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ARTICLE

Complex dynamics in a two-sector Solow–Swan Model

Olaf Schmitz[1] and Jan Wenzelburger[2]

[1]VPV Versicherungen, Mittlerer Pfad 19, 70499 Stuttgart, Germany
[2]Faculty of Business Studies and Economics, RPTU University of Kaiserslautern-Landau, Kaiserslautern, Germany
Corresponding author: Jan Wenzelburger; Email: jan.wenzelburger@rptu.de

Abstract

This article investigates the dynamics of a parsimonious extension of the standard neo-classical growth model introduced by Solow (1956) and Swan (1956) to a two-sector growth model in discrete time. We identify the economy's *propensity to invest* as a market mechanism that may cause endogenous business cycles and complex dynamics, provided that the elasticity of factor substitution in producing consumption goods is sufficiently small. The theoretical results are supported by numerical evidence for topological chaos and strange attractors.

Keywords: Two-sector growth models; factor-intensity reversals; endogenous business cycles; complex dynamics; two-sector models

JEL classifications: C30; C63; E13; E32

1. Introduction

Over half a century ago, Solow (1956) and Swan (1956) developed a one-sector growth model that serves as the foundation for all neo-classical growth models. The qualitative dynamics of this model are rather simple, as capital-to-labour ratios either converge monotonically to a unique steady state or become unbounded. The purpose of this article is to demonstrate that a parsimonious extension of the Solow–Swan model to a two-sector growth model with a constant propensity to save out of aggregate income allows for endogenous cycles and complex dynamics.

The literature has addressed the limitation that the Solow–Swan model in discrete time cannot explain for endogenous cycles in two primary ways: first, by endogenising the savings propensity, and second, by incorporating multiple production sectors. In a seminal paper Day (1982) showed that one-sector growth models can exhibit cyclical growth patterns if the savings function depends non-linearly on aggregate income. In his model the difference equation governing growth paths is, in essence, a logistic map, which is well known for its capacity to generate complex dynamics, see Devaney (1989). However, a number of contributions have shown that Day's ad hoc savings function is generally inconsistent with intertemporal utility maximisation, for example, see the survey by Boldrin and Woodford (1990). In hindsight, it is somewhat surprising that it took considerable time to recognise that Kaldor (1956, 1957) simple modification of the Solow–Swan model permits complex dynamics. Böhm & Kaas (2000) demonstrated that Kaldor's model can exhibit topological chaos as defined by Li and Yorke (1975) even with small variations in savings propensities, provided the distribution of factor incomes varies sufficiently. Cycles of any order and complex dynamics become possible both in optimal growth and OLG models with two sectors as soon as the single all-purpose good is replaced by two distinct goods, one for consumption and one for investment purposes, see Stachurski et al. (2012).

Most of the literature on two-sector growth models adopts the so-called *no-factor-intensity-reversal condition*, also referred to as *capital-intensity condition*, according to which the

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capital-to-labour ratio in one sector consistently exceeds that in the other for all wage-rental ratios. This assumption is crucial in the classical continuous time two-sector growth model with a constant savings propensity, which dates back to Uzawa (1961, 1963), Inada (1963), and Drandakis (1963), and is needed to address the existence of steady states. However, in continuous time, complex dynamic behaviour is precluded by the well-known Poincaré–Bendixson Theorem. In Galor (1992), the capital-intensity condition is essential for the existence and uniqueness of perfect-foresight equilibria. Some contributions allow for factor-intensity reversals but limit their focus to specific technologies, such as a Cobb–Douglas and Leontief production functions, for example, see Boldrin and Deneckere (1990) or Ralf (2001) and references therein. [Uzawa (1963), p. 109], in response to the Solow (1961) critique, acknowledged that ruling out factor-intensity reversals is *'required mainly for reasons of a mathematical nature and for which it seems to be difficult to give any economic justification'*. Unfortunately, the capital-intensity condition imposes significant restrictions on production functions and, aside from Cobb–Douglas functions, excludes already those with a constant elasticity of substitution (CES).

The aim of this article is to present a two-sector growth model without a priori restrictions on factor-intensity reversals, allowing for a general class of homogeneous production functions, including CES production functions. The model employs a novel parametrisation of the production-possibility frontier in terms of wage-rental ratios, as introduced by Ritschel and Wenzelburger (2024). The other features of the model are sparse and standard. The savings propensity remains constant, resulting in dynamics governed by a univariate map. We will demonstrate that the non-linear dependency of the economy's propensity to invest on capital-to-labour ratios is responsible for endogenous business cycles and complex dynamic behaviour. This market mechanism has yet to be identified as a cause of non-monotonic dynamics.

2. The model

This section develops a two-sector extension of the neo-classical growth model, introduced by Solow (1956) and Swan (1956). Two distinct sectors employ capital and labour to produce consumption goods for consumption needs and investment goods for investment purposes. Neither consumption of investment goods nor investment in consumption goods is possible. Markets are perfectly competitive, and capital and labour are perfectly mobile between sectors. The investment good is chosen as numéraire and the prices of labour, capital, and the consumption good in period t are denoted by w_t, r_t, and p_t, respectively.

Time is discrete. In every period $t = 0, 1, \ldots$, consumers supply capital K_t and labour L_t inelastically to the markets and receive factor income in return from which they save and invest a constant proportion $0 \leq s \leq 1$. The factor income per capita in terms of the investment good is $w_t + r_t k_t$, where $k_t := K_t/L_t$ is the economy-wide capital-to-labour ratio. Per capita, consumers invest $s[w_t + r_t k_t]$ units of the investment good and spend $(1-s)[w_t + r_t k_t]$ units on consumption.

2.1 Technology

The consumption-goods sector (x) and the investment-goods sector (y) operate time-invariant constant-returns-to-scale production technologies given by the production functions

$$f_j : \mathbb{R}_+ \to \mathbb{R}_+, \quad k^j \mapsto f_j(k^j), \quad j = x, y,$$

where k^j denotes the capital intensity in sector j. The sector-specific marginal products of labour and capital are $w_j(k^j) := f_j(k^j) - f_j'(k^j)k^j$ and $f_j'(k^j)$, respectively, so that the corresponding *marginal rates of technical substitution* (MRTS) take the form

$$\Omega_j(k^j) := \frac{w_j(k^j)}{f_j'(k^j)} = \frac{f_j(k^j)}{f_j'(k^j)} - k^j, \quad j = x, y. \tag{1}$$

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Assumption 1 (Technologies). *The technologies of the two production sectors satisfy the following.*

(i) Each production function $f_j : \mathbb{R}_+ \to \mathbb{R}_+$, $j = x, y$, is twice continuously differentiable, strictly increasing, $f_j' > 0$, and strictly concave, $f_j'' < 0$.

(ii) Each marginal rate of technical substitution $\Omega_j : \mathbb{R}_+ \to \mathbb{R}_+$, $j = x, y$, fulfils the boundary conditions

$$\lim_{k \to 0} \Omega_j(k) = 0 \quad \textit{and} \quad \lim_{k \to \infty} \Omega_j(k) = \infty.$$

Assumption 1 dates back to Drandakis (1963).[1] It follows from Assumption 1 (i) that the MRTS functions Ω_x and Ω_y are continuously differentiable and strictly increasing. Assumption 1 (ii) ensures that there exist continuously differentiable *relative factor demand functions* $\kappa_j : \mathbb{R}_+ \to \mathbb{R}_+$, $j = x, y$, defined by

$$\Omega_j(\kappa_j(\omega)) = \omega \quad \text{for all } \omega \in \mathbb{R}_+. \tag{2}$$

These functions determine the sector-specific capital-to-labour ratios $\kappa_j(\omega)$, given a wage-rental ratio ω. *Factor-intensity reversals* arise if Ω_x and Ω_y intersect. Specifically, if $\Omega_x(k) < \Omega_y(k)$, then

$$\kappa_y(\omega) < k < \kappa_x(\omega) \quad \text{for all } \omega \in \big(\Omega_x(k), \Omega_y(k)\big). \tag{3}$$

Vice versa, if $\Omega_x(k) > \Omega_y(k)$, then

$$\kappa_x(\omega) < k < \kappa_y(\omega) \quad \text{for all } \omega \in \big(\Omega_y(k), \Omega_x(k)\big). \tag{4}$$

The class of production functions with a constant elasticity of factor substitution (CES) satisfies Assumption 1. This class features factor-intensity reversals and, in general, violates one of the Inada (1963) conditions, a rather restrictive property typically assumed in the literature.

For any wage-rental ratio ω, the *production elasticity* in sector j is determined by a function $\varepsilon_j : \mathbb{R}_{++} \to \mathbb{R}$, defined by

$$\varepsilon_j(\omega) := \frac{f_j'(\kappa_j(\omega))\kappa_j(\omega)}{f_j(\kappa_j(\omega))}.$$

The *elasticity of factor substitution* in sector j is determined by the function $\sigma_j : \mathbb{R}_{++} \to \mathbb{R}_+$, defined by [2]

$$\sigma_j(\omega) := \frac{\kappa_j'(\omega)\omega}{\kappa_j(\omega)}. \tag{5}$$

Example 1 (CES production functions). *CES production functions $f_j : \mathbb{R} \to \mathbb{R}_+$ are defined by*

$$f_j(k) = \begin{cases} \dfrac{a_j}{b_j}\, k^{b_j} & \text{if } \sigma_j = 1 \\[2ex] \dfrac{a_j}{b_j}\left[b_j k^{\frac{\sigma_j - 1}{\sigma_j}} + (1 - b_j) \right]^{\frac{\sigma_j}{\sigma_j - 1}} & \text{if } \sigma_j \neq 1 \end{cases}, \tag{6}$$

where $a_j > 0$ scales the total factor productivity, $0 < b_j < 1$ stipulates the factor-income distribution, and $\sigma_j > 0$ determines the constant elasticity of factor substitution. The case of Cobb–Douglas production functions, $\sigma_j = 1$, is the only case in which both Inada conditions are satisfied. The MRTS

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functions $\Omega_j : \mathbb{R}_+ \to \mathbb{R}_+$, $j = x, y$ *take the form*

$$\Omega_j(k) = \frac{1 - b_j}{b_j} k^{\frac{1}{\sigma_j}}.$$

The relative factor demand functions $\kappa_j : \mathbb{R}_+ \to \mathbb{R}_+$, $j = x, y$ *are*

$$\kappa_j(\omega) = \left(\frac{b_j}{1 - b_j} \omega \right)^{\sigma_j}. \tag{7}$$

Unless $\sigma_x = \sigma_y$, *exactly one factor-intensity reversal occurs at*

$$\overline{\omega} := \left(\frac{b_x}{1 - b_x} \right)^{\frac{\sigma_x}{\sigma_y - \sigma_x}} \left(\frac{1 - b_y}{b_y} \right)^{\frac{\sigma_y}{\sigma_y - \sigma_x}}.$$

Specifically, for $\sigma_x < \sigma_y$,

$$\kappa_y(\omega) \gtreqless \kappa_x(\omega) \iff \omega \gtreqless \overline{\omega},$$

where $\overline{k} = \kappa_x(\overline{\omega}) = \kappa_y(\overline{\omega})$ *is the critical value at which the factor intensities revert.*

2.2 Temporary equilibria

A feasible allocation in period t is a list of relative factor inputs $(k_t^x, k_t^y, l_t^x, l_t^y) \geq 0$ that satisfies

$$l_t^x + l_t^y = 1 \quad \text{and} \quad l_t^x k_t^x + l_t^y k_t^y = k_t, \tag{8}$$

where $l_t^x := \frac{L_t^x}{L_t}$ and $l_t^y := \frac{L_t^y}{L_t}$ are the sector-specific labour shares, k_t^x and k_t^y the sector-specific capital-to-labour ratios, and $k_t = \frac{K_t}{L_t}$ is the economy-wide capital-to-labour ratio. A *temporary equilibrium* in period t is a feasible allocation $(k_t^x, k_t^y, l_t^x, l_t^y)$ together with prices (r_t, w_t, p_t) denominated in the investment good at which all markets clear and profits in both sectors a maximised.

