mathcal{L}$ let $\wp(\Delta)$ denote the *power set* of Δ. Accordingly, the set of all sets of propositions will be denoted by $\wp(\mathcal{L})$. Given a classical consequence relation $\vdash \subseteq \wp(\mathcal{L}) \times \mathcal{L}$ over $\mathcal{L}$, let $\mathrm{Cn}(\Delta) = \{\varphi \in \mathcal{L} \mid \Delta \vdash \varphi\}$ denote the set of logical consequences of Δ. A *probability distribution* Pr over $\mathcal{L}$ is a nonnegative function such that $\Pr(\varphi) = 1$ if φ is a tautology and $\Pr(\varphi_1 \vee \varphi_n) = \Pr(\varphi_1) + \Pr(\varphi_2)$ if φ_1 and φ_2 are inconsistent. Conditional probabilities are defined as fractions of unconditional probabilities as follows: $\Pr(\varphi_1|\varphi_2) = \Pr(\varphi_1 \wedge \varphi_2)/\Pr(\varphi_2)$.

§2. Measuring (in)coherence probabilistically.

2.1. Some extant probabilistic measures of coherence. Probabilistic measures of coherence quantify the coherence of a set of propositions on the basis of probabilistic information pertaining to the Boolean combinations of propositions in the given set.[2] According to Shogenji (1999), the coherence of a set of propositions is to be quantified in terms of the deviation from probabilistic independence: for a set $\Delta = \{\varphi_1, \ldots, \varphi_n\}$ of propositions, Shogenji's measure reads as follows:

$$\mathcal{D}(\Delta) = \frac{\Pr(\varphi_1 \wedge \ldots \wedge \varphi_n)}{\Pr(\varphi_1) \cdot \ldots \cdot \Pr(\varphi_n)}$$

Another proposal that has been put forward independently by Glass (2002) and Olsson (2002) focuses on the set-theoretic overlap between the propositions under consideration:

$$\mathcal{O}(\Delta) = \frac{\Pr(\varphi_1 \wedge \ldots \wedge \varphi_n)}{\Pr(\varphi_1 \vee \ldots \vee \varphi_n)}$$

[2] A note on terminology: talking about "degrees of coherence" in no way presupposes a qualitative decision as to whether a set under consideration is in fact coherent or incoherent. My use of this notion is a more technical one: when assigning different degrees of *coherence* to inconsistent sets, what is meant is that these sets might exhibit different degrees of incoherence.

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For both these approaches to measuring coherence refined generalizations have been put forward in the literature. These are based on the insight that in order to properly quantify the degree of coherence of a given set of propositions one has to take into account the various coherence degrees of subsets of the given set. Accordingly, the main idea of these refined measures is to consider a weighted average of all coherence values for all nonempty, nonsingleton subsets. For each set Δ let $[\Delta]^k$ denote the set of all subsets with exactly k elements, then there are $\theta_k = \binom{n}{k}$ sets within each $[\Delta]^k$. Now, one possibility for how to generalize Shogenji's measure $\mathcal{D}$ reads as follows (cf. Schupbach, 2011):[3]

$$\mathcal{RD}(\Delta) = \frac{1}{n-1} \sum_{2 \leq k \leq n} \sum_{\Delta_i \in [\Delta]^k} (\theta_k)^{-1} \log \mathcal{D}(\Delta_i)$$

A slightly different generalization has been proposed by Meijs (2006) for the overlap-based coherence measure $\mathcal{O}$. Let $\theta_{\geq i}$ be the sum of all θ_k for $i \leq k \leq n$, then the refined overlap measure reads as follows:

$$\mathcal{RO}(\Delta) = \frac{1}{\theta_{\geq 2}} \sum_{2 \leq k \leq n} \sum_{\Delta_i \in [\Delta]^k} \mathcal{O}(\Delta_i)$$

Obviously, both refined ways of generalizing the given measures are subset-sensitive in the sense that sets that are assigned the same initial coherence values on $\mathcal{D}$ or $\mathcal{O}$ might differ when assessed by the refined measures $\mathcal{RD}$ or $\mathcal{RO}$.

Finally, there is a family of measures that draws upon the idea that coherent propositions mutually support each other (Fitelson, 2003, 2004; Douven & Meijs, 2007; Roche 2013). These approaches utilize research on probabilistic measures of confirmation in order to quantify the amount of support one proposition conveys upon another. Accordingly, Douven & Meijs (2007) introduce a recipe for probabilistic measures of coherence based on arbitrary confirmation measure. In a nutshell, in order to assess the degree of coherence of a given set Δ of propositions, the recipe averages the mutual support of all pairs of nonempty, nonoverlapping subsets of Δ. Let $[\Delta]$ denote this set of pairs of nonempty, nonoverlapping subsets, then for some arbitrary confirmation measure $\mathfrak{s}$ the coherence measure $\mathcal{C}_\mathfrak{s}$ is defined as follows:

$$\mathcal{C}_\mathfrak{s}(\Delta) = \frac{1}{|[\Delta]|} \sum_{\langle \Gamma, \Gamma' \rangle \in [\Delta]} \mathfrak{s}\left(\bigwedge_{\varphi_i \in \Gamma} \varphi_i, \bigwedge_{\varphi_j \in \Gamma'} \varphi_j\right)$$

To illustrate, let $\Delta = \{\varphi, \psi\}$, then there are two pairs of sets of proposition to consider, viz. $\langle \{\varphi\}, \{\psi\} \rangle$ and $\langle \{\psi\}, \{\varphi\} \rangle$. Consequently, $\mathcal{C}_\mathfrak{s}(\Delta)$ is the average of the two degrees of confirmation $\mathfrak{s}(\varphi, \psi)$ and $\mathfrak{s}(\psi, \varphi)$.[4]

A probabilistic model of confirmation is based on one of the following qualitative explications of the notion of confirmation.[5] In terms of *incremental* confirmation, evidence ψ is supposed to confirm a hypothesis φ iff $\Pr(\varphi|\psi) > \Pr(\varphi)$, to disconfirm φ iff $\Pr(\varphi|\psi) < \Pr(\varphi)$ and to be irrelevant for φ iff $\Pr(\varphi|\psi) = \Pr(\varphi)$. On the other hand, scholars speak of *absolute* confirmation in case $\Pr(\varphi|\psi) > \Pr(\neg\varphi|\psi)$. Given the explication of

[3] As usual, we stipulate that $\log 0 = -\infty$.

[4] More generally, for each set Δ with n elements, $|[\Delta]| = \sum_{i=1}^{n-1} \binom{n}{i}(2^{n-i} - 1)$ (cf. Douven & Meijs, 2007).

[5] Usually, confirmation is taken to be a three-place relation between a hypothesis φ, a piece of evidence ψ and a specified amount of background knowledge χ. In the context of measuring coherence, however, reference to background knowledge is usually suppressed.

