Proceedings of the Royal Society of Edinburgh: Section A Mathematics, 1–17, *2026*
DOI:10.1017/prm.2025.10069

On approximation of convex functionals with a convexity constraint and general Lagrangians

Young Ho Kim
Department of Mathematics, Texas A&M University, College Station, Texas, United States and
Department of Mathematics, Indiana University, Bloomington, Indiana, United States (yhkim@tamu.edu)

(Received 6 May 2025; revised 5 August 2025; accepted 29 July 2025)

In this note, we prove that minimizers of convex functionals with a convexity constraint and a general class of Lagrangians can be approximated by solutions to fourth-order Abreu-type equations. Our result generalizes that of Le (Twisted Harnack inequality and approximation of variational problems with a convexity constraint by singular Abreu equations. *Adv. Math.* **434** (2023)) where the case of quadratically growing Lagrangians was treated.

Keywords: Convex functional; convexity constraint; linearized Monge–Ampère equation; Monge–Ampère equation; singular Abreu equation

2020 *Mathematics Subject Classification:* 35J35; 35J40; 35J96

1. Introduction and statement of the main result

In this note, we prove that minimizers of convex functionals with a convexity constraint and a general class of Lagrangians can be approximated by solutions to fourth-order Abreu-type equations. The problem of approximating minimizers to convex functionals with a convexity constraint by solutions of fourth-order Abreu-type equations has been studied by several authors [3, 6, 8–10, 12]. Previous results were proved either in two dimensions [12] or under a quadratic growth assumption on the Lagrangians [6, 8–10]. By replacing the quadratic term in the approximation scheme, we extend these results to the case with general Lagrangians in dimensions $n \geq 2$.

1.1. Variational problem with a convexity constraint

Let Ω and Ω_0 be bounded, smooth, convex domains in $\mathbb{R}^n$ $(n \geq 2)$ with $\Omega_0 \Subset \Omega$. Suppose $\varphi \in C^5(\overline{\Omega})$ is convex and $F = F(x, z, \mathbf{p}) : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a

© The Author(s), 2025. Published by Cambridge University Press on behalf of The Royal Society of Edinburgh. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (http://creativecommons.org/licenses/by/4.0), which permits unrestricted re-use, distribution and reproduction, provided the original article is properly cited.

https://doi.org/10.1017/prm.2025.10069 Published online by Cambridge University Press

smooth Lagrangian that is convex in the variables $\mathbf{p} \in \mathbb{R}^n$ and $z \in \mathbb{R}$. Consider the variational problem

$$\inf_{u \in \overline{S}[\varphi, \Omega_0]} \int_{\Omega_0} F(x, u(x), Du(x))\, dx \equiv \inf_{u \in \overline{S}[\varphi, \Omega_0]} J(u), \tag{1.1}$$

over the competitors u with a convexity constraint given by

$$\overline{S}[\varphi, \Omega_0] = \{u : \Omega \to \mathbb{R} \text{ convex}, \ u = \varphi \text{ on } \Omega \setminus \Omega_0\}. \tag{1.2}$$

An example of a variational problem of the form (1.1)–(1.2) is the Rochet–Choné model for the monopolist problem in economics [14]. In this model, the Lagrangian is given by $F(x, z, \mathbf{p}) = (|\mathbf{p}|^q/q - x \cdot \mathbf{p} + z)\gamma(x)$, for $q \in (1, \infty)$ and a nonnegative Lipschitz function γ.

Note that $\overline{S}[\varphi, \Omega_0]$ contains all convex functions in Ω_0, which have convex extensions that agree with a given convex function φ outside Ω_0. In addition to a Dirichlet boundary condition $u = \varphi$ on $\partial\Omega_0$, this constraint imposes, in some weak sense, a restriction on the gradient of u at the boundary of Ω_0. Consequently, it is hard to write a tractable Euler-Lagrange equation for the variational problem (1.1)–(1.2). Furthermore, variational problems of this type are difficult to handle in numerical schemes [2, 13]. Therefore, one may ask whether minimizers of these problems can be approximated by solutions of higher-order, well-posed equations.

To address these difficulties, Carlier and Radice [3] introduced an approximation scheme using solutions to the Abreu equations in the case when the Lagrangian $F = F(x, z)$ does not depend on the gradient variable $\mathbf{p} = (p_1, \cdots, p_n) \in \mathbb{R}^n$. This was extended by Le [8] to the case when F can be split into

$$F(x, z, \mathbf{p}) = F^0(x, z) + F^1(x, \mathbf{p}) \tag{1.3}$$

with suitable conditions on F^0 and F^1. In these schemes, penalizations of the form

$$J(v) + \frac{1}{2\varepsilon} \int_{\Omega \setminus \Omega_0} (v - \varphi)^2\, dx - \varepsilon \int_{\Omega} \log \det D^2 v\, dx \tag{1.4}$$

were introduced for small $\varepsilon > 0$. The idea behind the logarithmic penalization is that it should act as a good barrier for the convexity constraint (1.2) in problems like (1.1). Looking at the critical point u_ε of (1.4) where F is given by (1.3), we obtain an equation of the following form:

$$\begin{cases} \varepsilon U_\varepsilon^{ij} D_{ij} w_\varepsilon = f_\varepsilon := \left\{ \dfrac{\partial F^0}{\partial z}(x, u_\varepsilon) - \dfrac{\partial}{\partial x_i} \left(\dfrac{\partial F^1}{\partial p_i}(x, Du_\varepsilon) \right) \right\} \chi_{\Omega_0} \\ \qquad\qquad + \dfrac{1}{\varepsilon}(u_\varepsilon - \varphi)\chi_{\Omega \setminus \Omega_0} & \text{in } \Omega, \\ w_\varepsilon = (\det D^2 u_\varepsilon)^{-1} & \text{in } \Omega. \end{cases} \tag{1.5}$$

Here, $(U_\varepsilon^{ij})_{1 \le i,j \le n} = (\det D^2 u_\varepsilon)(D^2 u_\varepsilon)^{-1}$ is the cofactor matrix of the Hessian matrix $D^2 u_\varepsilon$ and χ_E is the characteristic function of the set E.

https://doi.org/10.1017/prm.2025.10069 Published online by Cambridge University Press

Note that (1.5) is a system of two equations, where one is a Monge–Ampère equation for u_ε:

$$\det D^2 u_\varepsilon = w_\varepsilon^{-1} \quad \text{in } \Omega, \tag{1.6}$$

and the other is a linearized Monge–Ampère equation for w_ε:

$$U_\varepsilon^{ij} D_{ij} w_\varepsilon = \varepsilon^{-1} f_\varepsilon \quad \text{in } \Omega. \tag{1.7}$$

Equation (1.7) is called a linearized Monge–Ampère equation because $U_\varepsilon^{ij} D_{ij}$ comes from linearizing the Monge–Ampère operator $\det D^2 u_\varepsilon$:

$$\det D^2(u_\varepsilon + tv) = \det D^2 u_\varepsilon + (U_\varepsilon^{ij} D_{ij} v)t + \cdots + (\det D^2 v)t^n.$$

As $w_\varepsilon = (\det D^2 u_\varepsilon)^{-1}$ is of second-order in u_ε, (1.5) is a fourth-order equation in u_ε. Because (1.5) is a system of these two equations, it is natural to consider second boundary value problems for (1.5) with Dirichlet boundary conditions on u_ε and w_ε, such as

$$u_\varepsilon = \varphi, \quad w_\varepsilon = \psi \quad \text{on } \partial\Omega. \tag{1.8}$$

When $\varepsilon^{-1} f_\varepsilon$ in (1.6)–(1.7) is replaced by -1,

$$U_\varepsilon^{ij} D_{ij}[(\det D^2 u_\varepsilon)^{-1}] = -1$$

is the Abreu equation [1] which appears in the problem of finding Kähler metrics of constant scalar curvature for toric manifolds [4, 5]. The term

$$\frac{\partial}{\partial x_i}\left(\frac{\partial F^1}{\partial p_i}(x, Du_\varepsilon)\right)$$

in (1.5) depends on $D^2 u_\varepsilon$, which is only guaranteed to be a matrix-valued measure under the assumption that u_ε is convex. Hence, (1.5) is called a *singular Abreu equation* [7–9, 12].

