2

One-dimensional solitons

Field theories in one spatial dimension provide a natural starting point for the study of solitons. Because of the simplifications that result from working in one dimension, many more calculations can be carried through explicitly. In addition, the topological considerations that play an important role in all dimensions are particularly easy to visualize in one-dimensional theories. Although the primary value of these theories is as toy models, some of the results we will obtain find application in the real world. First, there are condensed matter systems that can be treated as essentially one-dimensional, some of which support solitons. Second, some of the one-dimensional solitons that we will find can be trivially extended to higher dimensions, so that a localized one-dimensional soliton can become a planar domain wall in higher dimensions.

2.1 Kinks

The classic example of a soliton in one spatial dimension [1, 2] arises in a theory with a single scalar field ϕ and Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V(\phi), \tag{2.1}$$

where the scalar field potential

$$\begin{aligned} V(\phi) &= -\frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4 + \frac{\lambda}{4}v^4 \\ &= \frac{\lambda}{4}(\phi^2 - v^2)^2. \end{aligned} \tag{2.2}$$

Here m^2 and λ are both positive and

$$v = \sqrt{\frac{m^2}{\lambda}}. \tag{2.3}$$

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For later reference, note that in two spacetime dimensions the scalar field is dimensionless, so that the coupling constants are dimensionful. The dimensionless parameter that signals weak or strong coupling is the ratio λ/m^2.

The Lagrangian is invariant under the transformation $\phi \to -\phi$. However, this symmetry is spontaneously broken, with $V(\phi)$ having two degenerate minima, at $\phi = \pm v$. The constant term in Eq. (2.2), which has no effect on the dynamics, was chosen so that $V = 0$ at these minima. When the theory is quantized, these two minima correspond to two physically equivalent vacua. Choosing either one of them, say $\phi = v$, and then expanding in terms of the shifted field $\phi - v$, one finds that the theory has a single elementary scalar particle, with mass $\sqrt{2}\,m$.

The classical Euler–Lagrange equation of the theory is

$$\frac{d^2\phi}{dt^2} - \frac{d^2\phi}{dx^2} = -\lambda(\phi^2 - v^2)\phi. \tag{2.4}$$

We are particularly interested in static solutions, which satisfy

$$0 = \frac{d^2\phi}{dx^2} - \lambda(\phi^2 - v^2)\phi. \tag{2.5}$$

This is a nonlinear equation, and it may not be obvious from the outset that it has any nonsingular solutions other than the two constant vacuum solutions, $\phi(x) = v$ and $\phi(x) = -v$. To persuade ourselves that it does, note that Eq. (2.5) is also the condition for a configuration $\phi(x)$ to be a stationary point of the potential energy[1]

$$U[\phi(x)] = \int dx \left[\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + V(\phi)\right]. \tag{2.6}$$

This is not surprising. For a system whose kinetic energy is purely quadratic in time derivatives, the static solutions of the equations of motion are just the stationary points of the potential energy; the stable solutions are given by the local minima of U.

Thus, our task is to show that there are configurations other than the vacuum solutions that are local minima of $U[\phi(x)]$. To this end, consider a configuration, such as the one shown in Fig. 2.1, in which $\phi(\infty) = v$ and $\phi(-\infty) = -v$. Unless we have made a remarkably lucky choice, this configuration will not be a solution. This means that it can be smoothly varied in such a way as to lower its potential energy. Continuing this process until a minimum of U is reached will lead us to a static solution. Because a smooth variation cannot change the values of ϕ at spatial infinity, the solution we are led to will have $\phi(\infty) \neq \phi(-\infty)$, and so

[1] It is important to remember that the potential energy includes not just the contribution from the scalar field potential $V(\phi)$, but also that from the spatial gradient terms.

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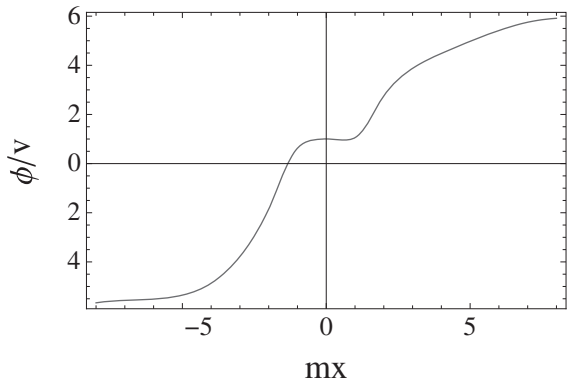


Fig. 2.1. A field configuration that cannot be smoothly deformed to a vacuum solution.

cannot be either of the vacuum solutions. It must instead be a nontrivial spatially varying solution; i.e., the soliton that we are seeking.[2]

Some important properties of this solution are revealed by a rescaling of variables. If we write $\phi = vf$ and $u = mx$, Eq. (2.5) becomes

$$0 = \frac{d^2 f}{du^2} - f(f^2 - 1), \tag{2.7}$$

with $f(\pm\infty) = \pm 1$, while the energy of this static solution takes the form

$$E = \frac{m^3}{\lambda} \int du \left[\frac{1}{2} \left(\frac{df}{du} \right)^2 + \frac{1}{4} (f^2 - 1)^2 \right]. \tag{2.8}$$

It is evident from Eq. (2.7) that $f(u)$ does not contain any explicit factors of m or λ. Hence, its spatial variation is characterized by a distance that is of order unity when measured in terms of u, and thus of order m^{-1} when measured in terms of x. Because the integral on the right-hand side of Eq. (2.8) is also independent of m and λ, it must be of order unity, so the solution has an energy of order m^3/λ.

This is much greater than the mass $\sqrt{2}\, m$ of the elementary scalar when the coupling is weak, and diverges in the limit $\lambda \to 0$. This fact, which is characteristic of solitons in field theory, explains why solitons are not encountered in ordinary perturbative approaches to quantum field theory.

[2] This argument is not rigorous, and must be used with care. Because the space of field configurations is not compact, there need not be any configuration that minimizes the potential energy. Although this does not happen here, we will encounter such a situation when we discuss multisoliton configurations later in this section.

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It is now time to tackle the field equation directly. Multiplying both sides of Eq. (2.5) by $d\phi/dx$ gives

$$0 = \frac{d}{dx}\left[\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 - \frac{\lambda}{4}(\phi^2 - v^2)^2\right]. \tag{2.9}$$

Hence, the quantity in brackets must be independent of x. Evaluating it at $x = \infty$, we see that it actually vanishes. It follows that

$$\frac{d\phi}{dx} = \pm\sqrt{\frac{\lambda}{2}}(\phi^2 - v^2). \tag{2.10}$$

Our boundary conditions require the upper sign. Straightforward integration then gives

$$\phi(x) = v\tanh\left[\frac{m}{\sqrt{2}}(x - x_0)\right], \tag{2.11}$$

where x_0 is a constant of integration. This solution, which is shown in Fig. 2.2, is known as the kink solution; x_0 can be viewed as specifying the position of the kink. The solution obtained by starting with the opposite boundary conditions and taking the lower sign in Eq. (2.10),

$$\phi(x) = -v\tanh\left[\frac{m}{\sqrt{2}}(x - x_0)\right], \tag{2.12}$$

is called the antikink.

The energy density, shown in Fig. 2.3, is the same for both solutions. It is concentrated within a region of width $\sim m^{-1}$ centered about x_0. Outside this region, the field is essentially indistinguishable from that in a vacuum. Although it is a different vacuum on opposite sides of the kink, this is not evident to a local

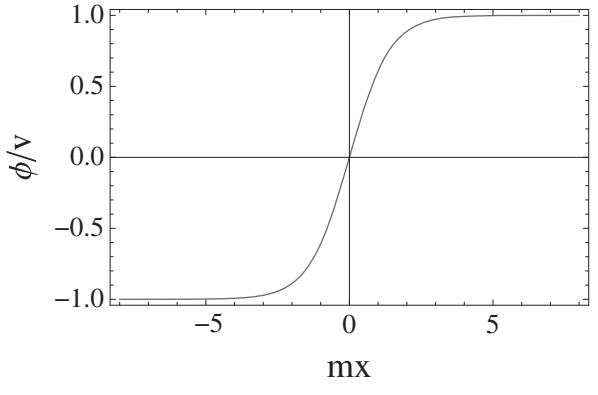


Fig. 2.2. The kink solution of Eq. (2.11), with $x_0 = 0$.

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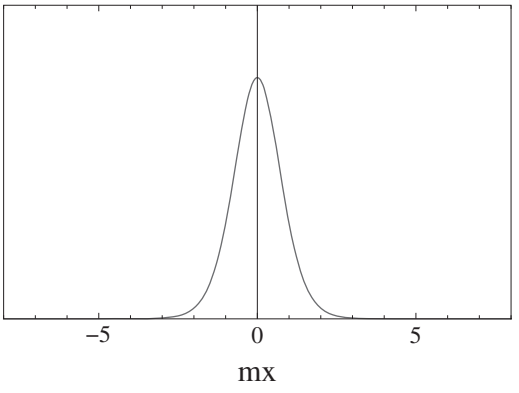


Fig. 2.3. The energy density of the kink solution with $x_0 = 0$.

observer. This localization of the energy suggests that the kink be interpreted as a kind of particle, with its mass given by the total energy of the static solution,

$$M_{\rm cl} = \frac{2\sqrt{2}}{3}\frac{m^3}{\lambda}. \tag{2.13}$$

The subscript here is intended to indicate that this is just the classical approximation to the mass. We will see in the next section that there are quantum corrections to the mass.