Since capital and labour are allowed to move frictionless, both sectors must pay the same factor prices. The first-order conditions for profit maximisation are

$$p_t f_x'(k_t^x) = f_y'(k_t^y) = r_t \quad \text{and} \quad p_t w_x(k_t^x) = w_y(k_t^y) = w_t. \tag{9}$$

Denote output per unit of aggregate labour input in sector $j = x, y$ by $x_t = l_t^x f_x(k_t^x)$ and $y_t = l_t^y f_y(k_t^y)$, respectively. Given the exogenous savings propensity $0 \leq s \leq 1$, the market-clearing condition in the investment-goods market is

$$s[w_t + r_t k_t] = l_t^y f_y(k_t^y) \tag{10}$$

and the market-clearing condition in the consumption-goods market is

$$(1 - s)[w_t + r_t k_t] = p_t l_t^x f_x(k_t^x). \tag{11}$$

Definition 1 (Temporary equilibrium). *Given* k_t, *a temporary equilibrium in period* t *is an allocation* $(k_t^x, k_t^y, l_t^x, l_t^y)$ *and a price vector* $(r_t, w_t, p_t) \gg 0$ *that satisfy (8)–(11).*

It follows from Assumption 1 that for any feasible allocation (8) at prices (r_t, w_t, p_t) that satisfies (9), aggregate factor income is equal to the value of aggregate output, that is,

$$w_t + r_t k_t = p_t x_t + y_t. \tag{12}$$

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A *temporary equilibrium* may now be characterised as follows. Denoting the wage-rental ratio by $\omega_t = \frac{w_t}{r_t}$, the first-order conditions (9) may be rewritten as

$$\Omega_x(k_t^x) = \Omega_y(k_t^y) = \omega_t \quad \text{and} \quad \frac{f_y'(k_t^y)}{f_x'(k_t^x)} = p_t. \tag{13}$$

The relative factor demand functions (2) fulfil the first equation in (13) by construction, so that $k_t^j = \kappa_j(\omega_t), j = x, y$ for any $\omega_t \in \mathbb{R}_+$. As a consequence, all equilibrium prices become functions of the wage-rental ratio, so that

$$r_t = f_y'(\kappa_y(\omega_t)), \quad w_t = w_y(\kappa_y(\omega_t)), \quad \text{and} \quad p_t = p_x(\omega_t) := \frac{f_y'(\kappa_y(\omega_t))}{f_x'(\kappa_x(\omega_t))}. \tag{14}$$

Given an economy-wide capital-to-labour ratio $k_t \in \mathbb{R}_+$, there are two cases, the '*generic*' case $\Omega_x(k_t) \neq \Omega_y(k_t)$ and the '*non-generic*' case $\Omega_x(k_t) = \Omega_y(k_t)$. Following Ritschel and Wenzelburger (2024), for each $k_t \in \mathbb{R}_+$, set

$$\Omega_{\min}(k_t) := \min\{\Omega_x(k_t), \Omega_y(k_t)\} \quad \text{and} \quad \Omega_{\max}(k_t) := \max\{\Omega_x(k_t), \Omega_y(k_t)\}.$$

For each $k_t > 0$ with $\Omega_x(k_t) \neq \Omega_y(k_t)$, define the two sector-specific *labour-share functions*

$$\ell_j(k_t, \cdot) : [\Omega_{\min}(k_t), \Omega_{\max}(k_t)] \to [0, 1], \quad j = x, y, \tag{15}$$

by setting

$$\ell_x(k_t, \omega) = \frac{\kappa_y(\omega) - k_t}{\kappa_y(\omega) - \kappa_x(\omega)} \quad \textit{and} \quad \ell_y(k_t, \omega) = \frac{k_t - \kappa_x(\omega)}{\kappa_y(\omega) - \kappa_x(\omega)},$$

respectively.[3] It is straightforward to verify that any list

$$\big(\kappa_x(\omega_t), \kappa_y(\omega_t), l_x(k_t, \omega_t), l_y(k_t, \omega_t)\big) \text{ with } \big(f_y'(\kappa_y(\omega_t)), w_y(\kappa_y(\omega_t)), p_x(\omega_t)\big),$$

where $\omega_t \in \big[\Omega_{\min}(k_t), \Omega_{\max}(k_t)\big]$, fulfils (8) and (9) and hence is a feasible factor allocation. The remaining market-clearing conditions (10) and (11) in the two goods markets are satisfied as follows. For any k_t with $\Omega_x(k_t) \neq \Omega_y(k_t)$, define the *excess demand function for investment goods* $T(k_t, \cdot) : \big[\Omega_{\min}(k_t), \Omega_{\max}(k_t)\big] \to \mathbb{R}$, by setting

$$T(k_t, \omega) := s\big[w_y(\kappa_y(\omega)) + f_y'(\kappa_y(\omega))k_t\big] - \ell_y(k_t, \omega) f_y(\kappa_y(\omega)). \tag{16}$$

Invoking Walras' law, the next lemma provides a characterisation of temporary equilibria.

Lemma 1 (Chracterisation of temporary equilibria). *Let Assumption 1 be satisfied. Then the following holds true.*

(i) If $\Omega_x(k_t) \neq \Omega_y(k_t)$, then

$$\big(\kappa_x(\omega_t), \kappa_y(\omega_t), \ell_x(k_t, \omega_t), \ell_y(k_t, \omega_t)\big) \textit{ with } \big(f_y'(\kappa_y(\omega_t)), w_y(\kappa_y(\omega_t)), p_x(\omega_t)\big)$$

is a temporary equilibrium if and only if there exists $\omega_t \in \big[\Omega_{\min}(k_t), \Omega_{\max}(k_t)\big]$ such that

$$T(k_t, \omega_t) = 0. \tag{17}$$

(ii) If $\Omega_x(k_t) = \Omega_y(k_t)$, then

$$(k_t, k_t, 1 - s, s) \textit{ with } \big(f_y'(k_t), w_y(k_t), p_x(\omega_t)\big),$$

where $\omega_t = \Omega_y(k_t)$, is a temporary equilibrium.

Lemma 1 states that a temporary equilibrium is determined by a wage-rental ratio $\Omega_{\min}(k_t) latbreak \leq \omega_t \leq \Omega_{\max}(k_t)$ for which the market for investment goods clears. In the non-generic case, the labour share in the investment-goods market is $l_t^y = s$.

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Proposition 1 (Existence of temporary equilibria). *Let Assumption 1 be satisfied and* $0 \leq s \leq 1$ *be an arbitrary but fixed savings propensity. Then there exists a uniquely determined continuous map* $\Omega : \mathbb{R}_+ \to \mathbb{R}_+$ *with the following properties.*

(i) For each $k \in \mathbb{R}_+$ *with* $\Omega_x(k) \neq \Omega_y(k)$*, the map* Ω *satisfies*

$$T(k, \Omega(k)) = 0 \quad \textit{and} \quad \Omega_{\min}(k) \leq \Omega(k) \leq \Omega_{\max}(k),$$

where the inequalities are strict whenever $s \in (0, 1)$.

(ii) For each $k \in \mathbb{R}_+$ *with* $\Omega_x(k) = \Omega_y(k)$*, the map* Ω *satisfies* $\Omega(k) = \Omega_y(k)$.

Since a temporary equilibrium in period t is uniquely determined by the wage-rental ratio $\omega_t = \Omega(k_t)$, will refer to Ω as the *temporary equilibrium map* of the economy. We show in Lemma 6, Appendix A.1 that Ω is differentiable.

Example 2 (Temporary equilibria for CES production functions). *Let* $k_t \neq \bar{k}$*, with* $\bar{k}$ *as defined in Example 1, be given and* ω_t *be the equilibrium wage-rental ratio. The equilibrium factor allocation obtains from inserting the corresponding capital-to-labour ratios (7) into the labour-share functions (15). For* $\sigma_x \neq 1$ *and* $\sigma_y \neq 1$*, the equilibrium wage and rental rate are*

$$\begin{aligned} w_y(\kappa_y(\omega_t)) &= \frac{a_y}{b_y}\left(b_y^{\sigma_y}\omega_t^{\sigma_y - 1} + (1-b_y)^{\sigma_y}\right)^{\frac{1}{\sigma_y - 1}} \\ f_y'(\kappa_y(\omega_t)) &= \frac{a_y}{b_y}\left(b_y^{\sigma_y} + (1-b_y)^{\sigma_y}\omega_t^{1-\sigma_y}\right)^{\frac{1}{\sigma_y - 1}}. \end{aligned} \tag{18}$$

The consumption-goods price is

$$p_x(\omega_t) = \frac{\frac{a_y}{b_y}\left(\alpha_y^{\sigma_y} + (1-b_y)^{\sigma_y}\omega_t^{1-\sigma_y}\right)^{\frac{1}{\sigma_y - 1}}}{\frac{a_x}{b_x}\left(b_x^{\sigma_x} + (1-b_x)^{\sigma_x}\omega_t^{1-\sigma_x}\right)^{\frac{1}{\sigma_x - 1}}}. \tag{19}$$

2.3 Capital accumulation

Applying the identity (2) twice, aggregate factor income per capita at any $(k, \omega) \in \mathbb{R}^2_{++}$ satisfies

$$w_y(\kappa_y(\omega)) + f_y'(\kappa_y(\omega))k = [\omega + k]f_y'(\kappa_y(\omega)) = \left[\frac{\omega + k}{\omega + \kappa_y(\omega)}\right] f_y(\kappa_y(\omega)). \tag{20}$$

The economy's *propensity to invest* in a temporary equilibrium may now be defined as a function of the economy-wide capital intensity $\mathcal{S} : \mathbb{R}_+ \to \mathbb{R}_+$ by setting[4]

$$\mathcal{S}(k) := s\left[\frac{\Omega(k) + k}{\Omega(k) + \kappa_y(\Omega(k))}\right]. \tag{21}$$

Market clearing requires the propensity to invest to be equal to the labour share of the investment-goods sector, so that

$$\mathcal{S}(k) = \begin{cases} \ell_y(k, \Omega(k)) & \text{if } \Omega_x(k) \neq \Omega_y(k) \\ s & \text{if } \Omega_x(k) = \Omega_y(k) \end{cases}. \tag{22}$$

Assuming that capital depreciates at the rate $0 \leq \delta \leq 1$ and that labour supply grows at the rate $n > -1$ so that $L_{t+1} = (1+n)L_t$, the capital-to-labour ratio in period $t+1$ is equal to the sum of depreciated capital and investments in period t. Given an initial capital-to-labour ratio $k_0 \geq 0$, the

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evolution of the capital-to-labour ratios $k_t = \frac{K_t}{L_t}$ is thus governed by the *time-one map* $G : \mathbb{R}_+ \to \mathbb{R}_+$, defined by

$$k_{t+1} = G(k_t) := \tfrac{1}{1+n}\big[(1-\delta)k_t + \mathcal{S}(k_t) f_y(\kappa_y(\Omega(k_t)))\big], \quad t \geq 0. \tag{23}$$

We will show next that our parsimonious growth model (23) allows for multiple steady states, cycles of any order, and complex dynamics. Since the maps f_y, κ_y, and Ω are all strictly increasing, all non-linearities of the map G must originate from the economy's propensity to invest $\mathcal{S}$.

Lemma 2 (Properties of $\mathcal{S}$). *The propensity to invest $\mathcal{S}$ takes the form*

$$\mathcal{S}(k) = \frac{s}{s + (1-s)h(\Omega(k))}, \quad k \in \mathbb{R}_{++}, \tag{24}$$

where $h : \mathbb{R}_{++} \to \mathbb{R}_+$ is defined by

$$h(\omega) := \frac{\omega + \kappa_y(\omega)}{\omega + \kappa_x(\omega)}. \tag{25}$$

Moreover, $\mathcal{S}'(k) < 0$ if and only if $h'(\Omega(k)) > 0$.