The general scheme is to first establish the existence of solutions $(u_\varepsilon)_{\varepsilon>0}$ to the second boundary value problem to Abreu-type equations of the form (1.5) with boundary conditions like (1.8), and then prove that after passing to a subsequence $\varepsilon_k \to 0$, solutions $(u_{\varepsilon_k})_k$ converge uniformly on compact subsets of Ω to a minimizer of the variational problem (1.1)–(1.2). Therefore, the solvability of the second boundary value problem for Abreu-type equations plays a critical role in approximating minimizers to convex functionals with a convexity constraint. For gradient-dependent Lagrangians, previous results were proved either in two dimensions [12] or under a quadratic growth assumption on the Lagrangian [6, 8–10]. By replacing the quadratic term in the approximation scheme, we extend the results to the case with general Lagrangians in dimensions $n \geq 2$; also see Remark 3.1.

1.2. The main result

In this note, we prove that the approximation scheme for the variational problem (1.1)–(1.2) using Abreu-type equations can be extended to a general class

https://doi.org/10.1017/prm.2025.10069 Published online by Cambridge University Press

of Lagrangians F that do not necessarily satisfy a quadratic growth assumption in dimensions $n \geq 2$. We achieve this by modifying the approximation scheme in (1.4).

Instead of a quadratic growth condition on the Lagrangian $F = F(x, z, \mathbf{p})$ in the $\mathbf{p}$ variable, we assume that F satisfies the following conditions:

(F1) F is smooth, and convex in variables $z \in \mathbb{R}$ and $\mathbf{p} \in \mathbb{R}^n$.

(F2) The derivatives of F satisfy the following growth estimates for $z \in \mathbb{R}$, $\mathbf{p} \in \mathbb{R}^n$:

$$\begin{aligned} \left|\frac{\partial F}{\partial z}(x,z,\mathbf{p})\right| + \left|\frac{\partial F}{\partial p_i}(x,z,\mathbf{p})\right| &\leq f_0(|z|)g_0(|\mathbf{p}|) \quad \text{for all } 1 \leq i \leq n, \\ 0 \leq (F_{p_ip_j}(x,z,\mathbf{p}))_{1\leq i,j\leq n} &\leq f_1(|z|)g_1(|\mathbf{p}|)I_n, \\ |F_{p_ix_i}(x,z,\mathbf{p})| &\leq f_2(|z|)g_2(|\mathbf{p}|), \\ |F_{p_iz}(x,z,\mathbf{p})| &\leq f_3(|z|)g_3(|\mathbf{p}|) \quad \text{for all } 1 \leq i \leq n. \end{aligned} \tag{1.9}$$

Here f_k, g_k $(0 \leq k \leq 3)$ are smooth, convex and increasing functions from $[0,\infty)$ to $[0,\infty)$, I_n is the $n \times n$ identity matrix, and repeated indices are summed.

The convexity assumptions on f_k, g_k are reasonable as any smooth, increasing growth function $\eta : [0,\infty) \to [0,\infty)$ can be replaced by

$$\widetilde{\eta}(x) := \int_0^{x+1} \eta(s)\, ds$$

which is convex, smooth, increasing and satisfies $\widetilde{\eta} \geq \eta$.

Now, we will introduce the modifications made to the approximating functional (1.4). The first modification comes from Le [9, 10]. Let ρ be a uniformly convex defining function of Ω, that is,

$$\{x \in \mathbb{R}^n \mid \rho(x) < 0\} = \Omega, \quad \rho = 0 \quad \text{on } \partial\Omega, \quad \text{and } D\rho \neq 0 \quad \text{on } \partial\Omega.$$

Now, for $\varepsilon > 0$, we set

$$\widetilde{\varphi}_\varepsilon(x) = \varphi(x) + \varepsilon^{\frac{1}{3n^2}}(e^{\rho(x)} - 1). \tag{1.10}$$

In the quadratic term

$$\frac{1}{2\varepsilon}\int_{\Omega\setminus\Omega_0} (u-\varphi)^2\, dx$$

from (1.4), we replace φ by $\widetilde{\varphi}_\varepsilon$. This makes the new function 'sufficiently' uniformly convex and makes it possible to handle Lagrangians F that are non-uniformly convex.

https://doi.org/10.1017/prm.2025.10069 Published online by Cambridge University Press

Furthermore, we replace the quadratic term

$$\frac{1}{2\varepsilon}\int_{\Omega\setminus\Omega_0}(u-\widetilde{\varphi}_\varepsilon)^2\,dx$$

again by

$$\frac{1}{\varepsilon}\int_{\Omega\setminus\Omega_0}G(u-\widetilde{\varphi}_\varepsilon)\,dx,$$

where G is a suitable function to be defined later. In Le [8–10], a quadratic growth assumption had to be imposed on F as the integral including the derivative $F^1_{p_ix_i}$ had to be bounded by the quadratic term in the approximation scheme; see [12, inequality (4.11)], [8, inequalities (2.4) and (4.15)], and [9, inequalities (1.9) and (3.12)]. In this note, this modification makes it possible to remove the quadratic growth assumption on F; also see Remark 2.3.

Because f_k, g_k are smooth, convex, increasing and nonnegative, if we define

$$H(x)=x(1+f_0(x)g_0(x)+f_2(x)g_2(x)+xf_3(x)g_3(x)), \tag{1.11}$$

then H is a convex, smooth, and increasing function from $[0,\infty)$ to $[0,\infty)$ with $H(x)\geq x$. Now, we define the convex function G by

$$G(x)=\int_0^{x^2}H(t)\,dt. \tag{1.12}$$

With these modifications to (1.4), the approximating functional used in this note will be

$$J_\varepsilon(u)=\int_{\Omega_0}F(x,u(x),Du(x))\,dx+\frac{1}{\varepsilon}\int_{\Omega\setminus\Omega_0}G(u-\widetilde{\varphi}_\varepsilon)\,dx-\varepsilon\int_\Omega\log\det D^2u(x)\,dx, \tag{1.13}$$

and our second boundary problem becomes

$$\begin{cases}\varepsilon U^{ij}_\varepsilon D_{ij}w_\varepsilon=f_\varepsilon\\ \qquad:=\left(\dfrac{\partial F}{\partial z}(x,u_\varepsilon,Du_\varepsilon)-\dfrac{\partial}{\partial x_i}\left(\dfrac{\partial F}{\partial p_i}(x,u_\varepsilon,Du_\varepsilon)\right)\right)\chi_{\Omega_0}\\ \qquad\quad+\dfrac{G'(u_\varepsilon-\widetilde{\varphi}_\varepsilon)}{\varepsilon}\chi_{\Omega\setminus\Omega_0} & \text{in }\Omega,\\ w_\varepsilon=(\det D^2u_\varepsilon)^{-1} & \text{in }\Omega,\\ u_\varepsilon=\varphi,\quad w_\varepsilon=\psi & \text{on }\partial\Omega.\end{cases} \tag{1.14}$$

Here $(U^{ij}_\varepsilon)_{1\leq i,j\leq n}$ is the cofactor matrix of D^2u_ε.