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Table 1. *Prominent Bayesian confirmation measures*

Confirmation measure	Advocate[6]
$d(\varphi, \psi) = \Pr(\varphi\|\psi) - \Pr(\varphi)$	Carnap (1962)
$r(\varphi, \psi) = \frac{\Pr(\varphi\|\psi)}{\Pr(\varphi)}$	Keynes (1921)
$s(\varphi, \psi) = \Pr(\varphi\|\psi) - \Pr(\varphi\|\neg\psi)$	Christensen (1999)
$k(\varphi, \psi) = \frac{\Pr(\psi\|\varphi)-\Pr(\psi\|\neg\varphi)}{\Pr(\psi\|\varphi)+\Pr(\psi\|\neg\varphi)}$	Kemeny and Oppenheim (1952)
$z(\varphi, \psi) = \begin{cases} \frac{\Pr(\varphi\|\psi)-\Pr(\varphi)}{1-\Pr(\varphi)} & \text{, if } \Pr(\varphi\|\psi) \geq \Pr(\varphi) \\ \frac{\Pr(\varphi\|\psi)-\Pr(\varphi)}{\Pr(\varphi)} & \text{, otherwise} \end{cases}$	Crupi et al. (2007)

incremental confirmation, any full-fledged confirmation model, $\mathfrak{s}(\varphi, \psi)$, ought to assess the *degree* of incremental confirmation between evidence and hypothesis. Obviously, this desideratum can be met in various ways. Table 1 lists a number of extant confirmation measures that have been proposed in this regard.[6] Beyond that, it is possible to generate a coherence measure based on the absolute concept of confirmation:[7]

$$f(\varphi, \psi) = 2 \cdot \Pr(\varphi|\psi) - 1$$

In what follows I will refer to the class of measures based on Douven and Meijs' recipe by the term "mutual support measures". Note that the considered class of measures is only a small subset of all possible measures in line with this approach. Further mutual support measures of coherence are discussed by Olsson & Schubert (2007) and Siebel & Wolff (2008). Furthermore, both Fitelson's (2003) and Roche's coherence measures are based on slight variants of two of the above confirmation measures and will be discussed in full generality in Section 2.3.

2.2. Are these measures proper measures of incoherence? The research on probabilistic measures of coherence has largely been concerned with evaluating consistent sets of propositions. However, in everyday situations we are often faced with inconsistent pieces of information that we have to evaluate. For example, imagine a situation in court where several pieces of information are delivered by independent witnesses. It might turn out that while there is a large corpus of testimonies that do fit together perfectly, there is a tiny piece of evidence that does not fit into the general picture formed on the basis of the coherent pieces of evidence (or is even self-contradictory). Still we might want to assess the incoherence of the overall set of testimonies. This suggests that in order to provide a full-fledged account of coherence, probabilistic measures should be able to cope with inconsistent sets of propositions as well.

This section classifies the extant coherence measures introduced in Section 2.1. with respect to their handling of inconsistent sets of propositions. To have a unified framework at our disposal, we distinguish the following three types of finite inconsistent sets.

[6] For further references see Crupi *et al*. (2007).

[7] Here, the variety of possible measures is limited: alternative proposals like $f'(\varphi, \psi) = \Pr(\varphi|\psi) - \Pr(\neg\varphi|\psi)$ or $f''(\varphi, \psi) = [\Pr(\varphi|\psi) - \Pr(\neg\varphi|\psi)] \,/\, [\Pr(\varphi|\psi) + \Pr(\neg\varphi|\psi)]$ are easily shown to be *ordinally equivalent* to f. This is not the case for any pair of confirmation measures listed in Table 1.

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One type of inconsistent sets that coherence measures should be able to deal with feature a nonempty consistent subset $\{\varphi_1, \ldots, \varphi_n\}$ and at least one outright contradiction $\bot$. Hence, Δ is an inconsistent set of *type 1* iff the inconsistency is exclusively due to the presence of a self-contradictory proposition $\bot$. Another type of inconsistent sets, that I will call *type 2*-inconsistent, is made up of a set of individually consistent but jointly inconsistent propositions $\{\varphi_1, \ldots, \varphi_n\}$. Eventually, *type 3*-inconsistent sets feature *both* at least one outright contradiction $\bot$ *and* a set $\{\varphi_1, \ldots, \varphi_n\}$ of individually consistent but jointly inconsistent propositions $\varphi_i \in \mathcal{L}$.[8] To illustrate the difference between these types of inconsistent sets, consider the following two sets $\Gamma_1 = \{\neg p_1 \vee p_2, p_3, p_1 \wedge \neg p_1\}$ and $\Gamma_2 = \{p_1, \neg p_1 \vee p_2, \neg p_2\}$. Due to the presence of $p_1 \wedge \neg p_1$, Γ_1 is type 1-inconsistent. Γ_2 is type 2-inconsistent because all individual members of Γ_2 are consistent and there is no outright contradiction present. Finally, their union $\Gamma_1 \cup \Gamma_2$ is type 3-inconsistent. The following theorems disclose the measures' treatment of inconsistent sets.[9]

THEOREM 2.1. *Both the Shogenji-measure $\mathcal{D}$ and its refined version $\mathcal{RD}$ are not defined for inconsistent sets of type 1 and type* 3. *All inconsistent sets of type 2 are awarded maximal incoherence both by $\mathcal{D}$ and $\mathcal{RD}$.*

Proof. All sets Δ of type 1 or 3 contain at least one contradiction $\bot$ among its member. Since $\Pr(\bot) = 0$, $\mathcal{D}(\Delta)$ and $\mathcal{RD}(\Delta)$ are not defined. For type 2-inconsistent sets, since $\Pr(\bigwedge_{\varphi\in\Delta}\varphi) = 0$ and $\prod_{\varphi\in\Delta}\Pr(\varphi) > 0$ (by *regularity*), we get $\mathcal{D}(\Delta) = 0$ and $\mathcal{RD}(\Delta) = -\infty$. □

THEOREM 2.2. *The overlap measures $\mathcal{O}$ and $\mathcal{RO}$ are defined for all inconsistent sets, given that there is at least one noncontradictory proposition. All these sets are awarded maximal incoherence by $\mathcal{O}$.*

Proof. Let Δ be an inconsistent set that contains at least one noncontradictory proposition, then $\Pr(\bigvee_{\varphi\in\Delta}\varphi) > 0$ and $\Pr(\bigwedge_{\varphi\in\Delta}\varphi) = 0$. Thus, $\mathcal{O}(\Delta) = 0$. If the requirement of at least one noncontradictory proposition is not met, $\Pr(\bigvee_{\varphi\in\Delta}\varphi) = 0$ and so $\mathcal{O}$ and $\mathcal{RO}$ are undefined. □

THEOREM 2.3. *Let $\mathcal{C}_\mathfrak{s}$ be a coherence measure in line with Douven and Meijs' recipe, where $\mathfrak{s}$ is either a confirmation measure of Table 1 or f, then $\mathcal{C}_\mathfrak{s}$ is undefined for all inconsistent sets of propositions except those of type* 2 *such that all subsets $\Delta' \subsetneq \{\varphi_1, \ldots, \varphi_n\}$ are consistent.*

Proof. Let Δ be an inconsistent set of type 2 such that all subsets $\Delta' \subsetneq \{\varphi_1, \ldots, \varphi_n\}$ are consistent, then all prior and posterior probabilities that enter the calculations for $\mathcal{C}_\mathfrak{s}$ are defined. On the other hand, if Δ is of type 1 or type 3, or there is an inconsistent subset $\Delta' \subsetneq \{\varphi_1, \ldots, \varphi_n\}$, some posterior probabilities like $\Pr(\varphi_i|\bot)$ or $\Pr(\bigwedge_{\varphi\in\mathfrak{A}\setminus\Delta'}\varphi|\bigwedge_{\psi\in\Delta'}\psi)$ are undefined. □

There is an important sense in which the measures measure $\mathcal{D}$, $\mathcal{RD}$ and $\mathcal{O}$ in its original shape differ both from the refined generalization $\mathcal{RO}$ and the mutual support measures of

[8] Note that this list of types of inconsistent sets is *not* exhaustive, because inconsistent singletons are missing. However, due to the fact that coherence is usually taken to be a relation, I dispense with a discussion of the coherence of inconsistent singletons.