In the proof of Lemma 2 in Appendix A.1, we will show with equation (50) that the function h may be rewritten as

$$h(\omega) = \frac{f_y(\kappa_y(\omega))}{p_x(\omega) f_x(\kappa_x(\omega))} = \frac{1 - \varepsilon_x(\omega)}{1 - \varepsilon_y(\omega)}, \quad \omega \in \mathbb{R}_{++}. \tag{26}$$

Thus, Lemma 2 implies that the propensity to invest is a decreasing function of the economy-wide capital-to-labour ratio whenever the investment-good output $f_y(\kappa_y(\omega))$ increases faster with an increase in the wage-rental ratios than the value of the consumption-good output $p_x(\omega) f_x(\kappa_x(\omega))$. Stated differently, a price effect is responsible for a decreasing propensity to invest. It follows from (22) that the equilibrium labour-share of the investment-goods sector must then be a non-increasing function of the capital-to-labour ratio.[5] The classical one-sector Solow–Swan growth model excludes such a nonlinearity because the propensity to invest is constant. With two sectors, investment is solely generated out of income earned in the investment-goods sector and (20) shows that factor incomes no longer add up to output of that sector. Therefore, a negative effect of an increase in k on capital income may dominate the positive effect on labour income.

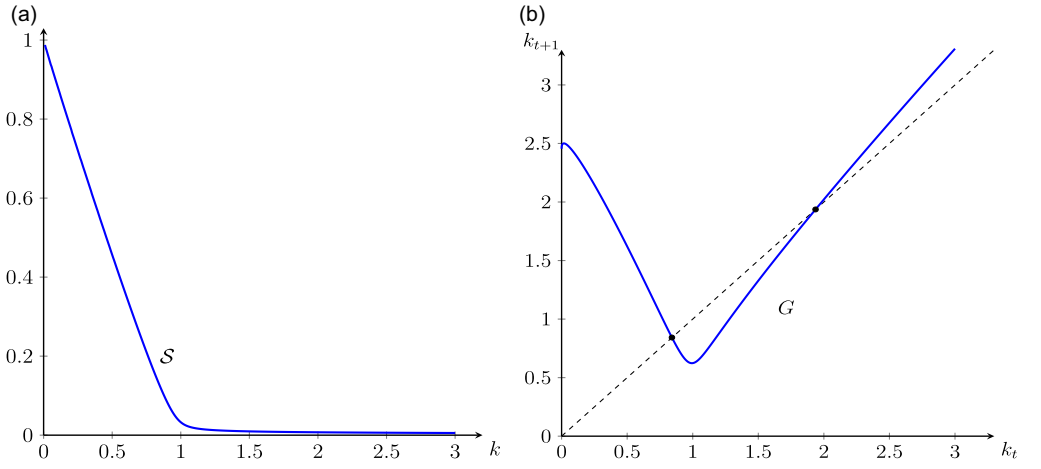
Example 3 (Propensity to invest for CES production functions). *Suppose that both production functions are of the CES type specified in Example 1. Inserting the capital intensities (7) into (25) yields*

$$h(\omega) = \frac{1 + \left(\frac{b_y}{1-b_y}\right)^{\sigma_y} \omega^{\sigma_y - 1}}{1 + \left(\frac{b_x}{1-b_x}\right)^{\sigma_x} \omega^{\sigma_x - 1}}. \tag{27}$$

Since by Lemma 6, Appendix A.1 the temporary equilibrium map Ω is strictly increasing, it follows from (27) that for $\sigma_x < 1 \leq \sigma_y$, the propensity to invest $\mathcal{S}$ is strictly decreasing with $\lim_{k \to 0} \mathcal{S}(k) = 1$. This case, therefore, has the potential for non-monotonic time-one maps G. For $\sigma_x \geq 1 \geq \sigma_y$, the propensity to invest $\mathcal{S}$ is non-decreasing. The corresponding time-one maps G will therefore be strictly increasing. In Section 3 below, it will be shown that G is strictly increasing whenever $\sigma_x \geq 1$. The analysis of the remaining case $\sigma_x, \sigma_y < 1$ is tedious and must be considered elsewhere.

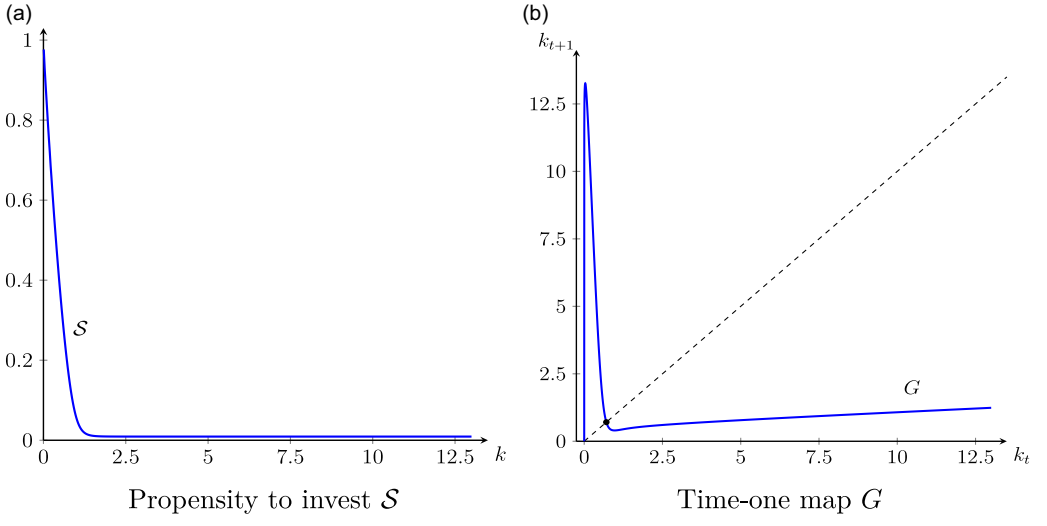
Example 3 is illustrated in Figures 1–3 with three specific parametrisations of CES production functions. These figures show that non-monotonic time-one maps G obtain whenever $\mathcal{S}$ is decreasing sufficiently fast. Figures 1 and 2 portray two parametrisations with non-monotonic

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a_x	b_x	σ_x	a_y	b_y	σ_y	n	δ	s	$\overline{k}$
1	0.5	0.025	5	0.4	1.5	0.1	0.69	0.03	1.0104

Figure 1. Case A. CES production functions, $\sigma_x < 1 < \sigma_y$.

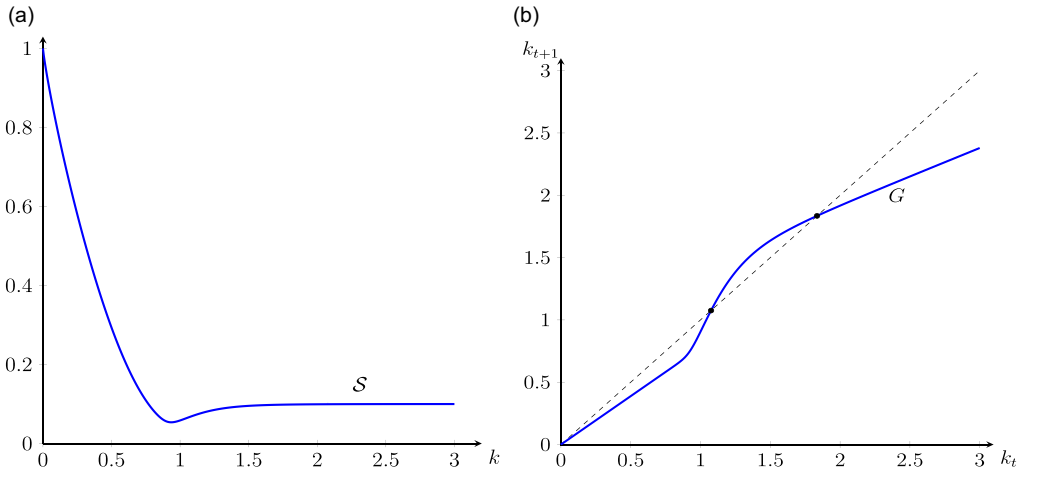


a_x	b_x	σ_x	a_y	b_y	σ_y	n	δ	s	$\overline{k}$
1	0.9	0.1	3	0.085	1	0	0.95	0.01	1.6622

Figure 2. Case B. CES production functions, $\sigma_x < \sigma_y = 1$.

time-one maps G; Figure 3 portrays a parametrisation in which $\mathcal{S}$ may be decreasing while G is nevertheless increasing. Since in all three cases $\sigma_x < \sigma_y$, we know from Example 1 that the investment-goods sector is more capital intensive for all k above the critical value $\overline{k}$. The figures suggest that G is increasing for $k > \overline{k}$ in all three cases.

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Propensity to invest $\mathcal{S}$ Time-one map G

a_x	b_x	σ_x	a_y	b_y	σ_y	n	δ	s	$\overline{k}$
1	0.01	0.025	3.5	0.99	0.2	0.1	0.5	0.1	0.7690

Figure 3. Case C. CES production functions, $\sigma_x < \sigma_y < 1$.

3. Steady states and long-run behaviour

In this section, we will investigate the existence, uniqueness, and stability of steady states and explore the scope for unbounded growth and development traps. To rule out uninteresting scenarios, we assume throughout the remainder of this article that $n + \delta > 0$.

A steady state is a capital-to-labour ratio k^* that satisfies $k^* = G(k^*)$. It follows from (23) that positive steady states $k^* > 0$ are solutions to the equation

$$\phi(k) = \frac{n+\delta}{s}, \tag{28}$$

where $\phi : \mathbb{R}_{++} \to \mathbb{R}_+$ is defined by

$$\phi(k) := \frac{\mathcal{S}(k)}{s} \cdot \frac{f_y\big(\kappa_y(\Omega(k))\big)}{k}. \tag{29}$$

The intermediate-value theorem guarantees the existence of $0 < k^* < \infty$ if

$$\lim_{k\to\infty} \phi(k) < \tfrac{n+\delta}{s} < \lim_{k\to 0} \phi(k). \tag{30}$$

In the proof of Theorem 1 in Appendix A.2, it will be shown that

$$f_y'(\kappa_y(\Omega(k))) < \phi(k) < \frac{1}{s} \frac{f_y(\kappa_y(\Omega(k)))}{\kappa_y(\Omega(k))} \quad \text{for all } k \in \mathbb{R}_{++}. \tag{31}$$

Since by Assumption 1, $\lim_{\omega\to 0} \kappa_y(\omega) = 0$ and $\lim_{\omega\to\infty} \kappa_y(\omega) = \infty$, and by Lemma 6, Appendix A.1, $\lim_{k\to 0} \Omega(k) = 0$ and $\lim_{k\to\infty} \Omega(k) = \infty$, the following existence result is intuitively clear.

Theorem 1 (Existence of positive steady states). *Let Assumption 1 be satisfied and assume that*

$$(i) \quad \lim_{k\to 0} f_y'(k) > \frac{n+\delta}{s} \qquad \textit{and} \qquad (ii) \quad \lim_{k\to\infty} \frac{f_y(k)}{k} < n + \delta.$$

Then there exists a positive steady state $k^* = G(k^*) > 0$.

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The existence conditions in Theorem 1 are sufficient but not necessary. They are fulfilled, in particular, by all production functions f_y that satisfy both Inada conditions.

Example 4 (Existence of positive steady states for CES production functions). *The limits of the marginal product of capital f_j' for CES production functions are*

$$\lim_{k\to 0} f_j'(k) = \begin{cases} a_j b_j^{\frac{1}{\sigma_j - 1}} & \text{if } \sigma_j < 1 \\ \infty & \text{if } \sigma_j \geq 1 \end{cases} \quad \text{and} \quad \lim_{k\to \infty} f_j'(k) = \begin{cases} 0 & \text{if } \sigma_j \leq 1 \\ a_j b_j^{\frac{1}{\sigma_j - 1}} & \text{if } \sigma_j > 1 \end{cases}. \tag{32}$$

Since $\lim_{k\to\infty} f(k)/k = \lim_{k\to\infty} f'(k)$, Theorem 1 is applicable.

The examples in Figures 1 and 3 suggest that there must exist scenarios with multiple positive steady states. Condition (28) implies that a positive steady state is uniquely determined if ϕ is strictly decreasing at any steady state $k^* > 0$. It turns out that the relative sizes of the production elasticities and the elasticity of factor substitution in producing consumption goods detemine the uniqueness of positive steady states. The following theorem generalises a well-known result in Drandakis (1963).