If the kink is to be interpreted as a particle, then there should also be solutions corresponding to moving kinks. Lorentz transforming the static solution gives

$$\phi(x,t) = v\tanh\left[\frac{m}{\sqrt{2}}\frac{(x-ut-x_0)}{\sqrt{1-u^2}}\right], \tag{2.14}$$

which describes a Lorentz-contracted kink moving with velocity u. The energy of this solution,

$$\begin{aligned} E &= \int dx\left[\frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 + \frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + V(\phi)\right] \\ &= \frac{M_{\rm cl}}{\sqrt{1-u^2}}, \end{aligned} \tag{2.15}$$

is precisely what is required for a particle with mass $M_{\rm cl}$ moving with velocity u.

The key element for establishing the existence of the kink was that $V(\phi)$ had multiple degenerate vacua, and that the field approached different vacua at the two points of spatial infinity. Because of this intertwining of the topology of the vacua with the topology of spatial infinity, the resulting solitons are known as topological solitons. We can define a topological current

$$J^\mu_{\rm top} = \frac{1}{2v}\,\epsilon^{\mu\nu}\partial_\nu\phi, \tag{2.16}$$

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where $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$ and $\epsilon^{01} = 1$. The antisymmetry of $\epsilon^{\mu\nu}$ guarantees that this current is divergenceless. Its normalization has been chosen so that the corresponding charge,

$$\begin{aligned} Q_{\text{top}} &= \int_{-\infty}^{\infty} dx J^0 = \frac{1}{2v} \int_{-\infty}^{\infty} dx\, \partial_1 \phi \\ &= \frac{1}{2v} \left[\phi(\infty) - \phi(-\infty)\right], \end{aligned} \tag{2.17}$$

is equal to 1 and –1 for the kink and the antikink, respectively. This charge is manifestly conserved, because no finite energy process can change the asymptotic values of the field. In contrast with the conserved charges obtained via Noether's theorem, the topological charge is not directly associated with a symmetry of the Lagrangian.

These methods can be applied to the case of an arbitrary potential $V(\phi)$ with two or more degenerate global minima.[3] As before, let us assume that an additive constant has been used to set $V = 0$ at these minima. Using the fact that $\phi(\pm\infty)$ must each be at one of the minima of the potential, we again find that the static field equation

$$0 = \frac{d^2\phi}{dx^2} - \frac{dV}{d\phi} \tag{2.18}$$

implies that

$$\frac{d\phi}{dx} = \pm\sqrt{2V(\phi)}. \tag{2.19}$$

This yields the useful virial identity

$$\int_{-\infty}^{\infty} dx \left[\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2\right] = \int_{-\infty}^{\infty} dx\, V(\phi) \tag{2.20}$$

that relates the gradient and potential terms in the Lagrangian.

Integrating Eq. (2.19) yields

$$x = \pm \int_{\phi_0}^{\phi} \frac{d\phi}{\sqrt{2V(\phi)}}, \tag{2.21}$$

where $\phi(0) = \phi_0$ is arbitrary. For a smoothly varying potential V, the integral on the right-hand side diverges as $\phi(x)$ approaches any of the global minima. Hence, as x ranges from $-\infty$ to ∞, ϕ must vary monotonically from one global minimum of V to an adjacent one. In particular, it cannot approach the same minimum at both $x = -\infty$ and $x = \infty$; i.e., there are no static solutions with vanishing topological charge and *a fortiori* no static solutions for theories with a single vacuum.

[3] The potential may also have other, higher, local minima.

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An alternative expression for the energy of a kink connecting vacua at ϕ_1 and ϕ_2 can be obtained with the aid of Eqs. (2.19) and (2.20). Assuming $\phi_2 > \phi_1$, we have

$$E = 2\int_{-\infty}^{\infty} dx\, V(\phi) = 2\int_{\phi_1}^{\phi_2} d\phi \left(\frac{d\phi}{dx}\right)^{-1} V(\phi) = \int_{\phi_1}^{\phi_2} d\phi\, \sqrt{2V(\phi)}. \tag{2.22}$$

One particular example of note is the theory [3, 4] with

$$V(\phi) = -\frac{m^4}{\lambda}\left[\cos\left(\frac{\sqrt{\lambda}}{m}\,\phi\right) - 1\right]. \tag{2.23}$$

This has become known as the sine-Gordon theory, the play on the term Klein–Gordon being suggested by its Euler–Lagrange equation,

$$0 = \frac{d^2\phi}{dt^2} - \frac{d^2\phi}{dx^2} + \frac{m^3}{\sqrt{\lambda}}\sin\left(\frac{\sqrt{\lambda}}{m}\,\phi\right). \tag{2.24}$$

The potential is periodic, with degenerate minima at $\phi = (2\pi m/\sqrt{\lambda})N \equiv Nv$, where N is any integer. Expanding about the minimum at $\phi = 0$, we have

$$V = \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\,\phi^4 + \frac{\lambda^2}{6!\,m^2}\,\phi^6 + \cdots. \tag{2.25}$$

From this we see that the mass of the elementary particle is m, while λ determines the strength of the interactions between these particles.

Static kink and antikink solutions connecting adjacent minima can be obtained via Eq. (2.19). In particular, the kink solution interpolating between the vacua at Nv and $(N+1)v$ is [4]

$$\phi(x) = Nv + \frac{2v}{\pi}\,\tan^{-1}\left[e^{m(x-x_0)}\right] \tag{2.26}$$

and has an energy

$$M_{\rm cl} = \frac{8m^3}{\lambda}. \tag{2.27}$$

Although the functional form of the sine-Gordon kink, Eq. (2.26), seems rather different from that of the ϕ^4 kink, Eq. (2.11), the two are actually remarkably similar. This is shown in Fig. 2.4a, where the two types of kinks are plotted together. The couplings and masses have been chosen so that $\tilde{\lambda}/\lambda = \pi^4/8$ and $\tilde{m}/m = \pi/\sqrt{8}$, with tildes indicating the sine-Gordon parameters. The similarity between the kinks can be understood by noting that the two potentials, shown in Fig. 2.4b, are quite close to each other (after a shift in ϕ) in the region between two adjacent minima.

Because the $\lambda\phi^4$ model has only two vacua, its topological charge can only take on the values -1, 0, or 1. Although multisoliton configurations are possible, they must be composed of alternating kinks and antikinks. By contrast, the

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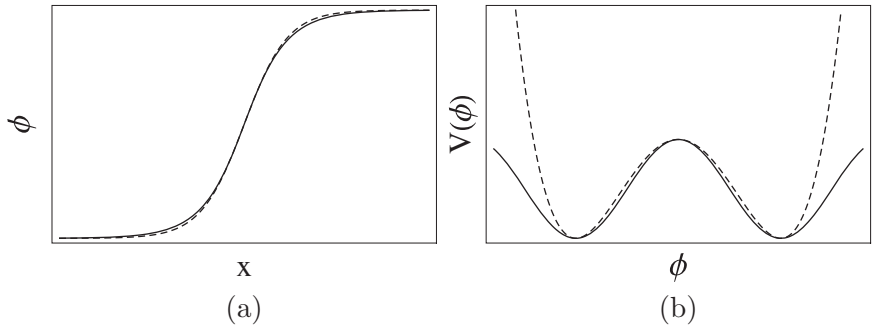


Fig. 2.4. A comparison of the kink solutions and potentials for the ϕ^4 (solid lines) and sine-Gordon (dashed lines) theories. The parameters are related as in the text, and the zeros of ϕ have been shifted so that the vacua of the two theories coincide.

sine-Gordon model admits configurations with more than one unit of topological charge. However, the discussion below Eq. (2.21) shows that these cannot include any static solutions. The physical explanation for this is that there is a repulsive force between kinks. Although this falls exponentially with the distance between the kinks, it never quite vanishes, so the potential energy of a two-kink configuration decreases monotonically as the kinks are drawn apart but never reaches a minimum at any finite separation.

2.2 Quantizing about the kink

In the context of quantum field theory, two questions naturally arise. First, to what extent is the explicit form of the classical solution meaningful in the quantum theory? Second, how is the existence of the classical soliton reflected in the states of the quantum theory?

In single-particle quantum mechanics it is meaningful to think of a particle as having a classical position if there is a state in which the uncertainties in the position and the velocity are both small compared to the relevant scales in the problem; this can be achieved if the particle mass is sufficiently large. The corresponding statement in quantum field theory is that we can meaningfully discuss a classical field profile if there is a state with sufficiently small uncertainties in both the field and its time derivative. This needs some explanation. If a quantum field is examined at very short distance scales the quantum fluctuations are enormous, and in fact diverge as the distance scale goes to zero. Nevertheless, this does not prevent us from treating electromagnetic fields as classical on macroscopic distances. This is possible because we can choose a distance scale L that simultaneously meets two criteria. First, it is small compared to the characteristic length scale of the classical field configuration, so that the latter is still evident even if it is only measured at a sequence of points separated by a distance L.

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Second, it is large enough that the quantum fluctuations of the field, averaged over a region of radius L, are small compared to the variation of the classical field over that region.