Our main result is the following theorem.

THEOREM 1.1. *Suppose Ω_0 and Ω are smooth and convex domains in $\mathbb{R}^n$ $(n\geq 2)$, where Ω is uniformly convex and $\Omega_0\Subset\Omega$. Let $\varphi\in C^5(\overline{\Omega})$, $\psi\in C^3(\overline{\Omega})$, φ is convex, and $\min_{\partial\Omega}\psi>0$. Let $F=F(x,z,\mathbf{p}):\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ satisfy (F1)–(F2). If*

https://doi.org/10.1017/prm.2025.10069 Published online by Cambridge University Press

$0 < \varepsilon < \varepsilon_0 < 1$, where ε_0 is a small number depending only on n, Ω, Ω_0, φ, ψ, f_k, and g_k, then the following are true.

(i) *The second boundary value problem (1.14) with G given by (1.11)–(1.12) has a uniformly convex $W^{4,s}(\Omega)$ solution u_ε for all $s \in (n, \infty)$.*
(ii) *Let $(u_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ be $W^{4,s}(\Omega)$ $(s > n)$ solutions to (1.14). Then, after passing to a subsequence $\varepsilon_k \to 0$, the sequence $(u_{\varepsilon_k})_k$ converges uniformly on compact subsets of Ω to a minimizer u of (1.1)–(1.2).*

REMARK 1.2. In Le [8–10], Lagrangians F that satisfy a quadratic growth condition in the $\mathbf{p}$ variable are considered. Compared to these results, Theorem 1.1 covers general Lagrangians in all dimensions $n \geq 2$ that do not necessarily have a quadratic growth in the $\mathbf{p}$ variable. One example of such a Lagrangian would be given by $F(x, z, \mathbf{p}) = e^{|\mathbf{p}|^2}$. This improvement comes from replacing the quadratic term in (1.4); see Remark 2.3.

The rest of this note is organized as follows. In § 2, we prove Theorem 1.1(i). In § 3, we prove Theorem 1.1(ii).

2. A priori estimates and existence of solutions

In this section, we prove Theorem 1.1(i) using degree theory and the a priori $W^{4,s}(\Omega)$ estimate in Proposition 2.1 below. The proof mostly follows Le [10, Section 2]. The main difference will be in proving the uniform L^∞ bound for u_ε in Lemma 2.2; see Remark 2.3.

PROPOSITION 2.1. *Suppose u_ε is a uniformly convex $W^{4,s}(\Omega)$ $(n < s < \infty)$ solution to (1.14), where F satisfies $(F1)$–$(F2)$ and G is defined by (1.11)–(1.12). If $0 < \varepsilon < \varepsilon_0 < 1$, where ε_0 is a small number depending only on n, Ω, Ω_0, φ, ψ, f_k, and g_k, then there is $C(\varepsilon) > 0$ such that*

$$\|u_\varepsilon\|_{W^{4,s}(\Omega)} \leq C(\varepsilon). \tag{2.1}$$

Fix $s \in (n, \infty)$. Throughout the section, u_ε will denote a uniformly convex $W^{4,s}(\Omega)$ solution to (1.14), and we will use numbered constants C_n to denote positive constants that do not depend on the solution u_ε but only on n, s, Ω, Ω_0, φ, ψ, f_k, and g_k. We will write C_n for constants that do not depend on ε, while for constants that depend on ε the dependency will be explicitly stated.

We start by getting an L^∞ bound for u_ε.

LEMMA 2.2. (Uniform L^∞ bound on u_ε). *If $0 < \varepsilon < \varepsilon_0$ where $\varepsilon_0 = \varepsilon_0(n, \Omega, \Omega_0, \varphi, \psi, f_k, g_k)$ is a small number satisfying $\varepsilon_0 < 1$, then*

$$||u_\varepsilon||_{L^\infty(\Omega)} < C_{14}. \tag{2.2}$$

Proof. Consider $\varepsilon < 1$. First, as u_ε is convex, we have an upper bound:

$$u_\varepsilon \leq \sup_{\partial\Omega} u_\varepsilon = \sup_{\partial\Omega} \varphi =: C_0 \quad \text{in } \Omega.$$

For the lower bound, we consider two cases as in Le-Zhou [12, pp.27–28].

https://doi.org/10.1017/prm.2025.10069 Published online by Cambridge University Press

Case 1. $u_\varepsilon(x_0) > \widetilde{\varphi}_\varepsilon(x_0) - 1$ for some $x_0 \in \Omega_0$. We have

$$\begin{aligned} u_\varepsilon(x_0) &> \inf_\Omega \widetilde{\varphi}_\varepsilon - 1 \geq -\sup_\Omega |\widetilde{\varphi}_\varepsilon| - 1 \\ &\geq -(\sup_\Omega |\varphi| + \sup_\Omega |e^\rho - 1| + 1) =: -C_1. \end{aligned}$$

Let $x \in \Omega \setminus \{x_0\}$ be arbitrary and set y to be the intersection of the ray $\overrightarrow{xx_0}$ and $\partial\Omega$. By the convexity of u_ε, we have

$$-C_1 \leq u_\varepsilon(x_0) \leq \frac{|x_0 - y|}{|x - y|} u_\varepsilon(x) + \left(1 - \frac{|x_0 - y|}{|x - y|}\right) u_\varepsilon(y). \tag{2.3}$$

Because $x_0 \in \Omega_0$ and $y \in \partial\Omega$, we have

$$\frac{|x_0 - y|}{|x - y|} \geq \frac{\operatorname{dist}(\Omega_0, \partial\Omega)}{\operatorname{diam}(\Omega)} > 0, \quad \text{and } |u_\varepsilon(y)| \leq \sup_{\partial\Omega} |\varphi|. \tag{2.4}$$

Combining (2.3) and (2.4) yields a lower bound for u_ε in Ω.

Case 2. $u_\varepsilon \leq \widetilde{\varphi}_\varepsilon - 1$ in Ω_0. We will use the following inequality [9, (3.6)]:

$$\int_{\partial\Omega} \varepsilon((u_\varepsilon)_\nu^+)^n \, dS \leq C_2 + \int_\Omega -f_\varepsilon(u_\varepsilon - \widetilde{\varphi}_\varepsilon) \, dx. \tag{2.5}$$

Here ν is the outer unit normal vector to $\partial\Omega$. Substituting f_ε from (1.14) and expanding the divergence term

$$\frac{\partial}{\partial x_i}\left(\frac{\partial F}{\partial p_i}(x, u_\varepsilon, Du_\varepsilon)\right) = F_{p_i x_i} + F_{p_i z} D_i u_\varepsilon + F_{p_i p_j} D_{ij} u_\varepsilon, \tag{2.6}$$

we find that the integral in the right-hand side of (2.5) becomes

$$\begin{aligned} \int_\Omega -f_\varepsilon(u_\varepsilon - \widetilde{\varphi}_\varepsilon) \, dx = &-\frac{1}{\varepsilon} \int_{\Omega \setminus \Omega_0} G'(u_\varepsilon - \widetilde{\varphi}_\varepsilon)(u_\varepsilon - \widetilde{\varphi}_\varepsilon) \, dx \\ &+ \int_{\Omega_0} -F_z(u_\varepsilon - \widetilde{\varphi}_\varepsilon) \, dx + \int_{\Omega_0} F_{p_i p_j} D_{ij} u_\varepsilon (u_\varepsilon - \widetilde{\varphi}_\varepsilon) \, dx \\ &+ \int_{\Omega_0} F_{p_i x_i}(u_\varepsilon - \widetilde{\varphi}_\varepsilon) \, dx + \int_{\Omega_0} F_{p_i z} D_i u_\varepsilon (u_\varepsilon - \widetilde{\varphi}_\varepsilon) \, dx. \end{aligned} \tag{2.7}$$

We estimate the terms in the right-hand side of (2.7) separately.