Theorem 2 (Uniqueness of positive steady states). *Under the hypotheses of Theorem 1, a positive steady state of G is uniquely determined if each $k^* = G(k^*) > 0$ with $\omega^* = \Omega(k^*)$ satisfies either*

$$(i) \quad \sigma_x(\omega^*) \geq \varepsilon_y(\omega^*) \quad or \quad (ii) \quad \varepsilon_x(\omega^*) \geq \varepsilon_y(\omega^*).$$

Since f_y is strictly concave, Condition (i) is automatically satisfied if the elasticity of substitution in the consumption-goods sector is always larger than unity, that is, if $\sigma_x(\omega) \geq 1$ for all $\omega > 0$. This case obtains for all CES production functions with $\sigma_x \geq 1$ and, in particular, for all Cobb–Douglas production functions.

Cycles and complex dynamics requires the time-one map G to be decreasing around a steady state. To address the question of whether or not such time-one maps are possible, we define an auxiliary function $g : \mathbb{R}_+ \to \mathbb{R}$ by setting

$$g(\omega) := \varepsilon_x(\omega)\big[1 - \sigma_x(\omega)\big] - \varepsilon_y(\omega). \tag{33}$$

Lemma 3 (Monotonicity of G). *Let Assumption 1 be satisfied. Then $G'(k) > 0$ if $\omega = \Omega(k)$ satisfies*

$$g(\omega) < \left(\frac{s}{1-s}\right) \frac{\sigma_y(\omega)\varepsilon_y(\omega)[1 - \varepsilon_y(\omega)]}{1 - \varepsilon_x(\omega)}. \tag{34}$$

Specifically, Condition (34) holds if one of the following two conditions is satisfied: either

$$(i) \quad \sigma_x(\omega) \geq 1 \quad or \quad (ii) \quad \varepsilon_y(\omega) \geq \varepsilon_x(\omega).$$

If, in addition, capital depreciates fully, $\delta = 1$, then Condition (34) is also necessary.

Since g is bounded from above and the r.h.s. in (34) is non-negative, Condition (34) is satisfied whenever the savings propensity s is sufficiently large or if $g(\omega) < 0$. Since $\kappa_y(\omega) \geq \kappa_x(\omega)$ if and only if $\varepsilon_y(\omega) \geq \varepsilon_x(\omega)$, Condition (ii) implies that G is increasing whenever the investment-goods sector is more capital intensive. In particular, if the consumption-goods sector is governed by a CES production function with $\sigma_x \geq 1$, it follows from Lemma 3 that the dynamics of the model are monotone.

The following stability result is an immediate consequence of Lemma 3.

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Theorem 3 (Asymptotic stability of positive steady states). *Under the hypotheses of Theorem 1, a positive steady state $k^* > 0$ of G is asymptotically stable if $\omega^* = \Omega(k^*)$ satisfies one of the following conditions: either*

$$(i) \quad \sigma_x(\omega^*) \geq 1 \quad or \quad (ii) \quad \varepsilon_y(\omega^*) \geq \varepsilon_x(\omega^*).$$

If, in addition, $\sigma_x(\omega) \geq 1$ for all $\omega > 0$, then k^* is asymptotically stable, globally on $\mathbb{R}_{++}$.

Theorem 3 implies that cycles and complex behaviour require a steady state k^* in which the substitution elasticity of the consumption-goods sector is less than unity or the consumption-goods sector is more capital intensive than the investment-goods sector. The second condition is well known, see Boldrin and Deneckere (1990) and Galor (1992).

Example 5 (Monotonicity with CES production functions). *We analyse the shape of G in Case A and in Case B, depicted in Figures 1 and 2, respectively. Since $\sigma_x < \sigma_y$, the investment-goods sector is more capital intensive for all $k > \bar{k}$, so that in both cases, the time-one map G is increasing for all $k > \bar{k}$. Inserting the expressions (68) provided in Appendix A.3, the map (33) takes the form*

$$g(\omega) = \frac{1-\sigma_x}{1+\left(\frac{1-b_x}{b_x}\right)^{\sigma_x}\omega^{1-\sigma_x}} - \frac{1}{1+\left(\frac{1-b_y}{b_y}\right)^{\sigma_y}\omega^{1-\sigma_y}}$$

and the r.h.s. of (34) the form

$$\chi(\omega) = \frac{\frac{s}{1-s}\sigma_y\left[1+\left(\frac{b_x}{1-b_x}\right)^{\sigma_x}\omega^{\sigma_x-1}\right]}{2+\left(\frac{1-b_y}{b_y}\right)^{\sigma_y}\omega^{1-\sigma_y}+\left(\frac{b_y}{1-b_y}\right)^{\sigma_y}\omega^{\sigma_y-1}}.$$

For $\sigma_x < 1 \leq \sigma_y$, the map g is strictly decreasing with $\lim_{\omega\to 0} g(\omega) \leq 1-\sigma_x$, while $\lim_{\omega\to 0}\chi(\omega) = \infty$ if, in addition, $\sigma_x + \sigma_y < 2$. This shows that in both cases, the time-one maps must be increasing in a neighbourhood of the origin.

A development trap occurs if the origin is an asymptotically stable steady state, so that there exists $\underline{k} > 0$ such that any growth path $k_{t+1} = G(k_t)$, $t = 0, 1, \ldots$ starting from some $k_0 < \underline{k}$ converges to 0. An economy admits unbounded growth if there exists $\underline{k} \geq 0$ such that the capital-to-labour ratios $k_{t+1} = G(k_t)$, $t = 0, 1, \ldots$ starting from some $k_0 \geq \underline{k}$ become arbitrarily large. Since

$$G(k) \gtreqless k \quad \Longleftrightarrow \quad \phi(k) \gtreqless \frac{n+\delta}{s}, \tag{35}$$

a necessary condition for the existence of a development trap is $\lim_{k\to 0}\phi(k) < \frac{n+\delta}{s}$, while a necessary condition for unbounded growth is $\lim_{k\to\infty}\phi(k) > \frac{n+\delta}{s}$. Observe that the two boundary functions in (31) are strictly decreasing. Together with (35), this observation yields two conditions under which no positive steady state exists.

Lemma 4. *Let Assumption 1 be satisfied. Then the following holds true.*

(i) Global development trap. If $\lim_{k\to 0} f_y(k)/k < n+\delta$, then $G(k) < k$ for all $k > 0$.

(ii) Global unbounded growth. If $\lim_{k\to\infty} f_y'(k) > \frac{n+\delta}{s}$, then $G(k) > k$ for all $k > 0$.

4. Qualitative dynamics for CES technologies

In this section, we illustrate the theoretical results of Section 3 with CES production functions. Analytical results are developed in Section 4.1, numerical results of a variety of simulation exercises are presented in Section 4.2.

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4.1 Steady states, development trap, and unbounded growth

For CES production functions, the map ϕ defined in (29) takes an analytically tractable form so that all theoretical results of Section 3 can be applied. The limits for the marginal products (32) suggest to distinguish between three cases: $\sigma_y < 1$, $\sigma_y > 1$, and $\sigma_y = 1$. For $\sigma_y \neq 1$, two mutually exclusive sets of model parameters $(b_x, \sigma_x, a_y, b_y, \sigma_y, n, \delta, s)$ turn out to be crucial. First, the case

$$\frac{n+\delta}{a_y b_y^{\frac{1}{\sigma_y - 1}}} < \begin{cases} s & \text{if } \sigma_x < 1 \\ \dfrac{s}{(1-b_x)s + b_x} & \text{if } \sigma_x = 1 \\ 1 & \text{if } \sigma_x > 1 \end{cases}. \tag{36}$$

Second, the case

$$\frac{n+\delta}{a_y b_y^{\frac{1}{\sigma_y - 1}}} > \begin{cases} s & \text{if } \sigma_x < 1 \\ \dfrac{s}{(1-b_x)s + b_x} & \text{if } \sigma_x = 1 \\ 1 & \text{if } \sigma_x > 1 \end{cases}. \tag{37}$$

Observe that the factor productivity a_x of the consumption-goods sector does not appear in the above expressions. From Example 2, we infer that any change in a_x affects solely output quantity and price of the consumption good.

Proposition 2. *Consider the case with CES production functions (6) and let $\sigma_y < 1$. Then the following holds true.*

(i) The origin is a steady state, $G(0) = 0$.

(ii) If the model parameters $(b_x, \sigma_x, a_y, b_y, \sigma_y, n, \delta, s)$ satisfy Condition (36), then the origin is unstable and there exists at least one positive steady state $G(k^*) = k^* > 0$. This steady state is the only positive steady state if $\sigma_x \geq 1$.

(iii) If the model parameters $(b_x, \sigma_x, a_y, b_y, \sigma_y, n, \delta, s)$ satisfy Condition (37), then the origin is asymptotically stable. In particular, if $a_y b_y^{\frac{1}{\sigma_y - 1}} < n + \delta$, then the origin is globally asymptotically stable and no positive steady state exists.

(iv) All growth paths are bounded.

Proposition 2 states that for $\sigma_y < 1$, a positive steady state exists whenever the savings propensity s and/or the factor productivity a_y are sufficiently large. Since $f_y(0) = 0$, capital is an essential input for producing investment goods so that the origin is a steady state. A development trap obtains whenever the savings propensity and/or the factor productivity a_y is too low so that Condition (37) is satisfied. Figure 3b portrays an example with an asymptotically stable origin and two positive steady states. Since Condition (37) holds, this example shows that Condition (36) is sufficient but not necessary for the existence of positive steady states. If the factor productivity is too low, then no positive steady state exists and the development trap is globally asymptotically stable.

Proposition 3. *Consider the case with CES production functions (6) and let $\sigma_y > 1$. Then the following holds true.*

(i) The origin is not a steady state as $G(0) = \frac{1}{1+n} f_y(0) > 0$.

(ii) If the model parameters $(b_x, \sigma_x, a_y, b_y, \sigma_y, n, \delta, s)$ satisfy Condition (37), then there exists at least one positive steady state $G(k^*) = k^* > 0$ and all growth paths are bounded. This steady state is uniquely determined if $\sigma_x \geq 1$.

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(iii) If the model parameters $(b_x, \sigma_x, a_y, b_y, \sigma_y, n, \delta, s)$ *satisfy Condition (36), then the model exhibits unbounded growth. If, in particular,* $a_y b_y^{\frac{1}{\sigma_y - 1}} > \frac{n+\delta}{s}$, *then no positive steady state exists and all growth paths are unbounded.*

Proposition 3 states that for $\sigma_y > 1$, a positive steady state exists whenever the savings propensity s and/or the factor productivity a_y is not too large. Condition (36) implies that except for the case $\sigma_x > 1$, unbounded growth occurs if the savings propensity s in relation to factor productivity a_y is sufficiently large. The origin is not a steady state, because capital is not essential for producing investment goods. The example portrayed in Figure 1b demonstrates that Condition (37) is sufficient but not necessary. The time-one map G has two positive steady states even though Condition (37) is violated and unbounded growth is possible.

Lemma 5. *Consider the case with CES production functions (6) and let* $\sigma_y = 1$. *Then the following holds true.*

(i) The origin $0 = G(0)$ *is an unstable steady state.*
(ii) There exists at least one positive steady state $G(k^*) = k^* > 0$.
(iii) All growth paths are bounded.

Lemma 5 states that neither a development trap nor unbounded growth is possible if the output of the investment-goods sector is determined by a Cobb–Douglas production function. This case is illustrated in Figure 2b, which depicts a time-one map G with a hump.

An immediate consequence of the last three results is the following corollary, which summarises conditions under which the dynamics are qualitatively indistinguishable from the dynamics of the classical Solow–Swan model.

Corollary 1. *Consider the case with CES production functions (6) and let* $\sigma_x \geq 1$. *Then* $G' > 0$. *If, in addition, one of the following three conditions,*

$$(i) \quad \sigma_y = 1, \quad (ii) \quad \sigma_y < 1 \text{ and } (36) \text{ holds}, \quad \text{or} \quad (iii) \quad \sigma_y > 1 \text{ and } (37) \text{ holds},$$

is satisfied, then there exists a uniquely determined positive steady state $G(k^*) = k^* > 0$, *which is asymptotically stable on* $\mathbb{R}_{++}$.