For both the $\lambda\phi^4$ and sine-Gordon kinks the variation of the field occurs primarily within a distance of order m^{-1}, so we want to use an averaging distance $L \ll m^{-1}$. Because $d\phi/dx \sim mv \sim m^2/\sqrt{\lambda}$, the change in the classical field over this distance is

$$(\Delta\phi_L)_{\rm cl} \sim L\,\frac{d\phi}{dx} \sim \frac{m^2 L}{\sqrt{\lambda}}. \tag{2.28}$$

To compare with this we can define a smeared quantum field, given in n spatial dimensions by

$$\phi_L(\mathbf{x}) = \frac{1}{(2\pi L^2)^{n/2}} \int d^n y\, e^{-(\mathbf{y}-\mathbf{x})^2/2L^2}\phi(\mathbf{x}). \tag{2.29}$$

An estimate of the quantum fluctuations of ϕ_L can be obtained by calculating them in the vacuum state of a free field with mass m. Thus, let us define

$$\begin{aligned}(\Delta\phi_L)^2_{\rm qu} &= \langle 0|\phi_L(0)^2|0\rangle \\ &= \frac{1}{(2\pi L^2)^n} \int d^n y\, d^n z\, e^{-\mathbf{y}^2/2L^2}\, e^{-\mathbf{z}^2/2L^2}\langle 0|\phi(\mathbf{y})\phi(\mathbf{z})|0\rangle.\end{aligned} \tag{2.30}$$

Using the fact that

$$\langle 0|\phi(\mathbf{y})\phi(\mathbf{z})|0\rangle = \int \frac{d^n k}{(2\pi)^n}\, \frac{e^{i\mathbf{k}\cdot(\mathbf{y}-\mathbf{z})}}{2\sqrt{k^2+m^2}}, \tag{2.31}$$

we find that

$$(\Delta\phi_L)^2_{\rm qu} = \int \frac{d^n k}{(2\pi)^n}\, \frac{e^{-\mathbf{k}^2 L^2}}{2\sqrt{k^2+m^2}}. \tag{2.32}$$

We are interested in the case where $L \ll m^{-1}$. From the behavior of the integral in this regime, we obtain

$$(\Delta\phi_L)_{\rm qu} \sim \begin{cases} \sqrt{\ln(1/mL)}, & n = 1, \\ L^{-(n-1)/2}, & n \geq 2. \end{cases} \tag{2.33}$$

For our one-dimensional kinks we set $n = 1$. Our requirement that $(\Delta\phi_L)_{\rm qu} \ll (\Delta\phi_L)_{\rm cl}$ then becomes

$$\sqrt{\ln(1/mL)} \ll \frac{m^2 L}{\sqrt{\lambda}}. \tag{2.34}$$

We can find an $L \ll m^{-1}$ that satisfies this condition as long as the dimensionless coupling λ/m^2 is small. This weak-coupling condition also ensures that the Compton wavelength $1/M_{\rm cl}$ of the kink is much smaller than its classical width $1/m$.

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This weak-coupling criterion extends to other one-dimensional solitons as well as to solitons in more spatial dimensions. If the coupling is strong, the classical soliton in general continues to be a solution of the field equations. However, its implications for the quantum theory are more difficult to untangle.

Let us therefore, at least for the time being, assume that we are in the weak-coupling regime. In this limit the classical soliton solution leads in a natural way to a family of quantum states with energies roughly equal to the classical soliton energy. These can be investigated using perturbative methods rather analogous to those that are used in studying the vacuum sector of the theory. Let us recall the standard procedure in that case.

We begin by finding the classical vacuum solution corresponding to a minimum of the potential. If there are several degenerate minima, we pick one, say $\phi = v$, as the starting point. If we define $\phi_{\rm vac} \equiv v$, we can write

$$\phi(x,t) = \phi_{\rm vac} + \eta(x,t) \tag{2.35}$$

and re-express the Lagrangian in terms of η. We then write

$$L = L_{\rm cl} + L_{\rm quad} + L_{\rm int}, \tag{2.36}$$

where

$$L_{\rm cl} = -\int dx\, V(\phi_{\rm vac}) \tag{2.37}$$

is the classical vacuum energy,

$$L_{\rm quad} = \int dx \left[\frac{1}{2}(\partial_\mu \eta)^2 - \frac{1}{2}V''(\phi_{\rm vac})\eta^2\right] \tag{2.38}$$

contains the terms quadratic in η, and the interaction term $L_{\rm int}$, which contains terms that are cubic and higher order in η, can be treated as a perturbation. There are no terms linear in η because we have expanded about a minimum of V.

We then change variables to diagonalize $L_{\rm quad}$. To do this, we find a set of normal modes $f_j(x)$ that obey

$$\left[-\frac{d^2}{dx^2} + V''(\phi_{\rm vac})\right] f_j(x) = \omega_j^2 f_j(x) \tag{2.39}$$

and correspond to periodic solutions

$$\eta(x,t) = f_j(x) e^{-i\omega_j t} \tag{2.40}$$

of the linearized field equations. Choosing the f_j to be orthonormal, we expand the field as

$$\eta(x,t) = \sum_j \left[c_j(t) f_j(x) + \text{h.c.}\right], \tag{2.41}$$

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where the $c_j(t)$ are operators and "h.c." represents the Hermitian conjugate term that must be added because η is a real field.[4] Substituting this expansion into Eq. (2.38) gives a sum of simple harmonic oscillator Lagrangians. After quantization, the energy eigenstates are labeled by the occupation numbers n_j of the various oscillators, with their energies being

$$E_{\rm quad} = \sum_j \left(n_j + \frac{1}{2} \right) \omega_j. \tag{2.42}$$

There is an immediate problem, because the sum of the zero-point energies is infinite. Putting aside for the moment the details of how to regulate this infinity, we subtract from the Lagrangian an appropriate constant, δE, whose value is chosen to make the energy of the vacuum finite (and perhaps zero). We then go on to include the effects of the interaction terms, working order by order in perturbation theory. These not only give new contributions to the energy of the vacuum—which, if gravity is neglected, is unobservable in any case—but also modify the splitting between the vacuum and the lowest one-particle state; i.e., the mass of the elementary scalar. In general, it turns out that this correction is also infinite, and must be compensated by the addition of divergent counterterms to the Lagrangian. Thus, we now have

$$L = L_{\rm cl} + L_{\rm quad} + L_{\rm int} - \delta E + L_{\rm ct}, \tag{2.43}$$

where $L_{\rm ct}$ contains the field-dependent counterterms, with coefficients chosen to satisfy appropriate renormalization conditions.

The energy of the vacuum is then

$$E_{\rm vac} = E_{\rm cl}(\phi_{\rm vac}) + \frac{1}{2} \sum_j \omega_j + \delta E + E_{\rm ct}(\phi_{\rm vac}) + \cdots, \tag{2.44}$$

where the ellipsis denotes terms that are at least first order in the coupling. Note that there can be an order unity contribution from $L_{\rm ct}$, even though the counterterm coefficients arise from perturbative diagrams, if $\phi_{\rm vac}$ is inversely proportional to the coupling.

We can now follow a similar procedure to obtain both the quantum corrections to the energy of a static kink and the spectrum of nearby states. Thus, we now write

$$\phi(x,t) = \phi_{\rm kink}(x) + \tilde{\eta}(x,t), \tag{2.45}$$

expand L in powers of $\tilde{\eta}$, and find the normal modes that diagonalize the quadratic Lagrangian. All the above equations apply, provided that we replace $\phi_{\rm vac}$ by $\phi_{\rm kink}$, and remember that the latter has a nontrivial x-dependence.

[4] In actuality, of course, the spectrum of normal frequencies includes a continuum portion, and the sums in this and subsequent formulas should be understood to include integrals where appropriate.

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It is important to recognize that the counterterms, including δE, have already been fixed by the calculations in the vacuum sector. We are considering a single theory, with a single Lagrangian, and so must use the same values for the parameters, either finite or infinite, in both the vacuum and the kink sectors. Thus, we have

$$\begin{aligned} M_{\rm kink} &= E_{\rm kink} - E_{\rm vac} \\ &= \Delta E_{\rm cl} + \Delta E_{\rm zp} + \Delta E_{\rm ct} + O(\lambda), \end{aligned} \tag{2.46}$$

where

$$\Delta E_{\rm cl} = E_{\rm cl}(\phi_{\rm kink}) - E_{\rm cl}(\phi_{\rm vac}) = M_{\rm cl} \tag{2.47}$$

is the classical kink mass,

$$\Delta E_{\rm zp} = \frac{1}{2}\sum_j \omega_j^{\rm kink} - \frac{1}{2}\sum_j \omega_j^{\rm vac} \tag{2.48}$$

is the sum of the shifts of the oscillator zero-point energies, and

$$\Delta E_{\rm ct} = E_{\rm ct}(\phi_{\rm kink}) - E_{\rm ct}(\phi_{\rm vac}) \tag{2.49}$$

is the contribution from the counterterms. Given the various divergent quantities that appear here, it is not at all obvious that the final result will be finite. Showing that it is indeed finite is a nontrivial test of our approach.