First, as $(F_{p_i p_j})_{1 \leq i,j \leq n}$ and $D^2 u_\varepsilon$ are nonnegative-definite, $F_{p_i p_j} D_{ij} u_\varepsilon \geq 0$. Since $u_\varepsilon \leq \widetilde{\varphi}_\varepsilon$, we get

$$\int_{\Omega_0} F_{p_i p_j} D_{ij} u_\varepsilon (u_\varepsilon - \widetilde{\varphi}_\varepsilon) \, dx \leq 0. \tag{2.8}$$

Next, we estimate $\int_{\Omega_0} -F_z(u_\varepsilon - \widetilde{\varphi}_\varepsilon)$. As u_ε is convex, we have the following gradient bound:

$$|Du_\varepsilon(x)| \leq \frac{\sup_{\partial\Omega} u_\varepsilon - u_\varepsilon(x)}{\operatorname{dist}(x, \partial\Omega)} \quad \text{for } x \in \Omega. \tag{2.9}$$

https://doi.org/10.1017/prm.2025.10069 Published online by Cambridge University Press

Therefore, for $x \in \Omega_0$, we have

$$|Du_\varepsilon(x)| \le \frac{|\sup_{\partial\Omega}\varphi| + ||u_\varepsilon||_{L^\infty(\Omega)}}{\operatorname{dist}(\Omega_0, \partial\Omega)} \le C_3(1 + ||u_\varepsilon||_{L^\infty(\Omega)}). \tag{2.10}$$

Because f_0 and g_0 are increasing functions, (1.9) and (2.10) give us

$$\begin{aligned}\int_{\Omega_0} -F_z(u_\varepsilon - \widetilde{\varphi}_\varepsilon) &\le \int_{\Omega_0} f_0\left(|u_\varepsilon(x)|\right) g_0\left(|Du_\varepsilon(x)|\right)\left(|u_\varepsilon(x)| + |\widetilde{\varphi}_\varepsilon(x)|\right) dx \\ &\le |\Omega_0| f_0\left(||u_\varepsilon||_{L^\infty(\Omega_0)}\right) g_0\left(||Du_\varepsilon||_{L^\infty(\Omega_0)}\right) \\ &\quad \times \left(||u_\varepsilon||_{L^\infty(\Omega_0)} + ||\widetilde{\varphi}_\varepsilon||_{L^\infty(\Omega_0)}\right) \\ &\le |\Omega_0| f_0\left(||u_\varepsilon||_{L^\infty(\Omega_0)}\right) g_0(S)\left(||u_\varepsilon||_{L^\infty(\Omega_0)} + ||\widetilde{\varphi}_\varepsilon||_{L^\infty(\Omega_0)}\right) \\ &\le S f_0(S) g_0(S), \end{aligned} \tag{2.11}$$

where

$$S = C_4(||u_\varepsilon||_{L^\infty(\Omega_0)} + 1) \tag{2.12}$$

for some large $C_4 > 0$. Other terms in the right-hand side of (2.7) can be estimated similarly using (1.9) and (2.10):

$$\begin{aligned}&\int_{\Omega_0} F_{p_i x_i}(u_\varepsilon - \widetilde{\varphi}_\varepsilon)\, dx \\ &\le n|\Omega_0| f_2\left(||u_\varepsilon||_{L^\infty(\Omega_0)}\right) g_2\left(||Du_\varepsilon||_{L^\infty(\Omega_0)}\right)\left(||u_\varepsilon||_{L^\infty(\Omega_0)} + ||\widetilde{\varphi}_\varepsilon||_{L^\infty(\Omega_0)}\right) \\ &\le S f_2(S) g_2(S), \end{aligned} \tag{2.13}$$

and

$$\begin{aligned}&\int_{\Omega_0} F_{p_i z} D_i u_\varepsilon (u_\varepsilon - \widetilde{\varphi}_\varepsilon)\, dx \\ &\le n|\Omega_0| f_3\left(||u_\varepsilon||_{L^\infty(\Omega_0)}\right) g_3\left(||Du_\varepsilon||_{L^\infty(\Omega_0)}\right) \|Du_\varepsilon\|_{L^\infty(\Omega_0)} \\ &\quad \times \left(||u_\varepsilon||_{L^\infty(\Omega_0)} + ||\widetilde{\varphi}_\varepsilon||_{L^\infty(\Omega_0)}\right) \\ &\le S^2 f_3(S) g_3(S). \end{aligned} \tag{2.14}$$

Combining (2.7), (2.8), (2.11), (2.13), (2.14) and (1.11), we obtain

$$\begin{aligned}&\int_\Omega -f_\varepsilon(u_\varepsilon - \widetilde{\varphi}_\varepsilon)\, dx \\ &\le -\frac{1}{\varepsilon}\int_{\Omega\setminus\Omega_0} G'(u_\varepsilon - \widetilde{\varphi}_\varepsilon)(u_\varepsilon - \widetilde{\varphi}_\varepsilon)\, dx + S(f_0(S)g_0(S) + f_2(S)g_2(S) + S f_3(S) g_3(S)) \\ &\le -\frac{1}{\varepsilon}\int_{\Omega\setminus\Omega_0} G'(u_\varepsilon - \widetilde{\varphi}_\varepsilon)(u_\varepsilon - \widetilde{\varphi}_\varepsilon)\, dx + H(S). \end{aligned} \tag{2.15}$$

We now estimate $H(S)$ using the following inequality [9, Corollary 2.2]:

$$||u_\varepsilon||_{L^\infty(\Omega)} \le C_5 + C_6 \int_{\Omega\setminus\Omega_0} |u_\varepsilon|\, dx. \tag{2.16}$$

https://doi.org/10.1017/prm.2025.10069 Published online by Cambridge University Press

From (2.12) and (2.16), we have

$$S \le C_4(1 + ||u_\varepsilon||_{L^\infty(\Omega)}) \le C_7 \int_{\Omega\setminus\Omega_0} (1 + |u_\varepsilon|)\, dx \le \frac{1}{|\Omega \setminus \Omega_0|} \int_{\Omega\setminus\Omega_0} C_8(1 + |u_\varepsilon|)\, dx. \tag{2.17}$$

Because H is convex, combining (2.17) with Jensen's inequality gives us

$$\begin{aligned} H(S) &\le H\left(\frac{1}{|\Omega \setminus \Omega_0|} \int_{\Omega\setminus\Omega_0} C_8(1 + |u_\varepsilon|)\, dx\right) \\ &\le \frac{1}{|\Omega \setminus \Omega_0|} \int_{\Omega\setminus\Omega_0} H(C_8(1 + |u_\varepsilon|))\, dx. \end{aligned} \tag{2.18}$$

Note that (1.12) implies

$$G'(u_\varepsilon - \widetilde{\varphi}_\varepsilon)(u_\varepsilon - \widetilde{\varphi}_\varepsilon) = 2(u_\varepsilon - \widetilde{\varphi}_\varepsilon)^2 H((u_\varepsilon - \widetilde{\varphi}_\varepsilon)^2). \tag{2.19}$$

As $|\widetilde{\varphi}_\varepsilon| \le C_1$, we have

$$(u_\varepsilon - \widetilde{\varphi}_\varepsilon)^2 \ge C_8(1 + |u_\varepsilon|) \tag{2.20}$$

when $|u_\varepsilon| > C_9 \ge C_1 + 1$. We consider the following cases:

(i) If $|u_\varepsilon| > C_9$. Because H is increasing, from (2.19) and (2.20) we have

$$G'(u_\varepsilon - \widetilde{\varphi}_\varepsilon)(u_\varepsilon - \widetilde{\varphi}_\varepsilon) \ge H((u_\varepsilon - \widetilde{\varphi}_\varepsilon)^2) \ge H(C_8(1 + |u_\varepsilon|)). \tag{2.21}$$

(ii) If $|u_\varepsilon| \le C_9$. We have

$$H(C_8(1 + |u_\varepsilon|)) \le H(C_8(1 + C_9)) =: C_{10}. \tag{2.22}$$

Combining (2.21) and (2.22) gives

$$H(C_8(1 + |u_\varepsilon|)) \le C_{10} + G'(u_\varepsilon - \widetilde{\varphi}_\varepsilon)(u_\varepsilon - \widetilde{\varphi}_\varepsilon) \quad \text{in } \Omega. \tag{2.23}$$

Therefore, from (2.18) we now have, for $\varepsilon < \varepsilon_0$, where $\varepsilon_0 = \varepsilon_0(n, \Omega, \Omega_0, \varphi, \psi, f_k, g_k)$ is small,

$$\begin{aligned} H(S) &\le \frac{1}{|\Omega \setminus \Omega_0|} \int_{\Omega\setminus\Omega_0} C_{10} + G'(u_\varepsilon - \widetilde{\varphi}_\varepsilon)(u_\varepsilon - \widetilde{\varphi}_\varepsilon)\, dx \\ &\le C_{10} + \frac{1}{2\varepsilon} \int_{\Omega\setminus\Omega_0} G'(u_\varepsilon - \widetilde{\varphi}_\varepsilon)(u_\varepsilon - \widetilde{\varphi}_\varepsilon)\, dx. \end{aligned} \tag{2.24}$$

Now, by combining (2.5), (2.15), and (2.24), we get

$$\int_{\partial\Omega} \varepsilon((u_\varepsilon)_\nu^+)^n\, dS + \frac{1}{2\varepsilon} \int_{\Omega\setminus\Omega_0} G'(u_\varepsilon - \widetilde{\varphi}_\varepsilon)(u_\varepsilon - \widetilde{\varphi}_\varepsilon)\, dx \le C_2 + C_{10}. \tag{2.25}$$

https://doi.org/10.1017/prm.2025.10069 Published online by Cambridge University Press

We are now ready to prove the L^∞ bound for u_ε. As $G'(x) = 2xH(x^2)$ and $H(x^2) \geq x^2$, (2.25) implies

$$\begin{aligned} C_{11} := (C_{10} + C_2)\varepsilon_0 &\geq \frac{1}{2}\int_{\Omega\setminus\Omega_0} G'(u_\varepsilon - \widetilde{\varphi}_\varepsilon)(u_\varepsilon - \widetilde{\varphi}_\varepsilon)\,dx \\ &\geq \int_{\Omega\setminus\Omega_0} (u_\varepsilon - \widetilde{\varphi}_\varepsilon)^4\,dx. \end{aligned} \tag{2.26}$$

From (2.16), we have

$$\begin{aligned} ||u_\varepsilon||_{L^\infty(\Omega)} &\leq C_5 + C_6\int_{\Omega\setminus\Omega_0} |u_\varepsilon|\,dx \\ &\leq C_5 + C_6|\Omega\setminus\Omega_0|\sup_\Omega|\widetilde{\varphi}_\varepsilon| + C_6\int_{\Omega\setminus\Omega_0}|u_\varepsilon - \widetilde{\varphi}_\varepsilon|\,dx \\ &\leq C_{12} + C_{13}\left(\int_{\Omega\setminus\Omega_0}(u_\varepsilon - \widetilde{\varphi}_\varepsilon)^4\,dx\right)^{1/4} \\ &\leq C_{14} \quad \text{by (2.26)}. \end{aligned}$$

The proof of the lemma is complete. □

Remark 2.3. In Le [8–10], the Lagrangian F was assumed to have a quadratic growth in the $\mathbf{p}$ variable. Especially, $|F_{p_i x_i}|$ was assumed to grow linearly in $\mathbf{p}$. Therefore, the integral in the left-hand side of (2.13) could be bounded by a quadratic term. In (2.13), this term cannot be bounded by a quadratic term because we are assuming a general growth assumption (1.9) for F. This is why we had to replace the quadratic term in (1.4) using G defined by (1.11)–(1.12) in (1.13).

Combining the L^∞ bound (2.2) with the gradient bound (2.10), we obtain the following corollary.

Corollary 2.4. *If $x \in \Omega_0$ and $0 < \varepsilon < \varepsilon_0$ where $\varepsilon_0 = \varepsilon_0(n, \Omega, \Omega_0, \varphi, \psi, f_k, g_k)$ is small, then we have*

$$|Du_\varepsilon(x)| \leq \frac{\sup_{\partial\Omega}|\varphi| + C_{14}}{\operatorname{dist}(\Omega_0, \partial\Omega)} =: C_{15}. \tag{2.27}$$

From now on, we fix $\varepsilon < \varepsilon_0$. Before we move on to the next step of the proof, we revisit the proof of (2.25) and note that the left-hand side can be bounded by a constant independent of ε, without having to assume that $u_\varepsilon \leq \widetilde{\varphi}_\varepsilon - 1$ in Ω_0.

Remark 2.5. In the proof of (2.25), the inequality $u_\varepsilon \leq \widetilde{\varphi}_\varepsilon - 1$ in Ω_0 was used to show (2.8). Having established the bounds (2.2) and (2.10), we can obtain an estimate for the left-hand side of (2.8) without the assumption.

https://doi.org/10.1017/prm.2025.10069 Published online by Cambridge University Press

From (1.9), (2.2) and (2.10), we have

$$0 \le (F_{p_ip_j})_{1\le i,j\le n} \le f_1(C_{14})g_1(C_{15})I_n,$$

and therefore

$$0 \le F_{p_ip_j} D_{ij}u_\varepsilon \le f_1(C_{14})g_1(C_{15})\Delta u_\varepsilon.$$

By the divergence theorem, (2.2) and (2.10), (2.8) can be replaced by

$$\begin{aligned}\int_{\Omega_0} F_{p_ip_j} D_{ij}u_\varepsilon(u_\varepsilon - \widetilde{\varphi}_\varepsilon)\,dx &\le \left(\|u_\varepsilon\|_{L^\infty(\Omega_0)} + \|\widetilde{\varphi}_\varepsilon\|_{L^\infty(\Omega_0)}\right) f_1(C_{14})g_1(C_{15}) \\ &\quad \times \int_{\Omega_0} \Delta u_\varepsilon\,dx \\ &\le C_{16}\int_{\Omega_0} \Delta u_\varepsilon\,dx = C_{16}\int_{\partial\Omega_0} (Du_\varepsilon\cdot\nu_0)\,dS \le C_{17},\end{aligned}$$

where ν_0 is the outer unit normal vector to $\partial\Omega_0$. Therefore, for u_ε not necessarily satisfying $u_\varepsilon \le \widetilde{\varphi}_\varepsilon - 1$ in Ω_0, we have instead of (2.25),

$$\int_{\partial\Omega} \varepsilon((u_\varepsilon)_\nu^+)^n\,dS + \frac{1}{2\varepsilon}\int_{\Omega\setminus\Omega_0} G'(u_\varepsilon - \widetilde{\varphi}_\varepsilon)(u_\varepsilon - \widetilde{\varphi}_\varepsilon)\,dx \le C_{18}. \tag{2.28}$$

Here C_{18} is a constant possibly larger than $C_2 + C_{10}$.