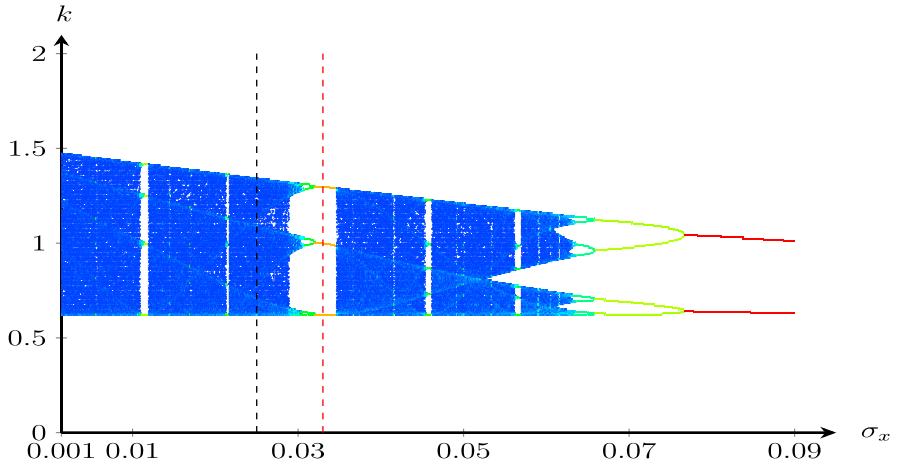
4.2 Numerical evidence for complex dynamics

The simulation results presented next provide numerical evidence for complex dynamics if the substitution elasticity σ_x is sufficiently small.[6]

Figure 1 shows that the time-one map G in Case A has at least two positive steady states, which are both unstable. By Proposition 3, the origin is not a steady state and unbounded growth obtains for all sufficiently large initial values. All growth paths $\{k_t\}_{t=0}^{\infty}$ with initial values k_0 below the larger steady state are bounded. For the initial value $k_0 = 1$, the bifurcation diagrams in Figures 4 and 5 illustrate how changes in the elasticities of factor substitution affect the qualitative dynamics. The growth path corresponding to $\sigma_x = 0.025$ in Figure 4 provides numerical evidence that the model exhibits non-periodic business cycles. For $\sigma_x = 0.033$, Figure 4 shows that the growth path starting in $k_0 = 1$ converges to a period-3 cycle. Similarly for $\sigma_y = 1.4$ instead of $\sigma_y = 1.5$, Figure 5 shows that the growth path starting in $k_0 = 1$ converges to a period-3 cycle as well. By Li and Yorke (1975), the period-3 cycles ensure the presence of topological chaos, implying that cycles of any order and irregular fluctuations exist.

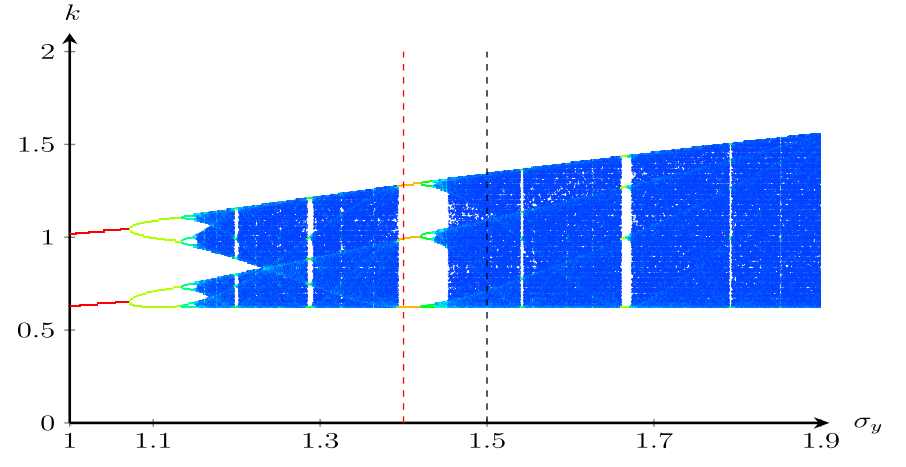
In many cases, however, only the period-3 cycle is numerically detected, because the other patterns exist only for a set of initial values of Lebesgue-measure zero. In contrast to topological chaos, numerical evidence for *observable chaos* is easier to detect. A model is said to exhibit observable

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a_x	b_x	σ_x	a_y	b_y	σ_y	n	δ	s	k_0
1	0.5	$[0.0014, 0.09]$	5	0.4	1.5	0.1	0.69	0.03	1

Figure 4. Case A. Bifurcation over the substitution elasticity σ_x.

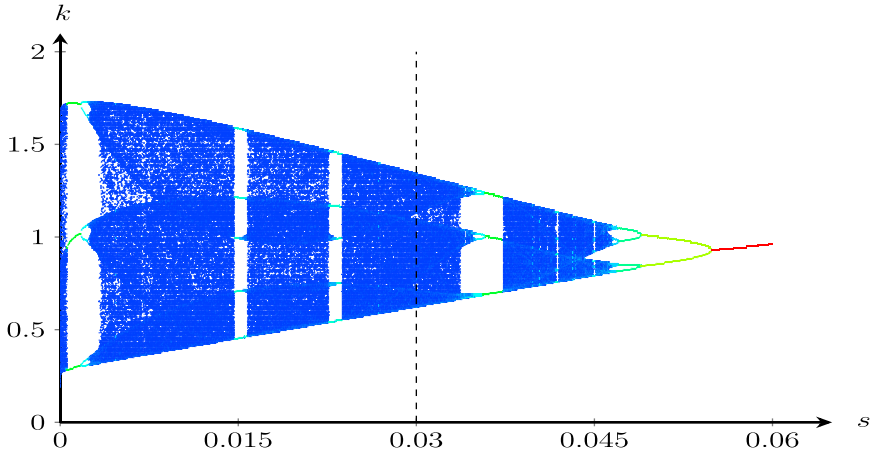


a_x	b_x	σ_x	a_y	b_y	σ_y	n	δ	s	k_0
1	0.5	0.025	5	0.4	$[1, 1.9]$	0.1	0.69	0.03	1

Figure 5. Case A. Bifurcation over the substitution elasticity σ_y.

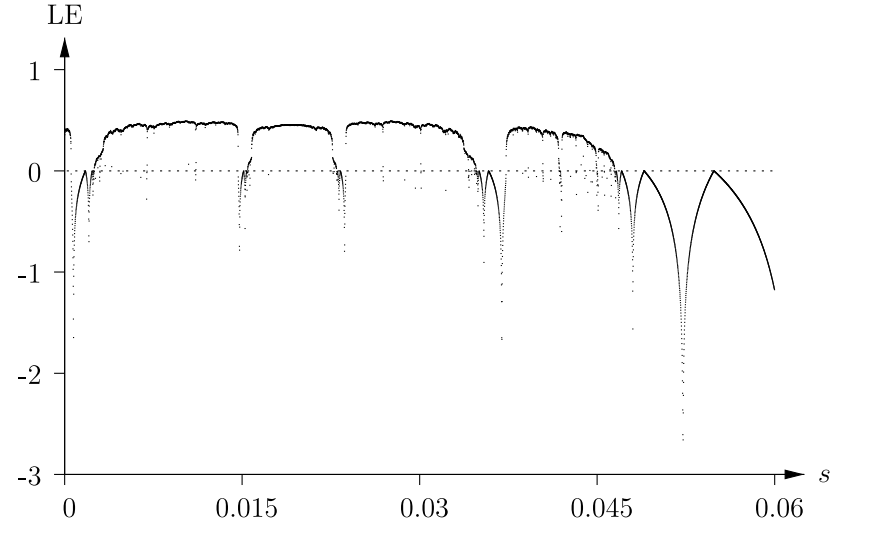
chaos if a so-called *strange attractor* is observed, for example, see Onozaki et al. (2000). These are numerically detected by computing Lyapunov exponents.[7] A positive Lyapunov exponent implies that the growth of capital-to-labour ratios depends sensitively on the initial capital-to-labour ratio. An attractor with such a property is called strange attractor. The bifurcation diagram in Figure 6 indicates that the savings propensity s has to be very small in order for strange attractors to exist. The Lyapunov exponents of the corresponding growth paths over the same range of savings propensities are depicted in Figure 7. The diagrams lead the the conjecture that savings propensities above 5.5% yield stable states, which confirms Lemma 3 and our earlier analysis in Example 5.

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a_x	b_x	σ_x	a_y	b_y	σ_y	n	δ	s	k_0
1	0.5	0.025	5	0.4	1.5	0.1	0.69	$[0.0, 0.06]$	1

Figure 6. Case A. Bifurcation over the savings propensity s.



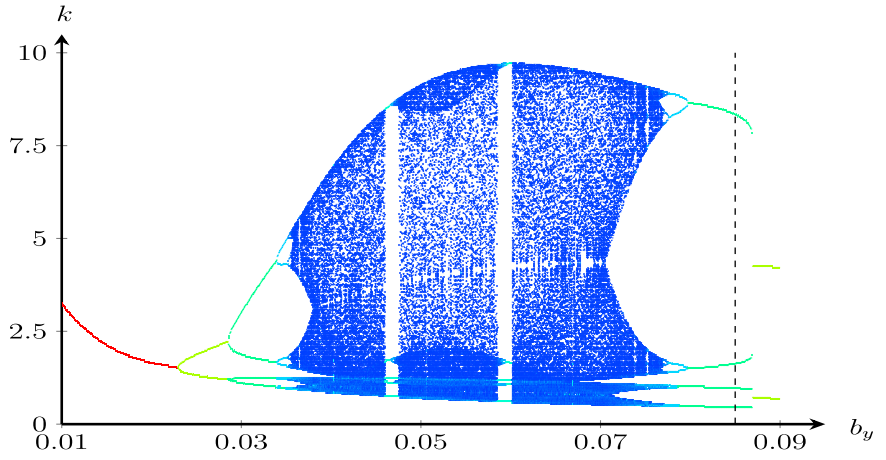
a_x	b_x	σ_x	a_y	b_y	σ_y	n	δ	s	k_0
1	0.5	0.025	5	0.4	1.5	0.1	0.69	$[0.0, 0.06]$	1

Figure 7. Case A. Lyapunov exponents.

In view of Lemma 5, the time-one map for case B depicted in Figure 2b has two steady states, a unique positive steady state and the origin. The dashed vertical line in the bifurcation diagram in Figure 8 shows that a period-4 cycle obtains for $b_y = 0.085$, indicating that both steady states may be unstable. This bifurcation diagram provides numerical evidence for topological chaos whenever the income distribution parameter b_y assumes values, for example, in the neighbourhood of 0.05.[8]

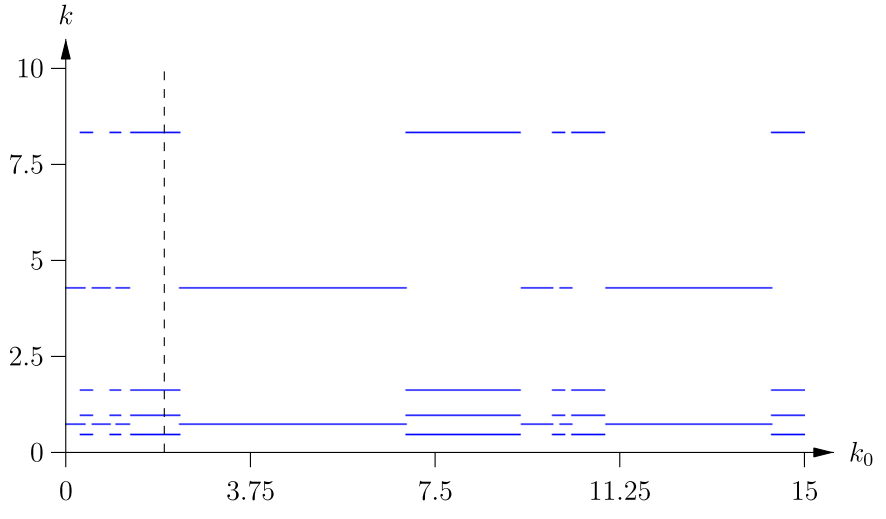
Figure 9 provides numerical evidence for the dependence on initial conditions. For initial capital-to-labour ratios k_0 ranging from 0 to 15, the corresponding growth paths either converge

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a_x	b_x	σ_x	a_y	b_y	σ_y	n	δ	s	k_0
1	0.9	0.1	3	$[0.01, 0.09]$	1	0	0.95	0.01	2

Figure 8. Case B. Bifurcation diagramm over $b_y \in [0.01, 0.09]$.