In order to see how this works, let us determine the leading quantum correction to the mass of the $\lambda\phi^4$ kink, following the approach of [1]. In the vacuum sector the normal modes are the familiar plane waves $f_k(x) = e^{ikx}$. The corresponding occupation numbers specify the number of particles with momentum k, where k ranges from $-\infty$ to ∞. Because the elementary particle mass is $\sqrt{2}\,m$, there is a continuous spectrum of frequencies

$$\omega(k) = \sqrt{k^2 + 2m^2} \tag{2.50}$$

beginning at $\sqrt{2}\,m$. Substituting the mode functions into Eq. (2.41), and following the standard procedure, one arrives at the usual perturbation theory in terms of Feynman graphs. There is only a single divergent integral, which arises from the graph in Fig. 2.5. A convenient choice of renormalization scheme is to cancel this exactly by taking

$$\mathcal{L}_{\rm ct} = \frac{1}{2}\delta m^2\,\phi^2, \tag{2.51}$$

with

$$\begin{aligned} \delta m^2 &= -3i\lambda \int \frac{d^2k}{(2\pi)^2}\,\frac{1}{k_\mu k^\mu - 2m^2 + i\epsilon} \\ &= \frac{3\lambda}{4\pi}\int_{-\infty}^{\infty} \frac{dk}{\sqrt{k^2+2m^2}} \\ &= \frac{3\lambda}{2\pi}\int_{0}^{\infty} \frac{dk}{\sqrt{k^2+2m^2}}. \end{aligned} \tag{2.52}$$

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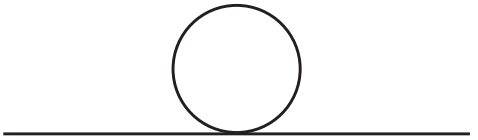


Fig. 2.5. The Feynman diagram that produces the only divergent integral in the $\lambda\phi^4$ theory.

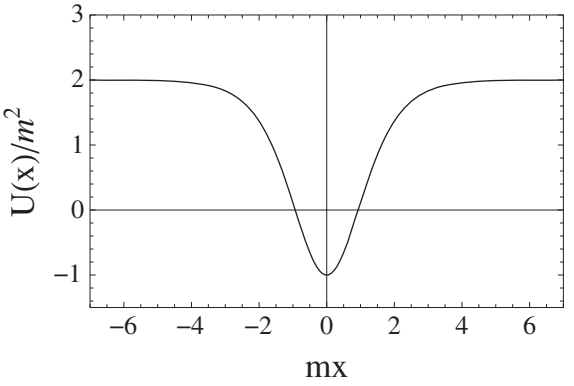


Fig. 2.6. The potential that enters the eigenvalue equation for the normal modes about the kink with $x_0 = 0$.

Turning now to the kink sector, we must first find the normal modes about the kink solution of Eq. (2.11). These modes are solutions of the eigenvalue equation

$$\left[-\frac{d^2}{dx^2} + U(x)\right] f_j(x) = \omega_j^2 f(x), \tag{2.53}$$

where

$$\begin{aligned} U(x) &= V''(\phi_{\rm cl}(x)) \\ &= -m^2 + 3\lambda\phi_{\rm cl}^2 \\ &= m^2\left\{2 - \frac{3}{\cosh^2[m(x-x_0)/\sqrt{2}]}\right\} \end{aligned} \tag{2.54}$$

is shown in Fig. 2.6. We see immediately that the spectrum of eigenvalues includes a continuum beginning at $U(\pm\infty) = 2m^2$. There may also be discrete eigenvalues.

Without even using the explicit form of the kink solution, we can show that there must be a "zero mode", with $\omega = 0$. To do this, we recall that $\phi_{\rm kink}(x)$ obeys the static field equation (2.18). Differentiating that equation with respect to x gives

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$$0 = -\frac{d^2}{dx^2}\left(\frac{d\phi_{\rm kink}}{dx}\right) + \frac{d}{dx}V'(\phi_{\rm kink}(x))$$
$$= -\frac{d^2}{dx^2}\left(\frac{d\phi_{\rm kink}}{dx}\right) + V''(\phi_{\rm cl}(x))\left(\frac{d\phi_{\rm kink}}{dx}\right), \tag{2.55}$$

which shows that

$$f_0(x) = \frac{d\phi_{\rm kink}}{dx} \tag{2.56}$$

is a solution of Eq. (2.53) with eigenvalue $\omega^2 = 0$.

We also learn a bit more. Because $d\phi_{\rm kink}/dx$ never vanishes, f_0 has no nodes. It is a familiar result from quantum mechanics that in a one-dimensional potential only the lowest eigenfunction has no nodes. Hence, in our problem there are no modes with $\omega^2 < 0$. This is fortunate, since an imaginary ω would correspond to an instability of the kink.

To proceed further, we have to actually solve the eigenvalue equation. Fortunately, Eq. (2.53) is equivalent to a Schrödinger equation with a potential $W(x) = \frac{1}{2}U(x)$, which turns out to be exactly soluble [5]. (This was one of the reasons for choosing the $\lambda\phi^4$ model as an example.) There is one more discrete eigenvalue, $\omega_1^2 = 3m^2/2$, with the (unnormalized) eigenfunction

$$f_1(x) = \frac{\sinh[m(x-x_0)/\sqrt{2}]}{\cosh^2[m(x-x_0)/\sqrt{2}]}. \tag{2.57}$$

The continuum modes are again labeled by a momentum k that runs over all positive and negative real values. The mode with momentum k has a frequency $\omega = \sqrt{k^2 + 2m^2}$ and corresponds to the eigenfunction

$$f_k(x) = e^{ikx}\Big\{3m^2\tanh^2[m(x-x_0)/\sqrt{2}] - m^2 - 2k^2$$
$$-3\sqrt{2}\,imk\tanh[m(x-x_0)/\sqrt{2}]\Big\}. \tag{2.58}$$

As $x \to \pm\infty$, this tends to

$$f_k(x) = \sqrt{4(m^2-k^2)^2 + 18m^2k^2}\; e^{i[kx \pm \frac{1}{2}\delta(k)]}, \tag{2.59}$$

where the phase shift $\delta(k)$ is

$$\delta(k) = -2\arctan\left(\frac{3\sqrt{2}}{2}\,\frac{mk}{m^2-k^2}\right). \tag{2.60}$$

We need to specify the branch of the arctangent. It is most convenient to choose this so that $\delta(k)$ tends to zero as $|k| \to \infty$. For positive k, the phase shift is monotonically decreasing, with $\delta(0^+) = 2\pi$ and $\delta(\infty) = 0$. For negative k, we set $\delta(-k) = -\delta(k)$. Note that this gives a discontinuity at $k = 0$.

Our expression for the kink energy contains divergent integrals of the zero-point energies of the continuum modes about the kink and the vacuum. These

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can be regulated by taking the one-dimensional space to have length a, and then imposing periodic boundary conditions on $\eta(x)$, with the limit $a \to \infty$ to be taken at the end of the calculation. This makes the spectra discrete and converts the integrals to sums, but the sums are still divergent. To make them finite, let us imagine that space consists of $2N$ points, so that there are only $2N$ normal modes, with N also to be eventually taken to infinity.

The modes about the vacuum are now labeled by discrete wavenumbers obeying

$$k_n^{\rm vac} a = 2\pi n, \qquad n = \pm 1, \pm 2, \ldots, \pm N. \tag{2.61}$$

The "continuum" modes about the kink satisfy

$$k_n^{\rm kink} a + \delta(k_n^{\rm kink}) = 2\pi n, \qquad n = \pm 2, \pm 3, \ldots, \pm N. \tag{2.62}$$

The labeling of these follows from our conventions for the phase shift. These imply that $k_n^{\rm kink} = -k_{-n}^{\rm kink}$ and that $k_{n+1}^{\rm kink} \approx k_n^{\rm vac}$ for $0 < n \ll ma$, while for $n \gg ma$

$$(k_n^{\rm kink} - k_n^{\rm vac})a = -\delta(k_n^{\rm vac}) + O(a^{-1}) \tag{2.63}$$

tends to zero. The absence of kink continuum modes with $n = \pm 1$ compensates for the two localized modes, with $\omega_0^{\rm kink} = 0$ and $\omega_1^{\rm kink} = \sqrt{3/2}\, m$, that have no counterparts in the vacuum sector.