[9] Here, and in what follows, we assume the probability measures to be *regular* so that only contradictions are awarded probability 0.

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coherence. Whereas the former assign a maximum degree of incoherence to *all* inconsistent sets (given that the respective measure is defined at all), the refined version and all mutual support measures are able to distinguish different degrees of incoherence for inconsistent sets (for a proof see Appendix 6.1.).

THEOREM 2.4. *Let* Φ *be either* $\mathcal{C}_{\mathfrak{s}}$ *for any confirmation measure* $\mathfrak{s} \in \{d, r, s, k, z, f\}$ *or* $\mathcal{RO}$*, then there are inconsistent sets* Δ, Γ *such that* $\Phi(\Delta) \neq \Phi(\Gamma)$.

The results stated in the former theorems are twofold. On the one hand, we proved that for each coherence measure there are inconsistent sets of propositions such that these measures are undefined. On the other hand, we saw that on the assumption of being defined, only based on measures along Douven and Meijs' recipe and the refined overlap measure $\mathcal{RO}$ one can differentiate the degree of coherence of various inconsistent sets. Since core intuitions concerning the degree of coherence of inconsistent sets seem to be heterogeneous, I will postpone discussion of this latter aspect to Section 3. Beforehand, the next subsection will outline two ways to overcome the shortcomings of partially defined measures.

2.3. A quick fix? Adapting measures. As was shown in the previous section, all coherence measures discussed in Section 2.1 manifest problems with respect to the evaluation of inconsistent sets of propositions because they are undefined for some of these sets. To remedy this disadvantage, the current section introduces refined versions of extant coherence measures in order to enable a ubiquitous application to inconsistent sets. Basically, we imitate an approach to measuring coherence that has initially been pursued by Fitelson (2003, 2004). When explicating coherence, Fitelson endorses a variant of the Kemeny-Oppenheim measure of factual support, k, that explicitly takes into account the limiting cases of entailment and inconsistency.

$$k'(\varphi, \psi) = \begin{cases} \frac{\Pr(\psi|\varphi)-\Pr(\psi|\neg\varphi)}{\Pr(\psi|\varphi)+\Pr(\psi|\neg\varphi)} & \text{, if} \quad \psi \nvdash \varphi \text{ and } \psi \nvdash \neg\varphi \\ +1 & \text{, if} \quad \psi \vdash \varphi \text{ and } \psi \nvdash \bot \\ -1 & \text{, if} \quad \psi \vdash \neg\varphi \end{cases}$$

To see why k' is superior to k with respect to the evaluation of inconsistent sets, consider the set $\Delta = \{\varphi, \bot\}$. Since $\bot \vdash \neg\varphi, k'(\varphi, \bot) = -1$ and so $\mathcal{C}_{k'}(\Delta)$ is not undefined anymore. The analogous approach can be carried out for all other measures in the set $\{d, r, s, z, f\}$.[10] Thus, let $\mathfrak{s}$ be a probabilistic measure of confirmation, then $\mathfrak{s}'$ is defined as follows:

$$\mathfrak{s}'(\varphi, \psi) = \begin{cases} \mathfrak{s}(\varphi, \psi) & \text{, if} \quad \psi \nvdash \neg\varphi \\ \min_{\mathfrak{s}} & \text{, otherwise} \end{cases}$$

The basic idea underlying these adapted confirmation measures is that confirmation by inconsistent information is the worst possible case and hence equals the minimum possible degree of support.[11]

[10] Since we focus on applications to inconsistent sets of propositions, we won't consider cases where $\psi \vdash \varphi$ separately. Measure f' has recently been advocated by Roche (2013).

[11] Admittedly, this is not the only reasonable choice for adapting confirmation measures. Another possibility is to assign a neutral degree of confirmation in case of inconsistent evidence (cf. Schippers & Siebel, 2014). All of the following theorems are independent of the chosen modification. If a measure's range is not bounded from below, we assume $min_{\mathfrak{s}}$ to be equal to the greatest lower bound.

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The missing coherence measures $\mathcal{D}$ and $\mathcal{O}$ can be adapted accordingly (see Brössel, 2014; Roche, 2013). Regarding the Shogenji-measure, problematic cases arose from all sets containing at least one contradiction. Consequently, we get the following variant:

$$\mathcal{D}'(\Delta) = \begin{cases} \mathcal{D}(\Delta) & \text{, if } \prod_{\varphi \in \Delta} \Pr(\varphi) > 0 \\ 0 & \text{, otherwise} \end{cases}$$

Finally, the overlap measure $\mathcal{O}$ was unable to cope with inconsistent sets that contain no consistent proposition at all. Therefore, the following alternative definition is given:

$$\mathcal{O}'(\Delta) = \begin{cases} \mathcal{O}(\Delta) & \text{, if } \Pr(\bigvee_{\varphi \in \Delta} \varphi) > 0 \\ 0 & \text{, otherwise} \end{cases}$$

The adapted versions of the refined deviation- and overlap-measures $\mathcal{RD}'$ and $\mathcal{RO}'$ are simply based on the adapted initial deviation- and overlap-measures. Given these variants, the following theorem is an immediate consequence of Theorems 2.1–2.4.

THEOREM 2.5. *All inconsistent sets* Δ *are awarded maximal incoherence according to* $\mathcal{D}'$, $\mathcal{RD}'$ *and* $\mathcal{O}'$. *On the other hand, let* Φ *be either* $\mathcal{RO}'$ *or* $\mathcal{C}_{\mathfrak{s}}$ *for any confirmation measure* $\mathfrak{s}' \in \{d', r', s', k', z', f'\}$, *then there are inconsistent sets* Δ, Γ *such that* $\Phi(\Delta) \neq \Phi(\Gamma)$.

Now that we have an approach for how to adapt coherence measures at hand, we can pass on to the second aspect mentioned at the end of Section 2.2. Accordingly, the next section tries to elucidate diverging intuitions regarding the degree of incoherence of inconsistent sets. It is argued that if we pay closer attention to the context in which coherence is assessed, we can relate those intuitions to different contexts of applying coherence measures.

§3. Inconsistency and incoherence: Diverging intuitions. This section systematically deals with issues concerning the relation between incoherence and inconsistency. In Section 3.1. I discuss the pros and cons of a position that takes inconsistent propositions to be maximally incoherent. Afterwards the more alleviated position, suggesting to keep apart different *degrees of incoherence* for inconsistent sets, is in focus. Finally, I outline a refined approach to measuring coherence that offers a framework in which both intuitions on inconsistency and incoherence can be accounted for.