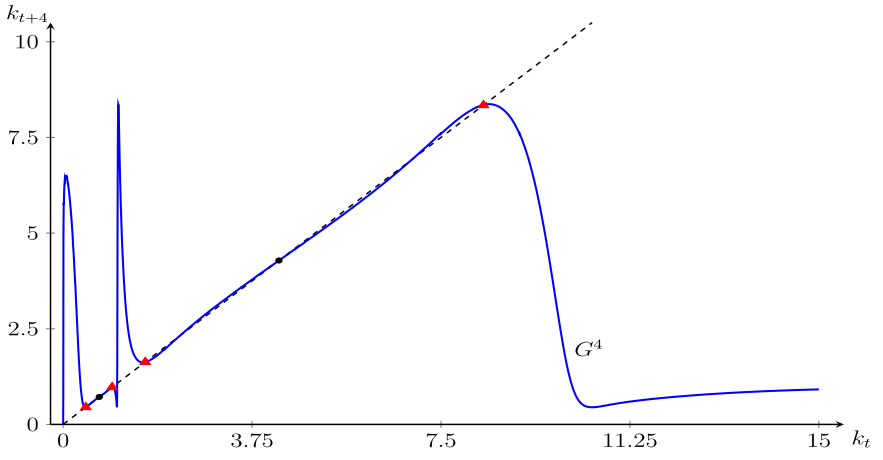


a_x	b_x	σ_x	a_y	b_y	σ_y	n	δ	s	k_0
1	0.9	0.1	3	0.085	1	0	0.95	0.01	$[0, 15]$

Figure 9. Case B. Dependence on initial conditions $k_0 \in [0, 15]$.

to a period-2 cycle or to a period-4 cycle. The shape of the fourth iterate G^4 of the time-one map G portrayed in Figure 10 provides some more insight into this phenomenon. A subcritical pitchfork bifurcation occurs when increasing the parameter b_y from 0.08 to 0.09. The period-2 cycle indicated by two dots, which already exists for $b_y = 0.08$, becomes asymptotically stable. Decreasing b_y from 0.09 to 0.08, an asymptotically stable period-4 cycle, marked by four triangles, emerges in addition to the existing period-2 cycle via a saddle-node bifurcation.

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a_x	b_x	σ_x	a_y	b_y	σ_y	n	δ	s
1	0.9	0.1	3	0.085	1	0	0.95	0.01

Figure 10. Case B. Fourth iterate of G.

5. Concluding remarks

This article demonstrates that variations in production and substitution elasticities between the production of two goods can lead to a non-linear propensity to invest. For CES production functions, this may result in a unimodal time-one map, reminiscent of a tent map, which exhibits well-known dynamic properties. Complex dynamics arise solely from factor price equalisation and income effects triggered by the equilibrium price of the investment good. This effect is absent in the one-sector model. Neither heterogeneities – such as variations in consumer characteristics – nor utility-maximising consumers are required to generate the observed qualitative dynamics of the model. The novel parametrisation of equilibrium factor allocations opens up the possibility for the exploration of multi-sectoral optimal growth and overlapping generations models with more general production functions. Ritschel and Wenzelburger (2025) address some aspects of technological transformation and fiscal policies.

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Notes

1 Leontief production functions do not satisfy Assumption 1 because they are not differentiable.

2 The marginal rate of technical substitution $\Omega_j(k)$ is the slope of the isoquant that runs through the point (kL, L), where $L > 0$ is arbitrary but fixed. Typically, the elasticity of substitution in (kL, L) is defined by $\widetilde{\sigma}_j(k) := \frac{\Omega_j(k)}{\Omega_j'(k)k}$.

Differentiating (2), it is straightforward to see that $\sigma_j(\omega) = \widetilde{\sigma}_j(\kappa_j(\omega))$.

3 In view of (3) and (4), the functions (15) are well defined except for $\Omega_x(k_t) = \Omega_y(k_t)$.

4 The importance of this map was first recognised by an associate editor.

5 The propensity to invest is constant whenever the two MRTS coincide, $\Omega_x(k) = \Omega_y(k)$ of all $k \in \mathbb{R}_+$.

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6 All bifurcation diagrams have been generated with the software package *Simtools*, see Schilling and Wenzelburger (2026). For each parameter value, the time-one map was iterated 5,000 times with the last 5% of the values being plotted.
7 Recall that a Lyapunov exponent describes the exponential growth rate at which two growth paths drift apart for an infinitesimally small difference in the initial conditions, that is, in initial capital-to-labour ratios. See Eckmann and Ruelle (1985) for details. The C++ source code of the program used to compute the Lyapunov exponent is available from the authors upon request.
8 Numerical evidence for observable chaos by means of Lyapunov exponents is available upon request.

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Appendix A. Mathematical Appendix

A.1 Proofs of Section 2

Proof of Lemma 1, *cf. Ritschel and Wenzelburger (2024).* (i) Let $\Omega_x(k_t) \neq \Omega_y(k_t)$. Because of (3) and (4), the labour-share functions (15) are well defined. It is straightforward to verify that

$$k_t = \ell_x(k_t, \omega)\kappa_x(\omega) + \ell_y(k_t, \omega)\kappa_y(\omega) \quad \text{and} \quad 1 = \ell_x(k_t, \omega) + \ell_y(k_t, \omega)$$

for all $\omega \in \left[\Omega_{\min}(k_t), \Omega_{\max}(k_t)\right]$. Hence, all factor allocations are feasible. For any price vector (14), both sectors maximise profits. Walras' law implies that the consumption-goods market clears if

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the investment-goods market clears, that is, if $T(k_t, \omega_t) = 0$. Hence, a temporary equilibrium exists whenever $\omega_t \in \left[\Omega_{\min}(k_t), \Omega_{\max}(k_t)\right]$ satisfies (17).

(ii) If $\Omega_x(k_t) = \Omega_y(k_t) = \omega_t$, then $\kappa_x(\omega_t) = \kappa_y(\omega_t) = k_t$. The first market-clearing condition in (8) is now satisfied whenever the second condition holds. Inserting ω_t into (14), both firms maximise profits at prices $r_t = f_y'(k_t)$, $w_t = w_y(k_t)$, and $p_t = \frac{f_y'(k_t)}{f_x'(k_t)}$. It now follows directly from $f_y(k_t) = w_y(k_t) + f_y'(k_t)k_t$ that (10) holds for $l_t^y = s$. Walras' law implies that the consumption-goods market clears for $l_t^x = 1 - s$. □

Proof of Proposition 1. *Step 1 (Existence).* Let $k_t > 0$ with $\Omega_x(k_t) \neq \Omega_y(k_t)$ be given and $0 < s < 1$ be arbitrary but fixed. Then $T(k_t, \omega) = 0$ if and only if

$$s[\omega + k_t] - \ell_y(k_t, \omega)[\omega + \kappa_y(\omega)] = 0. \tag{38}$$

Observe that

$$\ell_y\big(k_t, \Omega_x(k_t)\big) = 0 \quad \text{and} \quad \ell_y\big(k_t, \Omega_y(k_t)\big) = 1. \tag{39}$$

It follows from (38) and (39) that

$$T(k_t, \Omega_y(k_t)) < 0 < T(k_t, \Omega_x(k_t)).$$

Hence, there exists $\omega_t \in \big(\Omega_{\min}(k_t), \Omega_{\max}(k_t)\big)$ such that $T(k_t, \omega_t) = 0$.

Equations (38) and (39) imply that $\omega_t = \Omega_x(k_t)$ solves $T(k_t, \omega) = 0$ if $s = 0$ and $\omega_t = \Omega_y(k_t)$ solves $T(k_t, \omega) = 0$ if $s = 1$.

Step 2 (Uniqueness). A temporary equilibrium ω_t is given by a solution to

$$\widetilde{T}(k_t, \omega) := \kappa_x(\omega) - k_t + s\frac{\omega + k_t}{\omega + \kappa_y(\omega)}[\kappa_y(\omega) - \kappa_x(\omega)] = 0. \tag{40}$$

Uniqueness obtains if $\frac{\partial \widetilde{T}}{\partial \omega}(k_t, \omega_t) > 0$ for any ω_t that solves (40). Using the fact that in any temporary equilibrium

$$s\,\frac{\omega_t + k_t}{\omega_t + \kappa_y(\omega_t)} = \ell_y(k_t, \omega_t),$$

the partial derivative of $\widetilde{T}$ computes as

$$\begin{aligned}
\frac{\partial \widetilde{T}}{\partial \omega}(k_t, \omega_t) &= \left[1 - s\,\frac{\omega_t + k_t}{\omega_t + \kappa_y(\omega_t)}\right]\kappa_x'(\omega_t) + s\left[\frac{\omega_t + k_t}{\omega_t + \kappa_y(\omega_t)}\right]\left[\frac{\omega_t + \kappa_x(\omega_t)}{\omega_t + \kappa_y(\omega_t)}\right]\kappa_y'(\omega_t) \\
&\quad + s\,[\kappa_y(\omega_t) - k_t]\,\frac{\kappa_y(\omega_t) - \kappa_x(\omega_t)}{[\omega_t + \kappa_y(\omega_t)]^2} \\
&= \ell_x(k_t, \omega_t)\,\kappa_x'(\omega_t) + \ell_y(k_t, \omega_t)\left[\frac{\omega_t + \kappa_x(\omega_t)}{\omega_t + \kappa_y(\omega_t)}\right]\kappa_y'(\omega_t) \\
&\quad + s\,\ell_x(k_t, \omega_t)\left[\frac{\kappa_y(\omega_t) - \kappa_x(\omega_t)}{\omega_t + \kappa_y(\omega_t)}\right]^2.
\end{aligned}$$

For any $\Omega_{\min}(k_t) \leq \omega_t \leq \Omega_{\max}(k_t)$, at least one of the three terms in the last expression is strictly positive, while all of them are non-negative. Hence, $\frac{\partial \widetilde{T}}{\partial \omega}(k_t, \omega_t) > 0$.

Step 3 (Temporary equilibrium map). Observe first that the excess demand function T is only well defined on the open set

$$\mathcal{D} := \big\{(k, \omega) \in \mathbb{R}^2_+ \;\big|\; \Omega_{\min}(k) < \omega < \Omega_{\max}(k)\big\}.$$

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Fix any $k_0 \in \mathbb{R}_+$ with $\Omega_x(k_0) \neq \Omega_y(k_0)$. Employing the results of the previous steps, we may define a map $\omega_0 = \Omega(k_0)$ as the solution to $T(k_0, \omega) = 0$. Since $\Omega(k_0) \in \big(\Omega_{\min}(k_0), \Omega_{\max}(k_0)\big)$ and T is continuous, Ω is a continuous function locally around k_0. Since k_0 was arbitrary, Ω is continuous at any $k \in \mathbb{R}_+$ with $\Omega_x(k) \neq \Omega_y(k)$.

We extend the map Ω by setting $\Omega(\bar{k}) := \Omega_y(\bar{k})$ for any $\bar{k}$ with $\Omega_x(\bar{k}) = \Omega_y(\bar{k})$. Since $\bar{k} = \kappa_y(\Omega_y(\bar{k}))$ and $w_y(\bar{k}) + f_y'(\bar{k})\bar{k} = f_y(\bar{k})$, the investment-goods market clears for $I^y = s$. The continuity of Ω_x and Ω_y on $\mathbb{R}_+$ together with the local continuity of Ω imply that Ω is continuous on all of $\mathbb{R}_+$. □

The following properties of the temporary equilibrium map are important.