The difference in zero-point energies can be written as

$$\Delta E_{\rm zp} = \frac{1}{2}\left(\omega_0^{\rm kink} - \omega_{-1}^{\rm vac}\right) + \frac{1}{2}\left(\omega_1^{\rm kink} - \omega_1^{\rm vac}\right) + \sum_{n=2}^{N}\left(\omega_n^{\rm kink} - \omega_n^{\rm vac}\right). \tag{2.64}$$

The first two terms on the right-hand side are the contributions from the two localized modes about the kink and the two lowest vacuum modes. Up to terms of order $1/a$, they give

$$\frac{1}{2}\left(0 - \sqrt{2}\, m\right) + \frac{1}{2}\left(\frac{\sqrt{6}}{2}\, m - \sqrt{2}\, m\right) = \left(\frac{\sqrt{6}}{4} - \sqrt{2}\right) m. \tag{2.65}$$

The last term in Eq. (2.64) is the contribution from the kink continuum modes and the remainder of the vacuum modes, with the prefactor of $1/2$ canceled by a factor of 2 because the sum is only over positive values of n. To evaluate this continuum contribution, we use Eq. (2.63) and write

$$\begin{aligned}\sum_{n=2}^{N}\left(\omega_n^{\rm kink} - \omega_n^{\rm vac}\right) &= \sum_{n=2}^{N}\left[\sqrt{(k_n^{\rm kink})^2 + 2m^2} - \sqrt{(k_n^{\rm vac})^2 + 2m^2}\right] \\ &= -\sum_{n=2}^{N}\sqrt{(k_n^{\rm vac})^2 + 2m^2}\left\{\frac{k_n^{\rm vac}\delta(k_n^{\rm vac})}{[(k_n^{\rm vac})^2 + 2m^2]\, a} + O(a^{-2})\right\}.\end{aligned} \tag{2.66}$$

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If we now take the limits $N \to \infty$ and $a \to \infty$, this sum is converted to an integral, with

$$\frac{1}{a}\sum_{n=2}^{N} \to \frac{1}{2\pi}\int_0^\infty dk. \tag{2.67}$$

The right-hand side of Eq. (2.66) then becomes

$$\begin{aligned} &-\frac{1}{2\pi}\int_0^\infty dk\,\frac{k\delta(k)}{\sqrt{k^2+2m^2}} \\ &\quad = \frac{1}{2\pi}\int_0^\infty dk\,\sqrt{k^2+2m^2}\,\frac{d\delta}{dk} - \frac{1}{2\pi}\,\delta(k)\sqrt{k^2+2m^2}\Big|_{k=0}^{k=\infty}. \end{aligned} \tag{2.68}$$

The integral on the right-hand side of this equation is

$$\begin{aligned} &\frac{1}{2\pi}\int_0^\infty dk\,\sqrt{k^2+2m^2}\,\frac{d\delta}{dk} \\ &\quad = -\frac{3\sqrt{2}}{2\pi}m\int_0^\infty \frac{dk}{\sqrt{k^2+2m^2}} - \frac{3\sqrt{2}}{4\pi}m\int_0^\infty \frac{m^2\,dk}{\sqrt{k^2+2m^2}(k^2+\frac{1}{2}m^2)} \\ &\quad = -\frac{3\sqrt{2}}{2\pi}m\int_0^\infty \frac{dk}{\sqrt{k^2+2m^2}} - \frac{\sqrt{6}}{6}\,m, \end{aligned} \tag{2.69}$$

while the surface term gives

$$-\frac{1}{2\pi}\left[\lim_{k\to\infty}\delta(k)\sqrt{k^2+2m^2} - \lim_{k\to 0^+}\delta(k)\sqrt{k^2+2m^2}\right] = -\frac{3\sqrt{2}}{2\pi}\,m + \sqrt{2}\,m. \tag{2.70}$$

Adding the continuum contributions from Eqs. (2.69) and (2.70) to the contribution from the discrete modes in Eq. (2.65), we obtain

$$\Delta E_{\rm zp} = \left(\frac{\sqrt{6}}{12} - \frac{3\sqrt{2}}{2\pi}\right)m - \frac{3\sqrt{2}\,m}{2\pi}\int_0^\infty \frac{dk}{\sqrt{k^2+2m^2}}. \tag{2.71}$$

This is still divergent. However, we have not yet added the contributions from the counterterm Lagrangian, which are given by

$$\begin{aligned} \Delta E_{\rm ct} &= -\frac{1}{2}\delta m^2\int dx\,\left[\phi_{\rm kink}(x)^2 - \phi_{\rm vac}(x)^2\right] \\ &= \left(-\frac{1}{2}\right)\left[\frac{3\lambda}{2\pi}\int_0^\infty \frac{dk}{\sqrt{k^2+2m^2}}\right]\left(-\frac{2\sqrt{2}\,m}{\lambda}\right). \end{aligned} \tag{2.72}$$

This is of order unity, because the factor of λ in δm^2 is canceled by the $1/\lambda$ from $\phi^2_{\rm kink}$, and exactly cancels the divergence from the zero-point energies. Putting our results together, we have

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$$M_{\text{kink}} = \frac{2\sqrt{2}}{3}\frac{m^3}{\lambda} + \left(\frac{\sqrt{6}}{12} - \frac{3\sqrt{2}}{2\pi}\right) m + O\left(\frac{\lambda}{m}\right) \tag{2.73}$$

for the energy of the ground state in the kink sector.

This is the state with all of the oscillator modes in their ground state. There are also higher energy states in which some of the occupation numbers n_j are nonzero. For the continuum modes, these are scattering states containing elementary bosons moving in the presence of the kink. Quanta of the discrete mode with $\omega = \sqrt{3/2}\,m$ can be interpreted as elementary bosons bound to the kink. The remaining mode, that with $\omega = 0$, is a different matter. A naive use of Eq. (2.42) would imply that the energy was independent of the occupation number. This would give an infinite set of degenerate states, a puzzling result.

This mode also leads to pathologies when one tries to use perturbation theory to calculate the next-order contribution to the kink mass. Proceeding straightforwardly leads to a Feynman diagram expansion similar to the usual one, except that the propagators and the vertices are position-dependent. The problem is that there are infrared divergences that arise from the contribution of the $\omega = 0$ mode to the propagator.

The explanation for these difficulties is that an oscillator with zero frequency is not an oscillator at all, so the line of reasoning that leads to states labeled by occupation number breaks down. The resolution of these problems is the subject of the next section.

2.3 Zero modes and collective coordinates

The anomalies associated with the $\omega = 0$ mode are not peculiar to the $\lambda\phi^4$ kink. Equation (2.55) does not depend on the form of the potential, and thus shows that a kink solution in any potential has a zero mode. The physical significance of this mode can be seen by noting that

$$\frac{d\phi_{\text{kink}}}{dx} = -\frac{d\phi_{\text{kink}}}{dx_0}, \tag{2.74}$$

so that excitation of the zero mode corresponds to a shift of the kink position. Because the underlying theory is translationally invariant, this shift leaves the potential energy unchanged.

This mode can be handled by recognizing that it is not an oscillator mode, and replacing its oscillator coefficient $c_0(t)$ by a collective coordinate $z(t)$ that represents the position of the kink [6–12]. Instead of our previous mode expansion, we write

$$\begin{aligned}\phi(x,t) &= \phi_{\text{kink}}(x - z(t)) + \eta(x - z(t)) \\ &= \phi_{\text{kink}}(x - z(t)) + {\sum_j}' c_j(t) f_j(x - z(t)),\end{aligned} \tag{2.75}$$

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where the prime indicates that the summation is only over modes with nonzero ω. Here both the $c_j(t)$ and $z(t)$ are to be understood as quantum mechanical operators. We will also need the time derivative of this expression,

$$\dot{\phi}(x,t) = \dot{z}(t)\,\frac{d\phi_{\rm kink}(x-z(t))}{dz} + {\sum_j}' \left[\dot{c}_j(t) f_j(x-z(t)) + \dot{z}(t) c_j(t) \frac{\partial f_j(x-z(t))}{\partial z}\right]. \tag{2.76}$$

The next step is to substitute these expansions into the Lagrangian and to pick out the terms that are quadratic in c_j, $\dot{c}_j$, or $\dot{z}$. (The translation invariance guarantees that the Lagrangian does not depend on z.) After the spatial integration, the terms quadratic in the c_j or the $\dot{c}_j$ (and independent of $\dot{z}$) are just as before, except for the omission of the zero-mode contributions. The $\dot{\phi}^2$ term in the Lagrangian appears to give cross-terms

$$\dot{z}\dot{c}_j \int dx \frac{d\phi_{\rm kink}(x-z)}{dz} f_j(x-z) = -\dot{z}\dot{c}_j \int dx \frac{d\phi_{\rm kink}(x-z)}{dx} f_j(x-z) \tag{2.77}$$

involving $\dot{z}\dot{c}_j$, but these vanish because the nonzero modes that we have retained are orthogonal to the zero mode, $d\phi_{\rm kink}/dx$. Finally, the term quadratic in $\dot{z}$ is

$$\frac{1}{2}\dot{z}^2 \int dx \left(\frac{d\phi_{\rm kink}}{dz}\right)^2 = \frac{1}{2}\dot{z}^2 \int dx \left(\frac{d\phi_{\rm kink}}{dx}\right)^2 \equiv \frac{1}{2} A \dot{z}^2. \tag{2.78}$$

By making use of the virial identity of Eq. (2.20), A can be rewritten as

$$A = \frac{1}{2}\int dx \left(\frac{d\phi_{\rm kink}}{dx}\right)^2 + \int dx\, V(\phi_{\rm kink}) = M_{\rm cl}. \tag{2.79}$$

Putting all this together, we have

$$L_{\rm quad} = \frac{1}{2} M_{\rm cl}\,\dot{z}^2 + {\sum_j}' \left(\frac{1}{2}\dot{c}_j^2 - \frac{1}{2}\omega_j^2 c_j^2\right). \tag{2.80}$$

The collective coordinate appears only in a single term, which is just the standard kinetic energy of a nonrelativistic particle with mass $M_{\rm cl}$. Its conjugate momentum,

$$P = M_{\rm cl}\,\dot{z}, \tag{2.81}$$

is a conserved quantity, commuting with the Hamiltonian. Because z can range from $-\infty$ to ∞, P has a continuous spectrum of eigenvalues. The energy eigenstates $|P, \{n_j\}\rangle$ are labeled by a momentum and a set of occupation numbers. The states with all of the n_j vanishing correspond to a kink with momentum P. States with nonzero occupation numbers have elementary particles of the theory in addition to the kink.