Next, we prove the following estimates for f_ε in Ω_0.

LEMMA 2.6. (Estimates for f_ε in Ω_0). *There are positive constants $\widetilde{C}$ and D_* that depend on $n, \Omega_0, \Omega, \varphi, \psi, f_k$ and g_k such that*

$$-\widetilde{C} - D_*\Delta u_\varepsilon \le f_\varepsilon \le \widetilde{C} \quad \text{in } \Omega_0. \tag{2.29}$$

Proof. Expanding as in (2.6), from (1.14) we get

$$\begin{aligned} f_\varepsilon = {} & \frac{\partial F}{\partial z}(x, u_\varepsilon, Du_\varepsilon) \\ & - \left(F_{p_ix_i}(x, u_\varepsilon, Du_\varepsilon) + F_{p_iz}(x, u_\varepsilon, Du_\varepsilon)D_iu_\varepsilon + F_{p_ip_j}(x, u_\varepsilon, Du_\varepsilon)D_{ij}u_\varepsilon\right) \quad \text{in } \Omega_0. \end{aligned} \tag{2.30}$$

Combining (1.9), (2.2) and (2.27) yields

$$\left|\frac{\partial F}{\partial z}(x, u_\varepsilon, Du_\varepsilon)\right| \le f_0(||u_\varepsilon||_{L^\infty(\Omega_0)})g_0(||Du_\varepsilon||_{L^\infty(\Omega_0)}) \le f_0(C_{14})g_0(C_{15}), \tag{2.31}$$

$$|F_{p_ix_i}(x, u_\varepsilon, Du_\varepsilon)| \le f_2(||u_\varepsilon||_{L^\infty(\Omega_0)})g_2(||Du_\varepsilon||_{L^\infty(\Omega_0)}) \le f_2(C_{14})g_2(C_{15}), \tag{2.32}$$

$$\begin{aligned}|F_{p_iz}(x, u_\varepsilon, Du_\varepsilon)D_iu_\varepsilon| &\le f_3(||u_\varepsilon||_{L^\infty(\Omega_0)})g_3(||Du_\varepsilon||_{L^\infty(\Omega_0)})||Du_\varepsilon||_{L^\infty(\Omega_0)} \\ &\le f_3(C_{14})g_3(C_{15})C_{15},\end{aligned} \tag{2.33}$$

https://doi.org/10.1017/prm.2025.10069 Published online by Cambridge University Press

and

$$0 \le (F_{p_i p_j})_{1\le i,j\le n} \le f_1(||u_\varepsilon||_{L^\infty(\Omega_0)}) g_1(||Du_\varepsilon||_{L^\infty(\Omega_0)}) I_n \le f_1(C_{14}) g_1(C_{15}) I_n =: D_* I_n. \tag{2.34}$$

From (2.34), we get

$$0 \le F_{p_i p_j} D_{ij} u_\varepsilon \le D_* \Delta u_\varepsilon. \tag{2.35}$$

Combining (2.30)–(2.33) with (2.35) gives the desired inequality. □

Having established the L^∞ bound in Lemma 2.2 and the estimates for f_ε in Lemma 2.6, we can carry out the rest of the proof in Le [10, Section 2]. We include an outline of the proof below.

As in [10, Lemma 2.3], we have an upper bound for $\det D^2 u_\varepsilon$:

LEMMA 2.7. (Upper bound for $\det D^2 u_\varepsilon$). *There is $C_{19}(\varepsilon) > 0$ such that*

$$\det D^2 u_\varepsilon \le C_{19}(\varepsilon) \quad \text{in } \Omega.$$

From the upper bound in Lemma 2.7 and the boundary condition $u_\varepsilon = \varphi$ on $\partial\Omega$, we can construct suitable barriers to show the gradient estimate:

$$|Du_\varepsilon| \le C_{20}(\varepsilon) \quad \text{in } \Omega. \tag{2.36}$$

We need the following transformation of (1.14) into linearized Monge–Ampère equations with drifts (see [10, Lemma 2.4] and [7, Lemma 2.1]):

LEMMA 2.8. (Transformation of (1.14)). *Let $x_0 \in \overline{\Omega}$ be fixed. Define the following functions in $\overline{\Omega}$:*

$$\begin{aligned} F_\varepsilon^{x_0}(x) &:= \frac{D_* |x - Du_\varepsilon(x_0)|^2}{2\varepsilon}, \\ \eta_\varepsilon^{x_0}(x) &:= w_\varepsilon(x) e^{F_\varepsilon^{x_0}(Du_\varepsilon(x))}, \\ \mathbf{b}^{x_0}(x) &:= -(\det D^2 u_\varepsilon(x)) \frac{D_*}{\varepsilon} (Du_\varepsilon(x) - Du_\varepsilon(x_0)). \end{aligned}$$

Then, we have

$$U_\varepsilon^{ij} D_{ij} \eta_\varepsilon^{x_0} + \mathbf{b}^{x_0} \cdot D\eta_\varepsilon^{x_0} = \frac{f_\varepsilon + D_* \Delta u_\varepsilon}{\varepsilon} e^{F_\varepsilon^{x_0}(Du_\varepsilon(x))} \quad \text{in } \Omega.$$

Using the transformation in Lemma 2.8 in conjunction with the Aleksandrov–Bakelman–Pucci maximum principle, we obtain a lower bound for $\det D^2 u_\varepsilon$.

LEMMA 2.9. (Lower bound for $\det D^2 u_\varepsilon$). *There is $C_{21}(\varepsilon) > 0$ such that*

$$\det D^2 u_\varepsilon \ge C_{21}^{-1}(\varepsilon) \quad \text{in } \Omega.$$

Combining the bounds on $\det D^2 u_\varepsilon$ in Lemmas 2.7 and 2.9, the boundary condition $u_\varepsilon = \varphi$ on $\partial\Omega$, and the global $C^{1,\alpha}$ estimates for Monge–Ampère equation in [11, Proposition 2.6], we obtain global $C^{1,\alpha}$ estimates for u_ε.

https://doi.org/10.1017/prm.2025.10069 Published online by Cambridge University Press

LEMMA 2.10. (Global $C^{1,\alpha}$ estimates for u_ε). *There is* $\alpha_0(\varepsilon) \in (0,1)$ *such that*

$$\|u_\varepsilon\|_{C^{1,\alpha_0(\varepsilon)}} \leq C_{22}(\varepsilon).$$

Using the transformation in Lemma 2.8 and the one-sided pointwise Hölder estimates at the boundary for solutions to non-uniformly elliptic, linear equations with pointwise Hölder continuous drift [10, Proposition 2.7], we obtain Hölder estimates for w_ε at the boundary. The twisted Harnack inequality in [10, Theorem 1.3] gives Hölder estimates for w_ε in the interior.

By combining the Hölder estimates for w_ε in the interior and the boundary, and using the boundary localization theorem of Savin [15], we obtain global Hölder estimates for w_ε.

LEMMA 2.11. (Global Hölder estimates for w_ε). *There are constants* $\alpha_1(\varepsilon) \in (0,1)$ *and* $C_{23}(\varepsilon) > 0$ *so that*

$$||w_\varepsilon||_{C^{\alpha_1(\varepsilon)}(\overline{\Omega})} \leq C_{23}(\varepsilon).$$

Having established the global Hölder estimates in Lemma 2.11, we can prove the global $W^{4,s}(\Omega)$ estimate.