Lemma 6 (Properties of Ω). *Let the hypotheses of Assumption 1 be satisfied and the savings propensity $0 \leq s \leq 1$ be arbitrary but fixed. Then the temporary equilibrium map $\Omega : \mathbb{R}_+ \to \mathbb{R}_+$ has the following properties:*

(i) *Ω is twice continuously differentiable and strictly increasing. Moreover,*

$$\lim_{k \to 0} \Omega(k) = 0 \quad \textit{and} \quad \lim_{k \to \infty} \Omega(k) = \infty. \tag{41}$$

(ii) *The inverse of Ω is given by the map $\Omega^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$, defined by*

$$\Omega^{-1}(\omega) = \frac{\kappa_x(\omega)\kappa_y(\omega) + \omega[(1-s)\kappa_x(\omega) + s\kappa_y(\omega)]}{s[\omega + \kappa_x(\omega)] + (1-s)[\omega + \kappa_y(\omega)]}. \tag{42}$$

Proof of Lemma 6. *Step 1(Limits).* Since by Proposition 1, $\Omega(k) \in \big[\Omega_{\min}(k), \Omega_{\max}(k)\big]$ for all $k > 0$, the limits (41) are a direct consequence of Assumption 1 (ii).

Step 2 (Differentiablity). Let $\omega > 0$ with $\kappa_x(\omega) \neq \kappa_y(\omega)$ be arbitrary but fixed. Solving the temporary equilibrium condition

$$\widetilde{T}(k, \omega) = 0$$

defined in (40) for k, yields

$$k = \psi(\omega) := \frac{\kappa_x(\omega)\kappa_y(\omega) + \omega[(1-s)\kappa_x(\omega) + s\kappa_y(\omega)]}{s[\omega + \kappa_x(\omega)] + (1-s)[\omega + \kappa_y(\omega)]}. \tag{43}$$

Observe that $\psi(\omega) = \kappa_y(\omega)$ whenever $\kappa_y(\omega) = \kappa_x(\omega)$. It follows from the definition of Ω that $\Omega(\psi(\omega)) = \omega$ for all $\omega > 0$, so that the map ψ is the well-defined inverse of Ω on $\mathbb{R}_{++}$. Since ψ is differentiable, so is Ω.

Step 3 (Monotonicity). Implicit differentiation of Ω using (40), implies that

$$\Omega'(k) = \frac{-\frac{\partial \widetilde{T}}{\partial k}(k, \Omega(k))}{\frac{\partial \widetilde{T}}{\partial \omega}(k, \Omega(k))} \quad \text{for all } \; k > 0 \text{ with } \Omega_x(k) \neq \Omega_y(k). \tag{44}$$

It was shown in the proof of Proposition 1 that the denominator of (44) is positive. Since

$$\frac{\partial \widetilde{T}}{\partial k}(k, \Omega(k)) = -1 + s\frac{\kappa_y(\Omega(k)) - \kappa_x(\Omega(k))}{\kappa_y(\Omega(k)) + \Omega(k)} = -\left[\frac{\kappa_x(\Omega(k)) + \Omega(k)}{k + \Omega(k)}\right] < 0,$$

it follows that $\Omega'(k) > 0$ for all $k > 0$ with $\Omega_x(k) \neq \Omega_y(k)$. Since $\Omega_x' > 0$ and $\Omega_y' > 0$, the map Ω must be strictly increasing on $\mathbb{R}_+$. This establishes (i).

Step 3 (Invertibility). The expression for the inverse of Ω has been established in (43). □

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Proof of Lemma 2. Using (42), it is straightforward to see that

$$\Omega^{-1}(\omega)+\omega=\frac{[\omega+\kappa_x(\omega)][\omega+\kappa_y(\omega)]}{s[\omega+\kappa_x(\omega)]+(1-s)[\omega+\kappa_y(\omega)]}. \tag{45}$$

Inserting (45) into (21) and rearranging, the propensity to invest takes the form

$$\mathcal{S}(k)=\frac{s}{s+(1-s)\frac{\Omega(k)+\kappa_y(\Omega(k))}{\Omega(k)+\kappa_x(\Omega(k))}}. \tag{46}$$

Since by Lemma 6 $\Omega'>0$, we obtain

$$\mathcal{S}'(k)<0 \iff h'(\Omega(k))>0, \tag{47}$$

where h was defined in (25). Observe that $\Omega_j(\kappa_j(\omega))=\omega$ implies

$$\omega+\kappa_j(\omega)=\frac{f_j(\kappa_j(\omega))}{f_j'(\kappa_j(\omega))}, \quad j=x,y \tag{48}$$

and

$$\frac{\varepsilon_j(\omega)}{\kappa_j(\omega)}=\frac{\varepsilon_j(\omega)}{\omega}\frac{\Omega_j(\kappa_j(\omega))}{\kappa_j(\omega)}=\frac{1-\varepsilon_j(\omega)}{\omega}, \quad j=x,y. \tag{49}$$

Using the definition of p_x in (14), the last two equations yield

$$h(\omega)=\frac{\omega+\kappa_y(\omega)}{\omega+\kappa_x(\omega)}=\frac{f_y(\kappa_y(\omega))}{p_x(\omega)f_x(\kappa_x(\omega))}=\frac{\kappa_y(\omega)}{\varepsilon_y(\omega)}\frac{\varepsilon_x(\omega)}{\kappa_x(\omega)}=\frac{1-\varepsilon_x(\omega)}{1-\varepsilon_y(\omega)}. \tag{50}$$

□

A.2 Proofs of Section 3

Proof of Theorem 1. The existence of a positive steady state k^* follows from the intermediate-value theorem once the two inequalities in (30) are established.

Step 1. Using the fact that Ω is invertible by Lemma 6, we define the map

$$\varphi(\omega):=\phi(\Omega^{-1}(\omega))=\frac{\Omega^{-1}(\omega)+\omega}{\Omega^{-1}(\omega)}\cdot\frac{f_y(\kappa_y(\omega))}{\omega+\kappa_y(\omega)}, \quad \omega\in\mathbb{R}_{++}. \tag{51}$$

Inserting (42), (45), and (48) into (51) yields

$$\varphi(\omega)=\frac{f_y'(\kappa_y(\omega))}{s\frac{\kappa_y(\omega)}{\omega+\kappa_y(\omega)}+(1-s)\frac{\kappa_x(\omega)}{\omega+\kappa_x(\omega)}}=\frac{f_y'(\kappa_y(\omega))}{s\varepsilon_y(\omega)+(1-s)\varepsilon_x(\omega)}. \tag{52}$$

Since the two production elasticities are less than 1, we obtain

$$\phi(k)=\varphi(\Omega(k))>f_y'(\kappa_y(\Omega(k))) \quad \text{for all } k\in\mathbb{R}_{++}. \tag{53}$$

Taking limits, the first inequality in (30) now follows from Condition (i) and the fact that by Lemma 6, $\lim_{k\to 0}\Omega(k)=0$.

Step 2. The map ϕ may be rewritten as

$$\phi(k)=\frac{\mathcal{S}(k)\kappa_y(\Omega(k))}{k}\cdot\frac{f_y(\kappa_y(\Omega(k)))}{s\,\kappa_y(\Omega(k))}.$$

In the non-generic case $k\in\mathbb{R}_+$ with $\Omega_x(k)=\Omega_y(k)$, Proposition 1 and (46) imply that

$$\frac{\mathcal{S}(k)\kappa_y(\Omega(k))}{k}=\mathcal{S}(k)=s. \tag{54}$$

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In the generic case $k \in \mathbb{R}_+$ with $\Omega_x(k) \neq \Omega_y(k)$, inserting the labour-share function (15) into (22) yields

$$\frac{\mathcal{S}(k)\kappa_y(\Omega(k))}{k} = \frac{l_y(k,\Omega(k))\kappa_y(\Omega(k))}{k} = \frac{k\kappa_y(\Omega(k)) - \kappa_x(\Omega(k))\kappa_y(\Omega(k))}{k\kappa_y(\Omega(k)) - k\kappa_x(\Omega(k))}.$$

Since by Proposition 1 (i), $\Omega(k) \in \big[\Omega_{\min}(k), \Omega_{\max}(k)\big]$ it follows from the factor-intensity reversals (3) and (4) that

$$\frac{\mathcal{S}(k)\kappa_y(\Omega(k))}{k} \leq 1 \quad \text{for all } k \in \mathbb{R}_+ \text{ with } \Omega_x(k) \neq \Omega_y(k). \tag{55}$$

Combing (54) and (55) yields

$$\phi(k) < \frac{1}{s}\frac{f_y(\kappa_y(\Omega(k)))}{\kappa_y(\Omega(k))} \quad \text{for all } k \in \mathbb{R}_+. \tag{56}$$

Taking limits, the second inequality in (30) now follows from Condition (ii) and the fact that by Lemma 6, $\lim_{k\to\infty} \Omega(k) = \infty$. □

Proof of Theorem 2. If $\phi'(k) < 0$ for any steady state $k = G(k) > 0$, then a positive steady state is uniquely determined. Since by Lemma 6, the map Ω is strictly increasing, this condition holds if we can show that the map φ defined in (51) is strictly decreasing. Applying the product rule to (52), it follows that $\varphi'(\omega) < 0$ if and only if

$$\frac{f_y''(\kappa_y(\omega))\kappa_y'(\omega)}{s\varepsilon_y(\omega) + (1-s)\varepsilon_x(\omega)} < \frac{f_y'(\kappa_y(\omega))[s\varepsilon_y'(\omega) + (1-s)\varepsilon_x'(\omega)]}{[s\varepsilon_y(\omega) + (1-s)\varepsilon_x(\omega)]^2}$$

or, equivalently, if and only if

$$E_{f_y'}(\kappa_y(\omega))\frac{\sigma_y(\omega)}{\omega}[s\varepsilon_y(\omega) + (1-s)\varepsilon_x(\omega)] < s\varepsilon_y'(\omega) + (1-s)\varepsilon_x'(\omega). \tag{57}$$

The elasticities of substitution defined in (5) satisfy

$$\sigma_j(\omega) = \frac{\kappa_j'(\omega)\omega}{\kappa_j(\omega)} = \frac{\Omega_j(\kappa_j(\omega))}{\Omega_j'(\kappa_j(\omega))\kappa_j(\omega)} = \frac{1-\varepsilon_j(\omega)}{-E_{f_j'}(\kappa_j(\omega))},$$

so that

$$E_{f_y'}(\kappa_y(\omega)) = -\left(\frac{1-\varepsilon_j(\omega)}{\sigma_j(\omega)}\right). \tag{58}$$

Using (58), the chain rule yields

$$\varepsilon_j'(\omega) = \varepsilon_j(\omega)\big[1 - \varepsilon_j(\omega) + E_{f_j'}(\kappa_j(\omega))\big]\frac{\sigma_j(\omega)}{\omega} = \varepsilon_j(\omega)\big[1-\varepsilon_j(\omega)\big]\frac{\sigma_j(\omega)-1}{\omega}. \tag{59}$$

Inserting (58) and (59) now shows that (57) holds if and only if

$$0 < s\varepsilon_y(\omega)[1-\varepsilon_y(\omega)]\sigma_y(\omega) + (1-s)\varepsilon_x(\omega)\big[[1-\varepsilon_x(\omega)]\sigma_x(\omega) + \varepsilon_x(\omega) - \varepsilon_y(\omega)\big]. \tag{60}$$

Equation (60) is satisfied if the term in the large brackets on the r.h.s. of (60) is non-negative. This is the case if and only if

$$\sigma_x(\omega) \geq \frac{\varepsilon_y(\omega) - \varepsilon_x(\omega)}{1-\varepsilon_x(\omega)}. \tag{61}$$

Since

$$\varepsilon_y(\omega) > \frac{\varepsilon_y(\omega) - \varepsilon_x(\omega)}{1-\varepsilon_x(\omega)} \quad \text{for all } \omega > 0,$$

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it follows from (61) that $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly decreasing if either $\sigma_x(\omega) \geq \varepsilon_y(\omega)$ or $\varepsilon_x(\omega) \geq \varepsilon_y(\omega)$ holds for all $\omega \in \mathbb{R}_+$. Specifically, if either Condition (i) or Condition (ii) holds for any $\omega^* = \Omega(k^*)$ with $k^* = G(k^*) > 0$, then k^* is uniquely determined if it exists. □