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The order-unity corrections to the energy of the kink states arise from the zero-point energies of the oscillating modes. These are just as calculated in the previous section, so the energy of this state is

$$E = M_{\rm kink} + \frac{P^2}{2M_{\rm cl}} + O(P^4) + O(\lambda), \tag{2.82}$$

where $M_{\rm kink}$ is given by Eq. (2.73).

To the order that we have calculated, this is just the low-momentum expansion of the relativistic energy of a particle with momentum P and mass $M_{\rm kink}$. It is notable that this result depends crucially on the virial identity, which was needed to relate the coefficient of P^2 to the kink mass; we will encounter other examples of such apparently fortuitous identities in higher dimensions.

The interaction terms in the Lagrangian lead to several types of corrections to Eq. (2.82). First, there are $O(\lambda)$ and higher-order corrections to $M_{\rm kink}$. There are also corrections to the coefficient of P^2, so that $M_{\rm cl}$ is replaced by $M_{\rm kink}$. Finally, there are corrections proportional to P^4 and higher powers of P, just as required from the expansion of the relativistic energy $\sqrt{M_{\rm kink}^2 + P^2}$.

The zero mode we have studied here resulted from the fact that the Lagrangian had a translational symmetry that was not a symmetry of the kink. In general, we will find zero modes arising whenever a soliton breaks one or more symmetries of the Lagrangian. In all such cases, one introduces a corresponding collective coordinate whose conjugate momentum becomes nonzero when the zero mode is excited in a time-dependent manner [13]. The symmetry in question may be a spacetime symmetry, as in the present case, or it may be an internal symmetry, with the corresponding momentum being the appropriate conserved charge.

Although most zero modes are directly related to symmetries in this manner, this is not always the case. In particular, there are some theories, described in more detail in Chap. 8, in which solitons at rest do not interact with each other. These theories have multisoliton solutions that have a set of zero modes, with corresponding collective coordinates, for each individual soliton. Because the solitons do interact when in motion, the conjugate variables, corresponding to the momenta and charges of the component solitons, are not necessarily conserved.

2.4 Fermions and fermion zero modes

The analysis of the preceding two sections can be generalized in a straightforward manner to accommodate additional fields, assuming that the kink remains a solution of the classical equations. For the case of an added bosonic field, preserving the kink solution requires an appropriate choice of its interaction terms in the Lagrangian. Because fermionic fields always vanish classically, they never interfere with the existence of a classical soliton solution.

One does a normal mode analysis of the additional fields in the background defined by the kink solution. Continuum modes correspond to scattering states

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of particles moving in the presence of the kink, and discrete modes with nonzero frequency correspond to elementary particles bound to the kink. Any bosonic zero modes are usually related to internal symmetries, and are handled by the introduction of collective coordinates, as described above. However, one often finds fermionic zero modes, which have somewhat different consequences.

As a concrete example [14], consider adding a single fermion field to the ϕ^4 theory that gave rise to our kink solution, with the Lagrangian density being

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\lambda}{4}(\phi^2 - v^2)^2 + i\bar{\psi}\gamma^\mu \partial_\mu \psi - G\bar{\psi}\phi\psi. \tag{2.83}$$

The discrete $\phi \to -\phi$ symmetry of the purely bosonic theory is preserved if the fermion field is simultaneously transformed according to $\psi \to \gamma^5 \psi$, where $\gamma^5 \equiv -\gamma^0\gamma^1$.

In the vacuum sector with $\phi = v$ the normal modes of ψ correspond to periodic solutions of the Dirac equation

$$i\gamma^\mu \partial_\mu \psi - M\psi = 0, \tag{2.84}$$

where $M = Gv$ is the fermion mass. If we write

$$\psi(x,t) = \chi_j(x) e^{-i\omega_j t}, \tag{2.85}$$

then $\chi_j(x)$ must satisfy

$$\left[-i\gamma^0\gamma^1\partial_x + M\gamma^0\right]\chi_j = \omega_j \chi_j. \tag{2.86}$$

This has a continuous spectrum of plane wave solutions χ_q with $-\infty < q < \infty$ and $\omega_q = \sqrt{q^2 + M^2}$. In addition, there is a second set of eigenfunctions $\chi_q^c \equiv -i\gamma^1\chi_q$ with negative frequency $-\omega_q$. The field and its adjoint can be expanded in terms of these modes as

$$\begin{aligned} \psi(x,t) &= \sum_q \left[b_q(t)\chi_q(x) + d_q^\dagger(t)\chi_q^c(x)\right], \\ \bar{\psi}(x,t) &= \sum_q \left[b_q^\dagger(t)\bar{\chi}_q(x) + d_q(t)\bar{\chi}_q^c(x)\right]. \end{aligned} \tag{2.87}$$

Requiring that ψ and $\bar{\psi}$ obey the canonical anticommutation relations leads to the usual algebra for the b_q, the d_q, and their adjoints. States are labeled by sets of occupation numbers that can each be either zero or one. The creation operators $b_q^\dagger$ and $d_q^\dagger$ have been introduced in such a way that they both increase the energy by an amount ω_q, with the former creating positive-energy fermions and the latter positive-energy antifermions.

In the presence of a kink, the normal modes correspond to solutions of

$$\left[-i\gamma^0\gamma^1\partial_x + G\gamma^0\phi_{\text{kink}}(x)\right]\chi_j = \omega_j\chi_j. \tag{2.88}$$

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As in the vacuum sector, if χ_j is a solution with positive eigenvalue ω_j, then $\chi_j^c = -i\gamma^1\chi_j$ is a solution with eigenvalue $-\omega_j$. Thus, the modes with nonzero eigenvalues all come in pairs, corresponding to a fermion state and a charge conjugate antifermion state. However, if there are any modes with $\omega_j = 0$, it is possible that χ_j and $-i\gamma^1\chi_j$ might be linearly dependent, in which case there would not be a doubling of the mode. In fact, it is easy to see that the zero modes can be chosen to be eigenstates of $-i\gamma^1$. In two spacetime dimensions the spinor representation is two-dimensional, so $-i\gamma^1$ has two eigenvalues, 1 and -1, with two-component eigenvectors $s_\pm$. With $\omega_j = 0$, Eq. (2.88) is solved by [14]

$$\chi_0 = e^{\mp G\int_0^x dx'\,\phi_{\rm kink}(x')}\, s_\pm. \tag{2.89}$$

For the kink solution, which is positive at large positive x and negative at large negative x, this is a normalizable mode with the upper choice of signs; for an antikink, with the opposite behavior, the lower choice of signs gives a normalizable solution. Finally, a vacuum solution, with ϕ having the same sign at both $x = \infty$ and $x = -\infty$, has no normalizable zero modes, in agreement with the previous discussion. In all of these cases, the presence or absence of the zero mode does not depend on the detailed structure of the background ϕ field, but only its values at spatial infinity. Like the existence of the kink itself, these zero modes are closely tied to the topology.

With a zero mode present, the mode expansions of Eq. (2.88) are replaced by

$$\begin{aligned}\psi(x,t) &= a(t)\chi_0(x) + \sum_q \left[b_q(t)\chi_q(x) + d_q^\dagger(t)\chi_q^c\right],\\ \bar\psi(x,t) &= a^\dagger(t)\bar\chi_0(x) + \sum_q \left[b_q^\dagger(t)\bar\chi_q(x) + d_q(t)\bar\chi_q^c\right],\end{aligned} \tag{2.90}$$

where the summations are only over modes with nonzero eigenvalue. The commutation relations among the b_q, the d_q, and their adjoints is now supplemented by

$$\{a, a^\dagger\} = 1, \qquad \{a, a\} = \{a^\dagger, a^\dagger\} = 0, \tag{2.91}$$

with a and $a^\dagger$ anticommuting with all of the b_q and d_q. As in the vacuum sector, states can be labeled by occupation numbers that are either one or zero. For the modes with nonzero frequencies the interpretation of these occupation numbers as indicating the presence or absence of particles and antiparticles is just as before.

The situation is somewhat different for the zero mode. As with the other modes, the anticommutation relations imply the existence of two states that are connected by a and $a^\dagger$. However, because the two states are now degenerate in energy (and momentum), it is not really meaningful to say that one corresponds to the presence of a particle while the other does not. Instead, it seems more reasonable to simply say that all of the kink states have become doubly degenerate. If we indicate this extra degree of freedom by a + or −, the states with

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momentum P and vanishing occupation numbers for all bosonic modes and all nonzero fermionic modes are related by

$$\begin{aligned} a^\dagger|P-\rangle &= |P+\rangle, \\ a|P+\rangle &= |P-\rangle. \end{aligned} \tag{2.92}$$

One way to express this is to assign a fractional fermion number, $\pm 1/2$, to these two basic states, with integer increments then arising from the action of the $b_q^\dagger$ and $d_q^\dagger$ [14].