Proof of Proposition 2.1. From

$$\det D^2 u_\varepsilon = w_\varepsilon^{-1} \quad \text{in } \Omega, \quad u_\varepsilon = \varphi \quad \text{on } \partial\Omega,$$

the Hölder estimates in Lemma 2.11, and the global $C^{2,\alpha}$ estimates for the Monge–Ampère equation [15, 16], we get

$$||u_\varepsilon||_{C^{2,\alpha_1(\varepsilon)}(\overline{\Omega})} \leq C_{24}(\varepsilon).$$

Therefore, $U_\varepsilon^{ij} D_{ij}$ is an uniformly elliptic operator with Hölder continuous coefficients. Moreover, f_ε is bounded in the L^∞ norm. Thus, from

$$U_\varepsilon^{ij} D_{ij} w_\varepsilon = f_\varepsilon/\varepsilon \quad \text{in } \Omega, \quad w_\varepsilon = \psi \quad \text{on } \partial\Omega,$$

we obtain estimates for w_ε in $W^{2,s}(\Omega)$. The $W^{4,s}(\Omega)$ estimate in (2.1) follows. □

We are now ready to prove Theorem 1.1(i).

Proof of Theorem 1.1(i). From the a priori estimate (2.1) in Proposition 2.1, we can use Leray–Schauder degree theory as in Le [8, pp.2275–2276] to prove the existence of a uniformly convex $W^{4,s}(\Omega)$ solution u_ε to (1.14) for all $s \in (n, \infty)$. □

3. Convergence of solutions to a minimizer

In this section, we prove Theorem 1.1(ii) on the convergence of solutions of (1.14) to a minimizer of the variational problem (1.1)–(1.2). We will follow the proof in Le [8, 9].

https://doi.org/10.1017/prm.2025.10069 Published online by Cambridge University Press

Proof of Theorem 1.1(ii). By (2.2), the family $(u_\varepsilon)_{\varepsilon>0}$ of $W^{4,s}(\Omega)$ solutions to (1.14) satisfies, whenever $0<\varepsilon<\varepsilon_0$,

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \le C \tag{3.1}$$

for C independent of ε. Furthermore, for any $\Omega' \Subset \Omega$, we can combine (3.1) with the gradient bound (2.9) to obtain

$$\|Du_\varepsilon\|_{L^\infty(\Omega')} \le \widehat{C}(\Omega', \Omega). \tag{3.2}$$

From (3.1) and (3.2), by passing to a subsequence $\varepsilon_k \to 0$, we have

$$\begin{aligned} &u_{\varepsilon_k} \to u \quad \text{weakly in } W^{1,2}(\Omega_0), \quad \text{and} \\ &u_{\varepsilon_k} \to u \quad \text{uniformly on compact subsets of } \Omega, \end{aligned} \tag{3.3}$$

for some convex function u in Ω. Combining (1.12) with (2.28) yields

$$\begin{aligned} C_{18}\varepsilon_k &\ge \frac{1}{2}\int_{\Omega\setminus\Omega_0} G'(u_{\varepsilon_k}-\widetilde{\varphi}_{\varepsilon_k})(u_{\varepsilon_k}-\widetilde{\varphi}_{\varepsilon_k})\,dx \\ &= \int_{\Omega\setminus\Omega_0} H((u_{\varepsilon_k}-\widetilde{\varphi}_{\varepsilon_k})^2)(u_{\varepsilon_k}-\widetilde{\varphi}_{\varepsilon_k})^2\,dx \\ &\ge \int_{\Omega\setminus\Omega_0} (u_{\varepsilon_k}-\widetilde{\varphi}_{\varepsilon_k})^4\,dx. \end{aligned}$$

Therefore, $\int_{\Omega\setminus\Omega_0}(u_{\varepsilon_k}-\widetilde{\varphi}_{\varepsilon_k})^4\,dx \to 0$ as $k\to\infty$. Because $u_{\varepsilon_k}\to u$ uniformly on compact subsets of Ω and

$$\widetilde{\varphi}_{\varepsilon_k} = \varphi + \varepsilon_k^{\frac{1}{3n^2}}(e^\rho - 1) \to \varphi \quad \text{as } k\to\infty$$

uniformly in Ω, we have $u=\varphi$ in $\Omega\setminus\Omega_0$ and hence $u\in\overline{S}[\varphi,\Omega_0]$. Now, we show that u minimizes J in (1.1) among the competitors in $\overline{S}[\varphi,\Omega_0]$ by the following steps.

Step 1. First, we show that

$$\liminf_{k\to\infty} J(u_{\varepsilon_k}) \ge J(u). \tag{3.4}$$

From the convexity of F in z and $\mathbf{p}$, we have

$$\begin{aligned} J(u_{\varepsilon_k}) - J(u) &= \int_{\Omega_0} F(x,u_{\varepsilon_k},Du_{\varepsilon_k}) - F(x,u,Du)\,dx \\ &\ge \int_{\Omega_0} F_z(x,u,Du)(u_{\varepsilon_k}-u) + F_{\mathbf{p}}(x,u,Du)\cdot(Du_{\varepsilon_k}-Du)\,dx. \end{aligned} \tag{3.5}$$

Here $F_{\mathbf{p}} = (F_{p_1},\cdots,F_{p_n})$. Therefore, from (3.3) the right-hand side of (3.5) converges to 0 as $k\to\infty$, and the desired inequality (3.4) follows.

https://doi.org/10.1017/prm.2025.10069 Published online by Cambridge University Press

Step 2. Next, we show that if v is a convex function in $\overline{\Omega}$ satisfying $v = \varphi$ in a neighbourhood of $\partial\Omega$, then

$$J_\varepsilon(v) - J_\varepsilon(u_\varepsilon) \geq \varepsilon \int_{\partial\Omega} \psi U_\varepsilon^{\nu\nu} \partial_\nu(u_\varepsilon - \varphi)\, dS + \int_{\partial\Omega_0} (v - u_\varepsilon) F_{\mathbf{p}}(x, u_\varepsilon, Du_\varepsilon) \cdot \nu_0\, dS, \tag{3.6}$$

where $U_\varepsilon^{\nu\nu} = U_\varepsilon^{ij}\nu_i\nu_j$. Here ν and ν_0 are outer unit normal vectors to $\partial\Omega$ and $\partial\Omega_0$.

We use mollification as in Le [8, p.2277] to obtain a sequence of uniformly convex $C^3(\overline{\Omega})$ functions $(v_h)_{h>0}$ that satisfy, for all $k \leq 2$,

$$D^k v_h \to D^k v \quad \text{as } h \to 0 \quad \text{in a neighbourhood of } \partial\Omega. \tag{3.7}$$

Recall from (1.13) that

$$J_\varepsilon(v) = \int_{\Omega_0} F(x, v(x), Dv(x))\, dx + \frac{1}{\varepsilon}\int_{\Omega\setminus\Omega_0} G(v - \widetilde{\varphi}_\varepsilon)\, dx - \varepsilon \int_\Omega \log\det D^2 v(x)\, dx.$$

First, by [8, (5.9)] we have

$$\begin{aligned} &-\int_\Omega \log\det D^2 v_h\, dx + \int_\Omega \log\det D^2 u_\varepsilon\, dx \\ &\geq \int_\Omega \varepsilon^{-1} f_\varepsilon(u_\varepsilon - v_h)\, dx - \int_{\partial\Omega} D_i w_\varepsilon U_\varepsilon^{ij}(u_\varepsilon - v_h)\nu_j\, dS + \int_{\partial\Omega} \psi U_\varepsilon^{\nu\nu}\partial_\nu(u_\varepsilon - v_h)\, dS \end{aligned} \tag{3.8}$$