Proof of Lemma 3. The time-one map (23) may be written as

$$G(k) = \frac{1}{1+n}\big[(1-\delta)k + s\,H(\Omega(k))\big], \quad k \in \mathbb{R}_{++}\,,$$

where $H : \mathbb{R}_{++} \to \mathbb{R}_{++}$ is defined by

$$H(\omega) := \frac{f_y(\kappa_y(\omega))}{\omega + \kappa_y(\omega)}\big(\Omega^{-1}(\omega) + \omega\big).$$

Inserting (45) and (48), H takes the form

$$H(\omega) = \frac{f_y'(\kappa_y(\omega))\kappa_x(\omega)\kappa_y(\omega)}{s\varepsilon_y(\omega)\kappa_x(\omega) + (1-s)\varepsilon_x(\omega)\kappa_y(\omega)}. \tag{62}$$

By Lemma 6, $\Omega' > 0$. Hence $G'(k) > 0$ if $H'(\omega) > 0$. Taking the derivative of (62), $H'(\omega) > 0$ is equivalent to

$$\frac{\frac{\mathrm{d}}{\mathrm{d}\omega}\big(f_y'(\kappa_y(\omega))\kappa_x(\omega)\kappa_y(\omega)\big)\omega}{f_y'(\kappa_y(\omega))\kappa_x(\omega)\kappa_y(\omega)} > \frac{\frac{\mathrm{d}}{\mathrm{d}\omega}\big(s\varepsilon_y(\omega)\kappa_x(\omega) + (1-s)\varepsilon_x(\omega)\kappa_y(\omega)\big)\omega}{s\varepsilon_y(\omega)\kappa_x(\omega) + (1-s)\varepsilon_x(\omega)\kappa_y(\omega)}. \tag{63}$$

The l.h.s. of (63) can be rewritten as

$$\frac{\frac{\mathrm{d}}{\mathrm{d}\omega}\big(f_y'(\kappa_y(\omega))\kappa_x(\omega)\kappa_y(\omega)\big)\omega}{f_y'(\kappa_y(\omega))\kappa_x(\omega)\kappa_y(\omega)} = \sigma_x(\omega) + \sigma_y(\omega) - \big[1 - \varepsilon_y(\omega)\big]. \tag{64}$$

Using (59), the numerator on the r.h.s. of (63) may be expanded into the form

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\omega}\big(s\varepsilon_y(\omega)\kappa_x(\omega) + (1-s)\varepsilon_x(\omega)\kappa_y(\omega)\big)\omega \\
&\quad = s\varepsilon_y(\omega)\kappa_x(\omega)\big[\big[1-\varepsilon_y(\omega)\big](\sigma_y(\omega)-1) + \sigma_x(\omega)\big] \\
&\qquad + (1-s)\varepsilon_x(\omega)\kappa_y(\omega)\big[\big[1-\varepsilon_x(\omega)\big](\sigma_x(\omega)-1) + \sigma_y(\omega)\big].
\end{aligned} \tag{65}$$

Inserting (64) and (65) and rearranging, it is straightforward to see that Condition (63) holds if and only if

$$\begin{aligned}
&s\varepsilon_y(\omega)\kappa_x(\omega)\big[\big[1-\varepsilon_y(\omega)\big][\sigma_y(\omega)-1] + 1 - \varepsilon_y(\omega) - \sigma_y(\omega)\big] \\
&+ (1-s)\varepsilon_x(\omega)\kappa_y(\omega)\big[\big[1-\varepsilon_x(\omega)\big][\sigma_x(\omega)-1] + 1 - \varepsilon_y(\omega) - \sigma_x(\omega)\big] < 0.
\end{aligned} \tag{66}$$

Rearranging, (66) is equivalent to

$$(1-s)\big[\varepsilon_x(\omega)\big[1-\sigma_x(\omega)\big] - \varepsilon_y(\omega)\big] < s\,\sigma_y(\omega)\frac{\varepsilon_y(\omega)^2}{\kappa_y(\omega)}\frac{\kappa_x(\omega)}{\varepsilon_x(\omega)}. \tag{67}$$

Summarising, $H'(\omega) > 0$ if and only if (67) holds. The desired inequality (34) follows from inserting (49) into the r.h.s. of (67). For $\delta = 1$, $G'(k) > 0$ if and only if $H'(\Omega(k)) > 0$. □

Proof of Lemma 4. (i) The second inequality in (31) implies that

$$\lim_{k\to 0}\phi(k) < \lim_{k\to 0}\tfrac{1}{s}\tfrac{f_y(k)}{k} \quad \text{for all } 0 < s < 1.$$

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Since $k \mapsto f_y(k)/k$ is strictly decreasing, (31) also implies $\phi(k) < \frac{n+\delta}{s}$ for all $k > 0$ and all $0 < s < 1$. It follows from (35) that $G(k) < k$ for all $k > 0$.

(ii) The first inequality in (31) implies

$$\lim_{k\to\infty} \phi(k) > \lim_{k\to\infty} f_y'(k).$$

Since $k \mapsto f_y'(k)$ is strictly decreasing, (31) also implies $\phi(k) > \frac{n+\delta}{s}$ for all $k > 0$. It follows from (35) that $G(k) > k$ for all $k > 0$. □

A.3 Proofs of Section 4

Proof of Proposition 2. (i) By (6), we have $f_y(0) = 0$ for $\sigma_y \leq 1$ and from (22) we know that $0 \leq \mathcal{S}(k) \leq 1$ for all $k \in \mathbb{R}_{++}$. Hence, $G(0) = 0$.

(ii) The claim is established by verifying the steady-state conditions (30). Using the function φ defined in (52), the first condition in (30) is obtained as follows. The production elasticities are

$$\varepsilon_j(\omega) = \frac{1}{1 + \left(\frac{1-b_j}{b_j}\right)^{\sigma_j} \omega^{1-\sigma_j}}, \quad j = x, y. \tag{68}$$

The limits of the elasticities (68) are

$$\lim_{\omega\to 0} \varepsilon_j(\omega) = \begin{cases} 1 & \text{if } \sigma_j < 1 \\ b_j & \text{if } \sigma_j = 1 \\ 0 & \text{if } \sigma_j > 1 \end{cases} \quad \text{and} \quad \lim_{\omega\to\infty} \varepsilon_j(\omega) = \begin{cases} 0 & \text{if } \sigma_j < 1 \\ b_j & \text{if } \sigma_j = 1 \\ 1 & \text{if } \sigma_j > 1 \end{cases}. \tag{69}$$

Inserting (68) and using the limits (69), the limit of the denominator in (52) is

$$\lim_{\omega\to 0} \big[s\varepsilon_y(\omega) + (1-s)\varepsilon_x(\omega)\big] = \begin{cases} 1 & \text{if } \sigma_x, \sigma_y < 1 \\ s + (1-s)b_x & \text{if } \sigma_y < 1 = \sigma_x \\ s & \text{if } \sigma_y < 1 < \sigma_x \end{cases}. \tag{70}$$

Recall that $\lim_{\omega\to 0} \kappa_y(\omega) = 0$ and that by Lemma 6, $\lim_{k\to 0} \Omega(k) = 0$. Using the limits (32) of f_y', it now follows from (52) and (70) that

$$\lim_{k\to 0} \phi(k) = \lim_{\omega\to 0} \varphi(\omega) = \begin{cases} a_y b_y^{\frac{1}{\sigma_y - 1}} & \text{if } \sigma_x < 1 \\ a_y b_y^{\frac{1}{\sigma_y - 1}} [s + (1-s)b_x]^{-1} & \text{if } \sigma_x = 1 \\ a_y b_y^{\frac{1}{\sigma_y - 1}} s^{-1} & \text{if } \sigma_x > 1 \end{cases}. \tag{71}$$

Hence, the first condition in (30) is satisfied if and only if (36) holds.

The second condition in (30) follows from Theorem 1 and the limits (32), noting that for $\sigma_y < 1$,

$$\lim_{k\to\infty} \frac{f(k)}{k} = \lim_{k\to\infty} f'(k) = 0.$$

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This shows that a positive steady state k^* exists. For $\sigma_x \geq 1$, the positive steady state is uniquely determined by Theorem 2 (i). Since $\lim_{k\to 0} \phi(k) > \frac{n+\delta}{s}$, the origin is unstable as $G(k) > k$ for all sufficiently small $k > 0$.

(iii) If (37) holds, then (71) implies $\lim_{k\to 0} \phi(k) < \frac{n+\delta}{s}$ so that the origin is asymptotically stable as $G(k) < k$ for all sufficiently small $k > 0$. The global asymptotic stability of the origin follows from Lemma 4 (i) and the fact that for $\sigma_y < 1$,

$$\lim_{k\to 0} \frac{f(k)}{k} = \lim_{k\to 0} f'(k) = a_y b_y^{\frac{1}{\sigma_y - 1}}.$$

(iv) Since $\lim_{k\to\infty} \phi(k) = 0$, unbounded growth is ruled out because $G(k) < k$ for all sufficiently large k. □

Proof of Proposition 3. (i) Since $s > 0$ and $\sigma_y > 1$, we know from Example 3 that $\lim_{k\to 0} \mathcal{S}(k) = 1$. Moreover, we read off (6) that $f_y(0) > 0$. Hence $G(0) = \frac{1}{1+n} f_y(0) > 0$.

(ii) The claim is established by verifying the steady-state conditions (30). The first condition in (30) follows from Theorem 1 and the limits (32), noting that $\lim_{k\to 0} f'(k) = \infty$ if $\sigma_y > 1$. To establish the second condition in (30), we again use the function φ defined in (51). Inserting (68) and the limits (69), the limit of the denominator in (52) is

$$\lim_{\omega\to\infty} \left[s\varepsilon_y(\omega) + (1-s)\varepsilon_x(\omega) \right] = \begin{cases} 1 & \text{if } 1 < \sigma_x, \sigma_y \\ s + (1-s)b_x & \text{if } \sigma_x = 1 < \sigma_y \\ s & \text{if } \sigma_x < 1 < \sigma_y \end{cases}. \tag{72}$$

Since $\sigma_y > 1$, it follows from the limits (32) and (72) that

$$\lim_{k\to\infty} \phi(k) = \lim_{\omega\to\infty} \varphi(\omega) = \begin{cases} a_y b_y^{\frac{1}{\sigma_y - 1}} s^{-1} & \text{if } \sigma_x < 1 \\ a_y b_y^{\frac{1}{\sigma_y - 1}} [s + (1-s)b_x]^{-1} & \text{if } \sigma_x = 1 \\ a_y b_y^{\frac{1}{\sigma_y - 1}} & \text{if } \sigma_x > 1 \end{cases}. \tag{73}$$

Thus, the second condition in (30) is satisfied whenever (37) holds. By Theorem 2 (i), a positive steady state is unique if $\sigma_x \geq 1$. Using the limits (73), Condition (37) implies $\lim_{k\to\infty} \phi(k) < \frac{n+\delta}{s}$, implying that $G(k) < k$ for all sufficiently large k. Hence, all growth paths are bounded.

(iii) If $a_y b_y^{\frac{1}{\sigma_y - 1}} > \frac{n+\delta}{s}$, then it follows from the limits (73) and Lemma 4 (ii) that any growth path starting in some $k_0 > 0$ is unbounded. Since $G(0) > 0$, all growth paths are unbounded.

Proof of Lemma 5. Recall that f_y satisfies both Inada conditions if $\sigma_y = 1$.

(i) The existence of the origin as a steady state was established in the first part of the proof of Proposition 2. It follows from (31) and the Inada conditions on f_y that $\lim_{k\to 0} \phi(k) = \infty$, implying that $G(k) > k$ for all sufficiently small $k > 0$. Hence, the origin is unstable.

(ii) The existence of a positive steady state follows directly from (32) and Theorem 1.

(iii) The Inada conditions on f_y and (31) imply that $\lim_{k\to\infty} \phi(k) = 0$, implying that $G(k) < k$ for all sufficiently large k. Hence, all growth paths are bounded. □

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