3.1. Inconsistency entails maximal incoherence. A main feature of the classical consequence relation '$\vdash$' is that it is *explosive*: it validates $\{\varphi, \neg\varphi\} \vdash \psi$ for every pair of propositions $\langle \varphi, \psi \rangle$. In combination with *monotony* it follows that $\Delta \vdash \psi$ for all propositions ψ and all sets Δ such that $\{\varphi, \neg\varphi\} \subseteq \Delta$. But what is the impact of inconsistencies on coherence? Provided that inferential relations are among the key coherence boosting factors (BonJour, 1985, p. 93), these relations often abound in inconsistent sets. Imagine a set $\Delta = \{\varphi_1, \varphi_2, \varphi_3, \bot\}$ of four propositions, where three of them are consistent and the fourth is a contradiction. In every such set there are already 19 entailment relations due to the presence of only one inconsistency.[12] But what does it mean to say that $\varphi_1 \wedge \bot$ entails

[12] $\bot$ not only entails all propositions φ_1, φ_2 and φ_3 but also all conjunctions of these propositions. Furthermore, conjoining $\bot$ with one or more proposition φ_i again entails the remaining propositions or conjunctions thereof. Accounting for all possible combinations yields the mentioned number of 19 entailment relations.

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φ_2, given that it also entails $\neg\varphi_2$. How might the quality of being related like *this* be held to increase the coherence of the given set? It seems that in the presence of inconsistencies, *explosion* causes a loss of sensitivity for "real" coherence boosting inferential relations as compared to mere "empty" entailment relations. Accordingly, since there seems to be no way to sensitively compare different inconsistent sets with respect to their degree of coherence, one might argue, inconsistent sets should be awarded maximal incoherence without exception.[13]

This "maximal incoherence view" can be further corroborated by utilizing an argument that has been put forward by Moretti & Akiba (2007). Moretti and Akiba focus on the coherence of equivalent belief sets, where equivalence is defined as follows:

> We consider two belief sets, Δ and Δ', *equivalent* iff the believer who holds all beliefs $\varphi \in \Delta$ can obtain every belief $\psi \in \Delta'$, and *vice versa*, by simply deducing the propositional content of ψ from the propositional contents of $\varphi's$, and, conversely, by deducing the propositional content of φ from the propositional contents of $\psi's$. (2007, p. 76 f., emphasis Moretti and Akiba's)

Moretti and Akiba then argue that equivalent belief sets ought to be assigned equal degrees of coherence. On this presumption, *all* inconsistent sets of propositions must be equally incoherent and since it seems compelling to award the set $\{\varphi, \neg\varphi\}$ a maximum degree of incoherence, all other inconsistent sets should be maximally incoherent as well. This line of argument accords with our variants of Shogenji's, Schupbach's and Glass and Olsson's measure.

OBSERVATION 3.1. *Let* Δ *be inconsistent, then* $\mathcal{D}'(\Delta) = 0$, $\mathcal{O}'(\Delta) = 0$ *and* $\mathcal{RD}'(\Delta) = -\infty$.

However, the measures $\mathcal{D}$ and $\mathcal{O}$ underlying these variants have been proven to be highly problematic as candidate measures of coherence (see, e.g., Akiba, 2000; Bovens & Hartmann, 2003a; Fitelson, 2003, 2004; Schupbach, 2011). On the other hand, the refined version of $\mathcal{O}$, viz. Meijs' $\mathcal{RO}$, *fails* to satisfy the maximal incoherence view in case of inconsistency.

Note that the result for Schupbach's measure is exclusively due to the application of the logarithm and there seems to be no principled argument why there should not be some otherwise preferable way of generalizing Shogenji's measure in a subset-sensitive manner that do *not* entail the maximum incoherence view. One such possibility is by means of the following rescaled version of the Shogenji measure:

$$\widetilde{\mathcal{D}}(\Delta) = 1 - 2^{-\mathcal{D}(\Delta)}$$

If we plug this version of the Shogenji measure into Schupbach's recipe, we receive a subset-sensitive version of Shogenji's measure of coherence that assigns different degrees of coherence to inconsistent sets of propositions (proof omitted). Thus, it seems that the idea of Schupbach's measure is compatible with violating the maximum incoherence intuition. Accordingly, the only approaches to measuring coherence that are committed to this latter intuition are the subset-*in*sensitive versions of the deviation- and overlap measures.

[13] A position along these lines is endorsed by Bovens & Hartmann (2003b, p. 608).

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Thus, we find ourselves in a position where only one coherence measure that seems viable, viz. $\mathcal{RD}$, accords with the maximal incoherence intuition.[14] Here conclusions might be twofold: On the one hand, one can stick to the view that inconsistency inevitably entails maximal incoherence. On this presumption, Observation 3.1 might be considered a vindication of (our variant of) Schupbach's measure $\mathcal{RD}$. On the other hand, one can give way to a more alleviated position distinguishing different degrees of incoherence for inconsistent sets. This position will be considered in the next section.

3.2. Inconsistency does* not *entail maximal incoherence. To render plausible the view that we should keep apart different degrees of incoherence for inconsistent sets of propositions, think of a highly coherent set of testimonies where only one minor piece of evidence that has been testified by a source known to be less then fully reliable is inconsistent with the remaining corpus of evidence. It does seem that there is ample reason to grant this set a lower degree of incoherence than, for example, the set $\{\varphi, \neg\varphi\}$. One such reason is the following: while $\{\varphi, \neg\varphi\}$ contains nothing but two contrary statements, the large testimonial set might contain various coherence-boosting explanatory or inductive relations among its elements. Consequently, we might want to distinguish inconsistent sets of propositions with regard to their degree of coherence. A position along these lines is also taken by Siebel (2005).

> I do not deny that inconsistencies have a negative *influence* on coherence. It might even be the case that propositions are coherent only if they are consistent (cf. BonJour 1985, p. 95). But this does not entail that contradictory propositions make a system incoherent *to the maximum*. There may be further factors present which have a positive impact and thus partly compensate for inconsistencies. (2005, p. 339, emphasis Siebel's)

As we have seen, the vast majority of extant proposals to measuring coherence accord with Siebel in keeping apart different degrees of incoherence for inconsistent sets.[15] That is, let Φ be one of these measures, then there are sets Δ, Δ' such that both Δ and Δ' are inconsistent but nonetheless $\Phi(\Delta)$ and $\Phi(\Delta')$ differ.

We seem to have encountered a dilemma: On the one hand, the vast majority of coherence measures violates the maximal incoherence demand. On the other hand, there are a few measures that accord with it. Furthermore, we saw that there is ample reason to consider these latter measures inadequate or at least in need of improvement. Shall we conclude that the fact that all reasonable coherence measures are among those that violate the maximality demand entails the inadequacy of this very demand? But what about the intuition, substantiated by Moretti & Akiba (2007), that equivalent sets of propositions ought to be assigned equal degrees of (in)coherence?

In what follows it is shown that there is an approach to measuring coherence which helps to disconnect the diverging intuitions on the one hand and the quest for proper coherence measures on the other. More precisely, it is argued that these diverging intuitions might stem from different contexts of applying coherence measures. In a nutshell, these are the following: on the one hand, we can try to assess the coherence of an agent's set of beliefs; on the other hand, we can focus on sets of testimonies given by different witnesses.

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[14] However, Jonah Schupbach considers this property with respect to inconsistent sets a "problem" for his measure as scaled in his 2011 paper (personal communication).