One can also envision situations with more than one fermionic zero mode; this is often the case for solitons in higher dimensions. If there are N fermionic zero modes, each soliton state acquires a degeneracy of 2^N. A similar phenomenon arises in string theory, where the zero-frequency Ramond modes of the superstring produce the degeneracy needed to obtain a multiplet of fermionic states.

It is interesting to note that a concrete realization of fermion fractionalization is found in polyacetylene, a conducting material consisting of parallel chains of atoms. Because the motion of the electrons is primarily along these chains, it can be treated as a quasi-one-dimensional material. It has kink-like defects and fermion zero modes very much analogous to those of the relativistic field theories, although differing in that the polyacetylene electrons, since they are actually living in three dimensions, have two spin states. The fermion fractionalization is manifested through an unusual relation between the charge and spin at a kink [15].

2.5 Kinks in more spacetime dimensions

The kink solutions in two spacetime dimensions are, trivially, also solutions of the field equations in $D > 2$ spacetime dimensions, with the field being independent of all but one of the coordinates. The energy density is localized in one spatial direction, but invariant under translations in the orthogonal directions, so the kink can no longer be interpreted as a particle. Instead, it represents a boundary, or domain wall, separating two regions in which the field approaches different vacuum values.

The expression for the kink energy now finds an interpretation as the energy density on the domain wall. Although this has dimensions of $(\text{mass})^{D-1}$, the same expression suffices because the dimensionality of the coupling constant is also dimension-dependent.

The domain wall clearly breaks translation invariance in the direction orthogonal to the wall. In the directions parallel to the wall, not only translation invariance, but also Lorentz invariance, is preserved. This can be seen, for example, in the energy–momentum tensor, which for our scalar field theories takes the form

$$T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - \eta_{\mu\nu}\mathcal{L}, \tag{2.93}$$

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with $\eta_{\mu\nu} = \mathrm{diag}\,(1, -1, -1, \ldots, -1)$ being the Minkowskian metric in D spacetime dimensions. Let us take x^{D-1} to be the direction orthogonal to the wall. Using Eqs. (2.1) and (2.19), we find that

$$T_{\mu\nu} = \hat{\eta}_{\mu\nu} \left(\frac{\partial \phi}{\partial x^{D-1}} \right)^2, \tag{2.94}$$

where

$$\hat{\eta}_{\mu\nu} = \mathrm{diag}\,(1, -1, -1, \ldots, -1, 0) \tag{2.95}$$

can be viewed as the Minkowskian metric restricted to the first $D-1$ spacetime dimensions.

From this expression for $T_{\mu\nu}$ we see that there is an anisotropic pressure that vanishes in the x^{D-1} direction but that is negative, and equal in magnitude to the energy density, in all other directions. Thus, the wall preserves an SO($D-2$,1) subgroup of the original SO($D-1$,1) Lorentz symmetry. The fact that the pressure is negative has interesting gravitational consequences, which will be discussed in Chap. 7.

The normal mode analysis of the fluctuations of the fields about the kink is easily extended to fluctuations about the domain wall. Each two-dimensional kink mode with eigenvalue ω gives rise to a family of modes with frequencies $\sqrt{\omega^2 + \mathbf{k}_T^2}$, where $\mathbf{k}_T$ represents the wavenumbers in the spatial directions parallel to the wall. If the two-dimensional mode was a bound state mode localized about the kink, the D-dimensional mode can be interpreted as a particle of mass ω whose motion is confined to the domain wall. The translational zero mode goes over to a family of massless modes with frequency $|\mathbf{k}|$; these correspond to small deformations of the planar domain wall.

If there are fermions in the theory, a fermion zero mode about the kink is promoted to a massless particle localized on the domain wall. If the number of spacetime dimensions is odd, this particle has a definite chirality. To see this, note that in a spacetime with an even number $D = 2k$ dimensions, the natural generalization of the γ^5 of four-dimensional spacetime is

$$\Gamma = i^{k-1} \gamma^0 \gamma^1 \ldots \gamma^{D-1}, \tag{2.96}$$

which anticommutes with the γ^A ($A = 0, 1, \ldots, D-1$) and obeys

$$\Gamma^2 = I. \tag{2.97}$$

It thus has eigenvalues ± 1. Massless fermions can be described by spinors that are eigenvectors of Γ and have a chirality determined by the corresponding eigenvalue. By contrast, in an odd number of dimensions the last of the Dirac matrices is just i times the γ^5 analogue defined in one lower dimension. The product of all the γ^A is then proportional to the unit matrix, and there is no analogue of γ^5.

Now consider a domain wall extension of the kink solution, with the number D of spacetime dimensions being odd and the coordinate orthogonal to the wall

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being x^{D-1}. From the analysis of Sec. 2.4, we know that the fermion zero mode is an eigenvector of $i\gamma^{D-1}$, with the eigenvalue determined by whether the wall is built upon a kink or an antikink. But this is the γ^5 analogue in $D-1$ dimensions, so this mode, when viewed as a particle moving in the $(D-1)$-dimensional world volume of the domain wall, has a definite chirality determined by the kink or antikink nature of the wall.

This finds application in lattice gauge theory, where the domain wall fermion method yields four-dimensional chiral fermions by means of zero modes centered on a domain wall in five spacetime dimensions.

2.6 Multikink dynamics

The methods of Sec. 2.2 make it possible to study the interactions of a kink with the elementary particles of the theory, as long as the coupling constants are small enough for perturbative calculations to be valid. One would also like to be able to study the interactions between kinks, or between a kink and an antikink, including such processes as kink–kink and kink–antikink scattering, or the production of many elementary scalars via kink–antikink annihilation.

There are some limiting cases where this can be approached analytically [16–18]. For example, if two solitons are far apart, with not too large a relative velocity, the tail of one can be treated as a small perturbation on the other, making it possible to calculate the force between them and determine whether it is attractive or repulsive. However, the more general case of two kinks colliding at finite velocity is much more challenging

A case that is particularly amenable to analysis is the sine-Gordon theory. If two kinks collide, conservation of topological charge requires that the final state also contain two kinks. One might also expect some of the kink kinetic energy to be converted into radiation, which at the classical level would be manifested as small fluctuations of the ϕ field. However, this is not what happens. There is an exact classical solution [19] of the field equation,

$$\phi(x,t) = \frac{2v}{\pi}\,\tan^{-1}\left\{u\,\sinh\left(\frac{mx}{\sqrt{1-u^2}}\right)\left[\cosh\left(\frac{mut}{\sqrt{1-u^2}}\right)\right]^{-1}\right\}, \tag{2.98}$$

that describes two kinks with velocity u that are well separated at large negative t, collide at $t = 0$, and then re-emerge and move away from each other with the same velocity and asymptotically unchanged field profiles, as illustrated in Fig. 2.7. Thus, these are actually solitons in the more restricted sense of the word. The total energy of this solution is[5]

$$E = \frac{16m^3}{\lambda}\frac{1}{\sqrt{1-u^2}} = \frac{2M_{\rm cl}}{\sqrt{1-u^2}}, \tag{2.99}$$

[5] Verifying this result at arbitrary t is quite tedious. However, since energy is conserved it is sufficient to do the calculation for $t = 0$, where it is quite straightforward.

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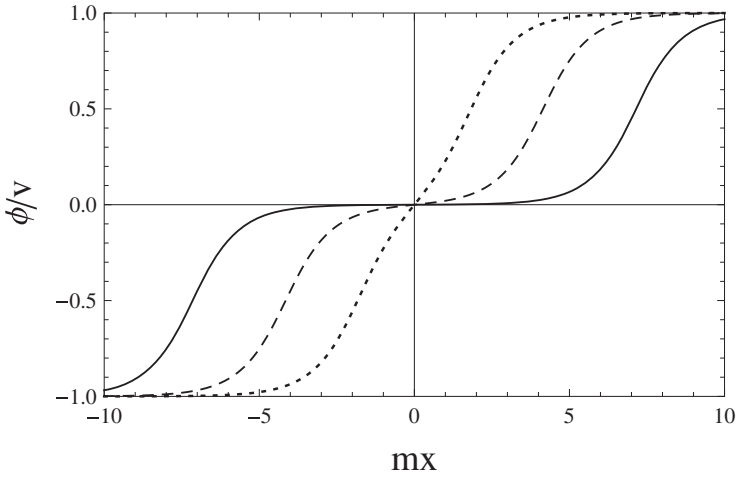


Fig. 2.7. The sine-Gordon two-kink solution for velocity $u = 0.3$. The profiles are for times $t = \pm 20m^{-1}$ (solid curve), $t = \pm 10m^{-1}$ (dashed curve), and $t = 0$ (dotted curve).

just as expected for two noninteracting solitons moving with velocity u. Because the solitons are identical, there is an ambiguity as to whether the kinks pass through each or instead collide and recoil backwards. Examination of the large-time behavior of Eq. (2.98) shows that there is a net time advance

$$\Delta t = -\frac{2\sqrt{1-u^2}}{mu} \ln |u|. \tag{2.100}$$

Since one can show that the long-range force between kinks is repulsive, Eq. (2.100) seems inconsistent with the former interpretation, but it is quite reasonable if the solution is interpreted as two solitons that approach each other but reverse direction before they meet.