Furthermore, using the convexity of F and integrating by parts, we get

$$\begin{aligned} &\int_{\Omega_0} F(x, v_h, Dv_h) - \int_{\Omega_0} F(x, u_\varepsilon, Du_\varepsilon) \\ &\geq \int_{\Omega_0} F_z(x, u_\varepsilon, Du_\varepsilon)(v_h - u_\varepsilon)\, dx + \int_{\Omega_0} F_{\mathbf{p}}(x, u_\varepsilon, Du_\varepsilon) \cdot D(v_h - u_\varepsilon)\, dx \\ &= \int_{\Omega_0} F_z(x, u_\varepsilon, Du_\varepsilon)(v_h - u_\varepsilon)\, dx - \int_{\Omega_0} \frac{\partial}{\partial x_i}\left(F_{p_i}(x, u_\varepsilon, Du_\varepsilon)\right)(v_h - u_\varepsilon)\, dx \\ &+ \int_{\partial\Omega_0} (F_{\mathbf{p}}(x, u_\varepsilon, Du_\varepsilon) \cdot \nu_0)(v_h - u_\varepsilon)\, dS. \end{aligned} \tag{3.9}$$

Finally, from the convexity of G, we can conclude

$$\frac{1}{\varepsilon}\int_{\Omega\setminus\Omega_0} G(v_h - \widetilde{\varphi}_\varepsilon)\, dx - \frac{1}{\varepsilon}\int_{\Omega\setminus\Omega_0} G(u_\varepsilon - \widetilde{\varphi}_\varepsilon)\, dx \geq \frac{1}{\varepsilon}\int_{\Omega\setminus\Omega_0} G'(u_\varepsilon - \widetilde{\varphi}_{\varepsilon_k})(v_h - u_\varepsilon)\, dx. \tag{3.10}$$

Combining (3.8)–(3.10), we get after cancellation,

$$\begin{aligned} J_\varepsilon(v_h) - J_\varepsilon(u_\varepsilon) \geq &-\varepsilon \int_{\partial\Omega} D_i w_\varepsilon U_\varepsilon^{ij}(u_\varepsilon - v_h)\nu_j\, dS \\ &+ \varepsilon \int_{\partial\Omega} \psi U_\varepsilon^{\nu\nu}\partial_\nu(u_\varepsilon - v_h)\, dS + \int_{\partial\Omega_0} (v_h - u_\varepsilon) F_{\mathbf{p}}(x, u_\varepsilon, Du_\varepsilon) \cdot \nu_0\, dS. \end{aligned} \tag{3.11}$$

https://doi.org/10.1017/prm.2025.10069 Published online by Cambridge University Press

By (3.7), the right-hand side of (3.11) converges to the right-hand side of (3.6) as $h \to 0$. Furthermore, we have from [8, (5.8)],

$$J_\varepsilon(v_h) \to J_\varepsilon(v) \quad \text{as } h \to 0.$$

Therefore, letting $h \to 0$ in (3.11) gives (3.6).

Step 3. We show that for any $v \in \overline{S}[\varphi, \Omega_0]$,

$$J(v) \geq \liminf_{k\to\infty} J(u_{\varepsilon_k}). \tag{3.12}$$

We use (3.6) to argue as in Le [9, pp.372–374] to obtain the following inequality:

$$J(v) \geq \liminf_{k\to\infty} J(u_{\varepsilon_k}) - \limsup_{k\to\infty} \left[{\varepsilon_k}^{(n-1)/n} \eta_{\varepsilon_k} + {\varepsilon_k}^{1/n} \eta_{\varepsilon_k}^{n-1} \right], \tag{3.13}$$

where

$$\eta_\varepsilon := \varepsilon^{1/n} \left(\int_{\partial\Omega} ((u_\varepsilon^+)_\nu)^n \, dS \right)^{1/n}.$$

From (2.28), η_ε is bounded independent of ε:

$$\eta_\varepsilon \leq C_{18}^{1/n},$$

and therefore (3.12) follows from (3.13).

Step 4. We now show the minimality of u. Combining (3.4) and (3.12) gives

$$J(v) \geq \liminf_{k\to\infty} J(u_{\varepsilon_k}) \geq J(u)$$

for all $v \in \overline{S}[\varphi, \Omega_0]$, which proves the minimality of u. The proof of the theorem is complete. □

REMARK 3.1. By the technique in [6], Theorem 1.1 is also true for $n = 1$. However, as mentioned in [6, Remark 3.2], the proof of Theorem 1.1(ii) requires an additional argument different from the ones used above.

Acknowledgements

The author would like to thank Professor Nam Q. Le for suggesting the problem and for the insightful guidance and support received during the work. The author also expresses his gratitude to the anonymous referee for carefully reading the note and providing constructive feedback.

The research of the author was supported by NSF grant DMS-2054686.

References

1 M. Abreu. Kähler geometry of toric varieties and extremal metrics. *Internat. J. Math.* **9** (1998), 641–651.

2 J. -D. Benamou, G. Carlier, Q. Mérigot and E. Oudet. Discretization of functionals involving the Monge-Ampère operator. *Numer. Math.* **134** (2016), 611–636.

https://doi.org/10.1017/prm.2025.10069 Published online by Cambridge University Press

3 G. Carlier and T. Radice. Approximation of variational problems with a convexity constraint by PDEs of Abreu type. *Calc. Var. Partial Differential Equations*. **58** (2019), 170, 13.

4 S. K. Donaldson. Scalar curvature and stability of toric varieties. *J. Diff. Geom*. **62** (2002), 289–349.

5 S. K. Donaldson. Interior estimates for solutions of Abreu's equation. *Collect. Math*. **56** (2005), 103–142.

6 Y. H. Kim. On the One-dimensional Singular Abreu Equations. *Appl. Math. Optim*. **90** Paper No. 35, (2024).

7 Y. H. Kim, N. Q. Le, L. Wang and B. Zhou. Singular Abreu equations and linearized Monge-Ampère equations with drifts. To appear. in *J. Eur. Math. Soc. (JEMS)*. https://doi.org/10.4171/jems/1548

8 N. Q. Le. Singular Abreu equations and minimizers of convex functionals with a convexity constraint. *Comm. Pure Appl. Math*. **73** (2020), 2248–2283.

9 N. Q. Le. On approximating minimizers of convex functionals with a convexity constraint by singular Abreu equations without uniform convexity. *Proc. Roy. Soc. Edinburgh Sect. A*. **151** (2021), 356–376.

10 N. Q. Le. Twisted Harnack inequality and approximation of variational problems with a convexity constraint by singular Abreu equations. *Adv. Math*. **434** (2023), 109325, 31.

11 N. Q. Le and O. Savin. Boundary regularity for solutions to the linearized Monge-Ampère equations. *Arch. Ration. Mech. Anal*. **210** (2013), 813–836.

12 N. Q. Le and B. Zhou. Solvability of a class of singular fourth order equations of Monge-Ampère type. *Ann. PDE*. **7** (2021), 13, 32.

13 J. M. Mirebeau. Adaptive, anisotropic and hierarchical cones of discrete convex functions. *Numer. Math*. **132** (2016), 807–853.

14 J. C. Rochet and P. Choné. Ironing, sweeping and multidimensional screening. *Econometrica*. **66** (1998), 783–826.

15 O. Savin. Pointwise $C^{2,\alpha}$ estimates at the boundary for the Monge-Ampère equation. *J. Amer. Math. Soc*. **26** (2013), 63–99.

16 N. S. Trudinger and X. J. Wang. Boundary regularity for the Monge-Ampère and affine maximal surface equations. *Ann. of Math. (2)*. **167** (2008), 993–1028.

https://doi.org/10.1017/prm.2025.10069 Published online by Cambridge University Press