[15] And so does Shogenji (2005), who anticipated the need to adapt his coherence measure for sets of inconsistent testimonies. His solution proposal is based on the pair-wise coherence of the elements of a given set of testimonies.

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The idea, then, is to argue that while inconsistent belief sets might be considered maximally incoherent, inconsistent sets of testimonies should not be so. Finally, we argue that these different contexts of measuring coherence can be accounted for within a refined approach to measuring coherence.

3.3. A refined approach to measuring coherence. According to most probabilistic coherence measures, containing contradictory propositions is not sufficient for a set to be maximally incoherent. Nevertheless, we seem to be well advised not to dispense with the "maximal incoherence intuition" without hesitation. Therefore in what follows we investigate how to account for this maximality intuition within an approach that is based on measures that violate this very intuition. In order to do so, we have to focus on what is the rationale for taking inconsistent sets to be maximally incoherent. In Section 3.1 it was outlined that one of the main reasons for taking each inconsistent set Δ to be maximally incoherent is its being equivalent to $\{\varphi, \neg\varphi\}$. Given that *this* set is maximally incoherent and presupposing that equivalent belief sets ought to be assigned equal degrees of (in)coherence, it follows that each inconsistent set ought to be judged maximally incoherent irrespective of any other possibly compensatory feature. This argument entertains the following two premises:

(i) $\{\varphi, \neg\varphi\}$ is maximally incoherent.
(ii) Equivalent sets ought to be awarded equal degrees of (in)coherence.

Condition (i) seems to be beyond reasonable doubt. Although it might be the case that not *all* inconsistent sets are on a par as regards their degree of coherence, each set containing only a pair of contradictory propositions seems to be one of the worst possible cases from a coherentist point of view. However, the following theorem shows that not all coherence measures mentioned before concur with this intuition (for a proof see Appendix 6.2.) .

THEOREM 3.2. *Let* $\Delta = \{p, \neg p\}$ *be a set of contingent contradictory propositions (such that* $0 < \Pr(p) < 1$*), then* Δ *is awarded a maximum degree of incoherence by* $\mathcal{D}'$, $\mathcal{O}'$, $\mathcal{RD}'$, $\mathcal{RO}'$ *and all measures* $\mathcal{C}_{\mathfrak{s}}$ *for* $\mathfrak{s} \in \{d', r', s', k', z', f'\}$.

The following corollary is a direct consequence of the proof of Theorem 3.2.

COROLLARY 3.3. *Let* $\Delta = \{p, \neg p\}$ *and* $\Gamma = \{q, \neg q\}$ *be two sets of contradictory propositions such that* $0 < \Pr(p) < \Pr(q) < 1$*, then* $\Phi(\Delta) = \Phi(\Gamma)$ *for all considered measures* Φ.

Thus, all considered measures agree with premises (i) and (ii) as long as only pairs of contradictory propositions are considered. But what about the general claim that (ii) is to be extended to *all* inconsistent sets, let alone to all equivalent sets? This, again, depends on the context under consideration. On the one hand, we might be interested in the coherence of witness reports given by independent witnesses. On the other hand, we might want to quantify the coherence of a (finite) belief set of a single agent. Therefore, when faced with an inconsistent set of propositions Δ, we need information regarding the status of Δ's elements. Given that all we know is that Δ is the belief set of an arbitrary agent, we might be inclined to conclude that it is maximally incoherent, as is the equivalent set $\Gamma = \{p_1, \neg p_1\}$. This 'bold' view of assessing coherence is in line with measures $\mathcal{D}$ and $\mathcal{O}$ and the idea that both sets are blatantly inconsistent on a qualitative level. However, if we instead assume that all of Δ's elements are testimonies given by independent witnesses, it seems reasonable to keep apart the degrees of coherence of Δ and Γ.

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This latter approach is in line with all other extant proposals. However, in contrast to different classes of coherence measures for different contexts of assessing coherence, it seems desirable to have a single class of measures that can account for intuitions in both contexts. In what follows we show that this is possible by means of a refined approach to measuring coherence (cf. Schippers, 2014). The basic idea underlying this proposal is that we have to keep apart different contexts in which coherence is to be assessed. Depending on whether the one context or the other has priority, it is argued, coherence measures should behave radically different with respect to "trivial" expansions like adding a tautology or adding a proposition that is already entailed by other elements in the set. To illustrate, consider the set $\Delta = \{\varphi_1, \varphi_2, \varphi_3\}$ and a further proposition φ_4 that is logically equivalent to φ_3. Assuming that Δ is Anne's belief set, it seems unreasonable to allow for Δ's coherence to be increased by merely adding the belief φ_4. More precisely, assume that Anne merely draws a conclusion from what she already believes, than inferring an increase in coherence "seems to amount to a 'cut-price offer"' since it allows "one to increase the coherence of one's doxastic system too simply, viz., just by inferring trivial consequences" (Siebel, 2005, 340). On the other hand, judged as a set of reports given by independent testimonies, it is clearly the case that another witness reporting equivalently on some issue raises the coherence of the set of reports. In order to account for both these intuitions without being committed to different coherence measures for each context, the notion of a *basic commitment* is introduced:

DEFINITION 3.4. *Let Δ be a set of propositions, then φ is an element of the set of basic commitments of Δ, $\mathcal{B}(\Delta)$, iff Δ entails φ and there is no fundamental formula ψ such that Δ entails ψ and $\Lambda(\psi) \subseteq \Lambda(\varphi)$.*

Here $\Lambda(\varphi)$ denotes the set of *literals* in φ, where a literal is either an atomic formula or the negation of an atomic formula. For example, $\Lambda(p_1 \wedge \neg p_2) = \{p_1, \neg p_2\}$ and $\Lambda(p_1 \wedge \neg p_1) = \{p_1, \neg p_1\}$. A formula is called *fundamental* if it is a disjunction of literals such that no atomic formula appears twice in it. To briefly illustrate the concept of a basic commitment, let $\Delta = \{p_1 \vee p_2, \neg p_2, p_3 \wedge \neg p_4\}$, then the basic commitments are $\mathcal{B}(\Delta) = \{p_1, \neg p_2, p_3, \neg p_4\}$.

To solve the problem of individuating beliefs, one may require that each belief set be in a canonical form that is unique for all equivalent belief sets. In this context, two belief sets Δ_1 and Δ_2 are equivalent ($\Delta_1 \equiv \Delta_2$) if $\mathrm{Cn}(\Delta_1) = \mathrm{Cn}(\Delta_2)$, i.e. if each element of Δ_1 is derivable from the elements of Δ_2, and vice versa. Now, in order to have a unique representative for such each class of equivalent sets at our disposal, one can resort to the set of all minimal equivalent sets of basic commitments.[16]

DEFINITION 3.5. *Let Δ be a set of propositions, then Γ is a minimal equivalent set of basic commitments for Δ iff (i) $\Gamma \subset \mathcal{B}(\Delta)$ such that $\Gamma \equiv \Delta$ and (ii) there is no $\Gamma' \subsetneq \Gamma$ such that $\Gamma' \equiv \Delta$.*

The degree of coherence of a set of beliefs Δ is then taken to be the average degree of coherence of all minimal equivalent sets of basic commitments. To illustrate, consider a set Δ and denote the class of minimal equivalent sets by $[\Delta]^{\equiv}$, then the coherence of Δ is defined as follows:

$$\mathfrak{C}_{\Phi}(\Delta) = \sum_{\Gamma \in [\Delta]^{\equiv}} \frac{\Phi(\Gamma)}{|[\Delta]^{\equiv}|}$$

16 For details, see Schippers (2014).

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It can be shown that the set of minimal equivalent sets of basic commitments for each inconsistent set is identical to the set of all sets of duals of literals $\{p, \neg p\}$, where p is an arbitrary atomic formula. Accordingly, the following observation is true for the canonical representation $[\Delta]^{\equiv}$ of an arbitrary inconsistent set $\Delta \subseteq \mathcal{L}$ (a proof is given in Appendix 6.3.):

OBSERVATION 3.6. *Let* Δ *be inconsistent, then* $[\Delta]^{\equiv} = \{\{p_1, \neg p_1\}, \ldots, \{p_n, \neg p_n\}\}$.

Consequently, in light of Corollary 3.3 every inconsistent set of propositions is awarded the same degree of incoherence. This result is in line with the maximal incoherence intuition.

OBSERVATION 3.7. *Let* Δ *be an inconsistent set of propositions, then* $\mathcal{C}_{\Phi}(\Delta)$ *is minimal for each coherence measure* Φ.

Now what about the contrary intuition suggesting a nonmaximum degree of incoherence? It seems that what people have in mind when expressing this intuition is a set of independent testimonies.[17] Accordingly, if we additionally require that each testimony is individuated *not only* be its propositional content but also by its source, then the above result does *not* apply to inconsistent sets of testimonies. This is because, given this assumption, it is simply not the case that two inconsistent sets of testimonies are equivalent in the above sense.[18]

So far, the focus is on whether (or which) inconsistent sets of propositions should be assigned the maximum degree of incoherence. However, while the notion of coherence is graded, consistency is viewed as an all-or-nothing affair. The next section provides a refined picture by relating the discussion of different degrees of incoherence to the discussion on different *degrees of inconsistency*.

§4. Degrees of inconsistency and incoherence. So far the discussion focused on whether inconsistent sets might be incoherent to different degrees or not. If we grant Siebel and others the claim that they are (at least when taken to be witness reports), the question arises what causes these different degrees of incoherence. This section tries to fruitfully combine the discussion on degrees of incoherence to the discussion on degrees of inconsistency.[19] Basically, the question is whether there is a covariation between the degree of inconsistency of a set of propositions and its degree of incoherence. Is it simply the case that of two inconsistent sets, one is more incoherent if it is inconsistent to a higher degree?

At first glance it might seem counterintuitive to try to distinguish degrees of inconsistency within classical logic. Since classical logics are explosive, all inconsistent sets seem to be on a par and it might seem impossible to say meaningfully that some contradictory set is worse than another. However, imagine a court case, in which three independent and equally reliable witnesses testify regarding the culprit of a robbery. All that is known for sure is that the culprit is one out of six suspects, each having the same probability of having committed the crime. Now, in scenario 1 witness i gives the following evidence A_i:

(A_1) The robbery was committed by suspect 1 or 2.
(A_2) The robbery was committed by suspect 2 or 3.
(A_3) The robbery was committed by suspect 1 or 3.

[17] See, for example, the discussion by Shogenji (2005).

[18] This distinction is similar to the testimonial system approach put forward by Olsson (2005) and others. For details, see, Schippers (2014).

[19] An overview of a variety of approaches to measuring the degree of inconsistency is given by Hunter & Konieczny (2005).

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In the second scenario, witness i is assumed to give the following evidence B_i:

(B_1) The robbery was committed by suspect 1 or 2.
(B_2) The robbery was committed by suspect 3 or 4.
(B_3) The robbery was committed by suspect 5 or 6.

Whereas both sets $\mathrm{A} = \{A_1, A_2, A_3\}$ and $\mathrm{B} = \{B_1, B_2, B_3\}$ are inconsistent in the narrow sense, it seems that A is *less* inconsistent than B: while in the second scenario each pair of testimonies is inconsistent, there is an overlap for each such pair in the first. In other words, all consistent pairs of propositions turn inconsistent when moving from A to B. Thus, B seems to be inconsistent to a higher degree when compared to A. In what follows, I introduce some basic explications of the degree of inconsistency of a set of propositions that comply with intuitions in the robber case.

Beforehand, we need the following additional formal preliminaries: for any set Δ let $\mathbf{Cons}(\Delta) = \{\Gamma \subseteq \Delta \mid \Gamma \nvdash \bot\}$ and $\mathbf{In}(\Delta) = \{\Gamma \subseteq \Delta \mid \Gamma \vdash \bot\}$ be the set of consistent and inconsistent subsets of Δ. In order to distinguish different degrees of inconsistency, we need a representation of an *atomic inconsistency*, which is the part of an inconsistent set that is to blame for the inconsistency. This is done by means of *minimal inconsistent subsets*: for each set Δ, Γ is a minimal inconsistent subset of Δ iff $\Gamma \in \mathbf{In}(\Delta)$ and there is no $\Sigma \in \mathbf{In}(\Delta)$ such that $\Sigma \subsetneq \Gamma$. In other words, Γ is an inconsistent subset of Δ such that all genuine subsets of Γ are consistent. Denote the set of all minimal inconsistent subsets of Δ by $\mathbf{MI}(\Delta)$. To illustrate, consider the following set $\Delta = \{p_1 \wedge p_2, \neg p_1, \neg p_2 \vee p_3, \neg p_3\}$. As can easily be seen, the set Δ has two minimal inconstent subsets, viz. $\mathbf{MI}(\Delta) = \{\{p_1 \wedge p_2, \neg p_1\}, \{p_1 \wedge p_2, \neg p_2 \vee p_3, \neg p_3\}\}$.

An *inconsistency measure* is required to satisfy the following formal constraints:

1. $0 \leq \mathrm{i}(\Delta) \leq 1$ for all $\Delta \in \wp(\mathcal{L})$
2. If $\mathrm{i}(\Delta) = 0$, then $\Delta \in \mathbf{Cons}(\mathcal{L})$

A set Δ is said to be less inconsistent than another set Γ iff $\mathrm{i}(\Delta) < \mathrm{i}(\Gamma)$. Furthermore, consistent sets are assigned the minimum possible inconsistency value 0. In what follows I introduce two particular inconsistency measures.[20]

The counting-based approach In the first sense, a set is the more inconsistent, the more inconsistent subsets there are in relation to the set's cardinality. Hence, the cardinality of the set of minimal inconsistent subsets, $\mathbf{MI}(\Delta)$, might seem to yield a reasonable benchmark to compare different degrees of inconsistent sets of propositions that share the same cardinality. However, the *incompatibility ratio* defined as $\mathrm{i}_{\mathrm{IR}}(\Delta) = |\mathbf{MI}(\Delta)|/|\Delta|$ (Hunter & Konieczny, 2005) does not satisfy our desideratum that $0 \leq \mathrm{i}_{\mathrm{IR}} \leq 1$. Even though $\mathrm{i}_{\mathrm{IR}}(\Delta) \geq 0$ for all $\Delta \in \wp(\mathcal{L})$, there are sets that are assigned a degree of inconsistency above 1. One such example is the following set

$$\Gamma = \{p_1 \wedge p_2 \wedge p_3 \wedge p_4, p_1 \wedge p_2 \wedge \neg p_3 \wedge \neg p_4, p_1 \wedge \neg p_2 \wedge p_3 \wedge \neg p_4, \neg p_1 \wedge \neg p_2 \wedge p_3 \wedge p_4\}$$

All pairs of propositions in Γ are obviously inconsistent; hence, $\mathrm{i}_{\mathrm{IR}}(\Gamma) = 3/2 > 1$. Alternatively, I suggest the following measure for the counting-based approach to measuring inconsistency:

$$\mathrm{i}_1(\Delta) = \left(\theta_{\geq 1}\right)^{-1} \cdot |\mathbf{In}(\Delta)|$$

[20] For a variety of refined approaches to measuring inconsistency cf. Hunter & Konieczny (2005).

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Hence, Δ is less inconsistent than Γ iff $\mathrm{i}_1(\Delta) < \mathrm{i}_1(\Gamma)$. If Δ is consistent, then $|\mathbf{MI}(\Delta)| = 0$ and therefore $\mathrm{i}_1(\Delta) = 0$ (provided that Δ is nonempty). Additionally, a set is assigned the maximum inconsistency value 1 just in case all propositions are inconsistent. Now we return to the robber-case considered beforehand. We immediately get the desired result: $\mathrm{i}_1(\mathrm{A}) = 1/7 < 4/7 = \mathrm{i}_1(\mathrm{B})$. Hence, according to the counting-based approach, A is indeed less inconsistent than B.

The origin-based approach The origin-based approach on measuring inconsistency abstracts away from the number of inconsistencies and instead brings the *severity* of the inconsistency into focus. According to this approach, the degree of inconsistency is proportional to the number of propositions needed in order to derive the inconsistency. Thus, containing an outright contradiction $p \wedge \neg p$ turns a set maximally inconsistent, irrespective of other factors like the set's cardinality or the presence of other minimal inconsistent sets. The corresponding measure reads as follows: For $\mathbf{MI}(\Delta) = \{\Delta_1, \ldots, \Delta_n\}$, let $\delta_\Delta = \min\left\{|\Delta_i| \;\middle|\; \Delta_\mathrm{i} \in \mathbf{MI}(\Delta)\right\}$, then

$$\mathrm{i}_2(\Delta) = (\delta_\Delta)^{-1}$$

If Δ is consistent, we assume that δ_Δ tends to infinity; hence $\mathrm{i}_2(\Delta) = \lim_{\delta_\Delta \to \infty} \frac{1}{\delta_\Delta} = 0$. On the other hand, if $\delta_\Delta = 1$, so that an outright contradiction is present in Δ, $\mathrm{i}_2(\Delta) = 1$. Returning to our robber case, we find that i_2 complies with intuitions, too: Since $\delta_\mathrm{A} = 3$ and $\delta_\mathrm{B} = 2$, $\mathrm{i}_2(\mathrm{A}) = 1/3 < 1/2 = \mathrm{i}_2(\mathrm{B})$.[21]

Both measures yield a total order on $\wp(\mathcal{L})$: for each measure i let $\Delta \leq_\mathrm{i} \Gamma$ iff $\mathrm{i}(\Delta) \leq i(\Gamma)$. The corresponding relations "$=_{\mathrm{i}_j}$" and "$<_{\mathrm{i}_j}$" for $1 \leq j \leq 2$ are defined as usual. It is important to note, however, that the orders generated by these approaches might differ considerably. To illustrate, consider the following sets of propositions: $\Delta = \{p_1, \neg p_1 \vee p_2, p_2, p_3 \wedge \neg p_3\}$, $\Gamma = \{p_1 \wedge p_2 \wedge p_3, p_1 \wedge \neg p_2 \wedge p_3, p_1 \wedge p_2 \wedge \neg p_3\}$ and $\Sigma = \{p_1, p_2 \wedge \neg p_2, \neg p_1\}$. We get the following inconsistency orders: (i) $\Delta <_{\mathrm{i}_1} \Gamma <_{\mathrm{i}_1} \Sigma$ and (ii) $\Gamma <_{\mathrm{i}_2} \Delta =_{\mathrm{i}_2} \Sigma$.

Now that we have two inconsistency measures at hand, we can scrutinize the relationship between inconsistency and incoherence in yet another perspective. So far it was shown that it is possible to keep apart inconsistent sets with respect to their degree of incoherence. But is there a straightforward relationship between the degree of inconsistency and the degree of incoherence? More precisely, does the following claim (†) hold?

(†) For all Δ, Γ: *If* Δ is more inconsistent than Γ, *then* Δ is also more incoherent than Γ.

According to (†), a higher degree of inconsistency inevitably entails a higher degree of incoherence. As the following theorem reveals, none of the coherence measures considered in Section 2 satisfies (†) neither for i_1 nor for i_2 (for a proof see Appendix 6.4.).

THEOREM 4.1. *Let $\mathcal{C}$ be one of the variants of extant coherence measures proposed in Section 2, then $\mathcal{C}$ violates* (†) *for* $\mathrm{i} \in \{\mathrm{i}_1, \mathrm{i}_2\}$.

Theorem 4.1 is completely in line with Siebel's (2005) suggestion that "inconsistencies have a negative *influence* on coherence, [. . .] , [but] there may be factors present which have a positive impact and thus partly compensate for inconsistencies." (Siebel, 2005, p. 339, emphasis Siebel's). Thus, allowing for factors to partly compensate for inconsistencies, the measures behavior is appropriate: given two sets Δ, Γ, where Δ is inconsistent to a higher

[21] This approach is akin to the probabilistic approach of Knight (2002).

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degree than Γ, it might well be the case that this higher degree of inconsistency is offset against a larger number of coherence-boosting factors in Δ.

§5. Conclusion. This paper scrutinized the relationship between inconsistency and incoherence. After a brief assessment of some a number of extant proposals, Section 2 proposed variants of extant coherence measures in order to allow for evaluating inconsistent sets with respect to their degree of incoherence. Utilizing a refined approach to measuring coherence, it was shown afterward that it is possible to reconcile two diverging intuitions concerning the degree of incoherence of inconsistent sets of propositions: the "maximal incoherence view" and the more alleviated position that two inconsistent sets need not be assigned the same degree of incoherence. Finally, Section 4 examined the relationship between degrees of incoherence and degrees of inconsistency. Based on two straightforward measures of the degree of inconsistency it was shown that it is *not* the case that a higher degree of inconsistency necessarily entails a higher degree of incoherence. It would, of course, be interesting to check whether this negative result can be generalized to other inconsistency measures. This endeavor is left for future research.

Moreover, by showing how to quantify the degree of incoherence of sets of propositions, this paper only provides the basis for an in-depth comparison of the plethora of coherence measures. One option to go ahead is to consider various test cases involving inconsistent sets with intuitively different degrees of incoherence and compare the measures' performance. Considered as a methodological toolkit, test cases are a generally accepted instrument for assessing the adequacy of probabilistic measures of coherence (Akiba, 2000; Bovens & Hartmann, 2003a; Siebel, 2004; Glass, 2005; Meijs, 2006; Siebel & Wolff 2008; Schupbach, 2011). However, so far these test cases focus largely on the comparison of *consistent* sets of propositions regarding their respective degree of coherence. Two notable exceptions are Siebel (2005) and Roche (2013). Since both papers focus on the comparison of *extant* coherence measures, the main complaint is that the majority of these measures are undefined for the cases under consideration. Consequently, a re-evaluation based on the refined coherence measures introduced in this paper would seem beneficial.

§6. Appendix.

6.1. Proof of theorem 2.4. Let $\Delta = \{p, \neg p\}$ and $\Gamma = \{p, \neg p \vee q, \neg q\}$ and assume a uniform distribution over the set of state descriptions $\mathfrak{S}(p, q) = \{p \wedge q, p \wedge \neg q, \neg p \wedge q, \neg p \wedge \neg q\}$. As was shown in Section 3 (Theorem 3.2), all considered coherence measures award a minimum degree of coherence to all sets of contradictory propositions. Furthermore, it is easy to show that there is at least one nonminimal confirmatory relation within Γ for each of these coherence measures. $\mathcal{RO}$ is obviously identical to $\mathcal{O}$ for all pairs of propositions and therefore assigns a minimum degree of coherence to Δ but not to Γ. This already entails the desired conclusion.

6.2. Proof of theorem 3.2. Let $\Delta = \{p, \neg p\}$ be a set of propositions such that $0 < \Pr(p) < 1$, then both posterior probabilities $\Pr(p|\neg p)$ and $\Pr(\neg p|p)$ are defined and equal to zero. Consequently, since $\Pr(p) \cdot \Pr(\neg p) > 0$ and $\Pr(p \vee \neg p) > 0$ measures $\mathcal{D}'$, $\mathcal{O}'$, $\mathcal{RD}'$, $\mathcal{RO}'$, $\mathcal{C}_{r'}$ and $\mathcal{C}_{f'}$ equal the minimum possible value zero. Furthermore, $d'(p, \neg p) = d'(\neg p, p) = s'(p, \neg p) = s'(\neg p, p) = -1$ so that $\mathcal{C}_{d'}$ and $\mathcal{C}_{s'}$ equal the minimum possible value -1. The same holds true by definition for Fitelson's measure $\mathcal{C}_{k'}$ and for $\mathcal{C}_{z'}$ (note that both priors of p and $\neg p$ exceed the respective posteriors). □

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6.3. Proof of observation 3.6. Let $\mathcal{B}(\Delta)$ be the set of basic commitments for an inconsistent set Δ, then $\Gamma \subseteq \mathcal{B}(\Delta)$ is a minimal equivalent set of basic commitments if (i) $\Gamma \equiv \Delta$ and (ii) there is no $\Gamma' \subsetneq \Gamma$ such that $\Gamma' \equiv \Delta$. Now, it is by definition the case that $\{p_i, \neg p_i\} \subseteq \mathrm{Cn}(\Delta)$ for all atomic propositions p_i with $1 \leq i \leq n$. Furthermore, it can easily be seen that even $\{p_i, \neg p_i\} \subseteq \mathcal{B}(\Delta)$ for all $1 \leq i \leq n$ and since there is no proper subset $\Sigma \subsetneq \{p_i, \neg p_i\}$ such that $\Sigma \equiv \Delta$, we conclude that each set $\{p_i, \neg p_i\}$ is a minimal equivalent set of fundamental formulae for each inconsistent set. On the other hand, it is straightforward to see that there can be no other minimal equivalent set. Hence, for every inconsistent set Δ, $[\Delta]^{\equiv} = \{\{p_1, \neg p_1\}, \ldots, \{p_n, \neg p_n\}\}$. □

6.4. Proof of theorem 4.1. Consider the sets $\Delta = \{p_1, \neg p_1\}$, $\Gamma = \{p_1, p_2, p_1 \wedge \neg p_1\}$ and $\Sigma = \{p_1, \neg p_1 \wedge p_2, \neg p_1 \wedge p_2 \wedge p_3\}$ and assume a uniform prior distribution over the set of state descriptions $\mathfrak{S}(p_1, p_2, p_3)$, where $\varphi \in \mathfrak{S}(p_1, p_2, p_3)$ iff $\varphi = \pm p_1 \wedge \pm p_2 \wedge \pm p_3$ and "$\pm$" is either "$\neg$" or empty. In order to prove Theorem 4.1, we distinguish the following cases:

Inconsistency measure i_1: On the one hand, by Theorem 3.2 we infer that $\mathcal{C}_{\mathfrak{s}}(\Delta)$ is minimal for all coherence measures except $\mathcal{C}_{d'}$. This, however, is not the case for Σ: first of all, both refined measures $\mathcal{RD}'$ and $\mathcal{RO}'$ assign nonminimal coherence values to Σ. This is due to the fact that $\neg p_1 \wedge p_2$ and $\neg p_1 \wedge p_2 \wedge p_3$ are not probabilistically independent (in case of $\mathcal{RD}'$) and are assigned a positive degree of overlap (by $\mathcal{RO}'$).

Furthermore, since $\neg p_1 \wedge p_2 \wedge p_3$ entails $\neg p_1 \wedge p_2$, $\mathfrak{s}(\neg p_1 \wedge p_2, \neg p_1 \wedge p_2 \wedge p_3)$ is nonminimal for all measures $\mathfrak{s}$. This is due to the fact that all support measures assign positive degrees of support if the evidence entails the (nontautologous) hypothesis. Hence, since the calculated degrees of coherence simply average all support values, it must be the case that $\mathcal{C}_{\mathfrak{s}}(\Delta) < \mathcal{C}_{\mathfrak{s}}(\Sigma)$ for all $\mathfrak{s} \in \{d', r', s', k', z', f'\}$.

On the other hand, Σ is assigned a higher degree of inconsistency by i_1:

$$\mathrm{i}_1(\Delta) = 1/3 < 3/7 = \mathrm{i}_1(\Sigma)$$

Consequently, even though Σ is assigned a *higher* degree of inconsistency according to i_1, it is assigned a *lower* degree of incoherence by all considered coherence measures. This establishes Theorem 4.1 for i_1.

Inconsistency measure i_2: Again, by Theorem 3.2 we know that $\mathcal{C}_{\mathfrak{s}}(\Delta)$ is minimal for all coherence measures except $\mathcal{C}_{d'}$. Yet, this is not the case for Γ. First of all, since p_1 and p_2 are probabilistically independent (and compatible) on distribution Pr, $\mathcal{RD}'$ and $\mathfrak{s}(p_1, p_2)$ is nonextreme for all support measures. Hence, we infer that $\mathcal{C}_{\mathfrak{s}}(\Delta) < \mathcal{C}_{\mathfrak{s}}(\Gamma)$ for all $\mathfrak{s} \in \{d', r', s', k', z', f'\}$.
Finally, since $\Pr(p_1 \wedge p_2)$ and $\Pr(p_1 \vee p_2)$ are nonzero, they are assigned a positive degree of overlap. Hence $\mathcal{RO}'(\Gamma) > 0$.

On the other hand, Γ is awarded a higher degree of inconsistency according to i_2:

$$\mathrm{i}_2(\Delta) = 1/2 < 1 = \mathrm{i}_2(\Gamma)$$

Thus, though being inconsistent to a *higher* degree, Γ is nonetheless awarded a *lower* degree of incoherence. Consequently, this counterexample establishes Theorem 4.1 for i_2. □

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