In a kink–antikink collision there is no topological impediment to annihilation of the two, but here again there is an analytic solution [19],

$$\phi(x,t) = \frac{2v}{\pi} \tan^{-1}\left\{\frac{1}{u} \sinh\left(\frac{mut}{\sqrt{1-u^2}}\right)\left[\cosh\left(\frac{mx}{\sqrt{1-u^2}}\right)\right]^{-1}\right\}, \tag{2.101}$$

that describes two solitons that collide, pass through each other, and then emerge undeformed, as shown in Fig. 2.8. There is again a net time advance given by Eq. (2.100). Since one can show that the long-range interaction between a kink and an antikink is attractive, the appearance of an advance makes sense. The energy for this case is also given by Eq. (2.99).

The fact that the kink–antikink interaction is attractive suggests that bound states might exist. Indeed, continuing Eq. (2.101) to imaginary u, with $u = is$, gives

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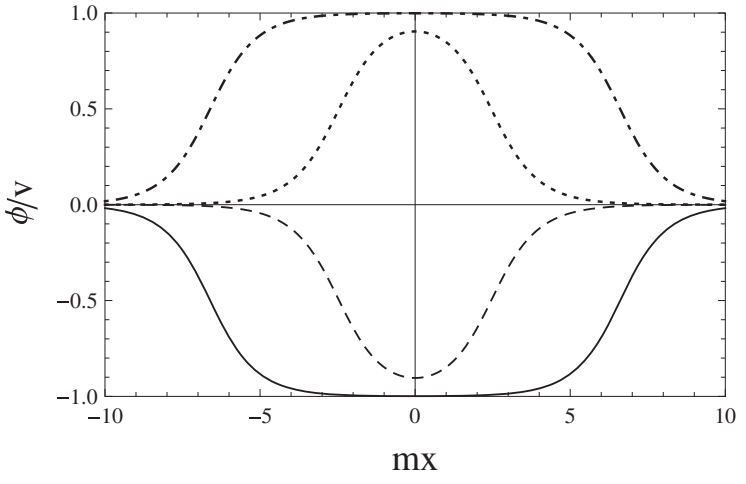


Fig. 2.8. The sine-Gordon solution for a left-moving kink and a right-moving antikink, with velocity $u = 0.3$. The profiles are for times $t = -20m^{-1}$ (solid curve), $t = -5m^{-1}$ (dashed curve), $t = 5m^{-1}$ (dotted curve), and $t = 20m^{-1}$ (dashed-dotted curve). At $t = 0$, $\phi(x) = 0$ for all x.

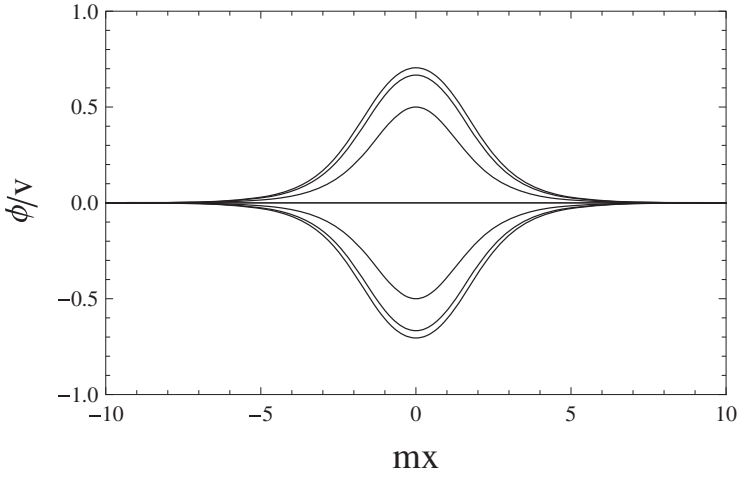


Fig. 2.9. The sine-Gordon breather solution for $s = 0.5$. The curves show the field profile at equally spaced times, one-twelfth of a period apart.

$$\phi(x,t) = \frac{2v}{\pi}\,\tan^{-1}\left\{\frac{1}{s}\,\sin\left(\frac{mst}{\sqrt{1+s^2}}\right)\left[\cosh\left(\frac{mx}{\sqrt{1+s^2}}\right)\right]^{-1}\right\}. \qquad (2.102)$$

This solution, known as the breather or doublet, can be interpreted as a kink and an antikink bound together and oscillating as illustrated in Fig. 2.9. The energy of the breather is

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$$E = \frac{2M_{\rm cl}}{\sqrt{1+s^2}} < 2M_{\rm cl}. \tag{2.103}$$

When the theory is quantized, this continuous family of breather solutions goes over to a discrete set of states. Dashen, Hasslacher, and Neveu [20, 21] showed that the energy levels can be calculated from

$$G(E) = \mathrm{Tr}\,\frac{1}{H-E} = i\,\mathrm{Tr}\int_0^\infty dT\, e^{i(E-H)T}. \tag{2.104}$$

The quantity e^{-iHT} can be evaluated as a path integral

$$\mathrm{Tr}\, e^{-iHT} = \int [d\phi]\, e^{iS(T)}, \tag{2.105}$$

where the integration is over paths with period T and the one-dimensional space is treated as being a circle of circumference L, with L taken to ∞ at the end of the calculation. In the stationary phase approximation the path integral is dominated by the classical periodic solutions; i.e., solutions in which the breather goes through an integral number of complete oscillations and traverses space an integral number of times in time T. In the leading approximation, the breather states have masses M determined by the condition that

$$S(\tau(M)) + M\tau(M) = 2\pi n, \tag{2.106}$$

where $\tau(M)$ is the period of a breather with energy $E = M$. Since for one degree of freedom an integral over one cycle gives

$$\oint p\, dq = \oint p\,\dot{q}\, dt = \oint (H+L)\, dt, \tag{2.107}$$

this can be seen as a generalization to field theory of the Bohr–Sommerfeld quantization condition.

For the solution of Eq. (2.102), with $\tau = 2\pi\sqrt{1+s^2}/ms$ and M given by Eq. (2.103), one finds that

$$S(\tau) = \frac{32\pi m^2}{\lambda}\left\{\cos^{-1}\left(\frac{2\pi}{m\tau}\right) - \left[\left(\frac{m\tau}{2\pi}\right)^2 - 1\right]^{-1}\right\}. \tag{2.108}$$

The second quantity in the curly brackets is equal to $-M\tau(M)$. Applying Eq. (2.106) then gives

$$M_n^{(0)} = \frac{16m^3}{\lambda}\,\sin\left(\frac{n\lambda}{16m^2}\right), \tag{2.109}$$

with the proviso that n be such that the argument of the sine is less than $\pi/2$.

A better approximation is obtained by including the effects of the fluctuations about the classical path. These are analogous to the order-unity zero-point

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corrections to the static ϕ^4 kink that were calculated in Sec. 2.2. It turns out that these can be calculated in closed form for the sine-Gordon breather [21]. Remarkably, their only effect is to replace $\gamma \equiv \lambda/m^2$ by

$$\gamma' = \frac{\lambda}{m^2}\left(1 - \frac{\lambda}{8\pi m^2}\right)^{-1}. \tag{2.110}$$

In fact, the first quantum corrections to the sine-Gordon kink mass can be obtained by the same substitution. Thus, the kink has a mass

$$M_{\text{kink}} = \frac{8m^3}{\lambda} - \frac{m}{\pi} + O(\lambda/m) = \frac{8m}{\gamma'} + O(\lambda/m), \tag{2.111}$$

while to the same order the breather states have masses

$$M_n = \frac{16m}{\gamma'}\sin\left(\frac{n\gamma'}{16}\right), \qquad n = 1, 2, \ldots, < \frac{8\pi}{\gamma'}. \tag{2.112}$$

For weak coupling, $\lambda/m^2 \ll 1$, we have

$$M_1 = m\left[1 - \frac{1}{6}\left(\frac{\lambda}{16m^2}\right)^2 + O(\lambda^3/m^6)\right], \tag{2.113}$$

so that in the limit $\lambda \to 0$ the lowest breather state becomes degenerate with the elementary boson of the theory. In fact, it *is* the elementary boson, just seen in a different guise. The higher excited states, with masses

$$M_n = M_1\left[n - \frac{1}{6}(n^3 - n)\left(\frac{\lambda}{16m^2}\right)^2 + O(\lambda^3/m^6)\right], \tag{2.114}$$

can be understood as bound states of n elementary bosons. As a check of this statement, Dashen, Hasslacher, and Neveu did a perturbative calculation of the binding energy of two elementary sine-Gordon bosons. Working to $O(\lambda^4)$, they obtained

$$\frac{2M_1 - M_2}{M_1} = \left(\frac{\lambda}{16m^2}\right)^2 + \frac{4}{\pi}\left(\frac{\lambda}{16m^2}\right)^3 + \left(\frac{12}{\pi^2} - \frac{1}{12}\right)\left(\frac{\lambda}{16m^2}\right)^4 + O(\lambda^5/m^{10}), \tag{2.115}$$

in perfect agreement with Eq. (2.112).

The special features of the sine-Gordon dynamics—the absence of energy loss or distortion in kink–kink and kink–antikink scattering, and the existence of dissipationless time-dependent breather solutions (as well as the ability to obtain such explicit results in closed form)—are due to the fact that the theory is a completely integrable system, with an infinite number of nontrivial conserved quantities. To explore other theories, such as the $\lambda\phi^4$ model, one must generally resort to numerical methods, even at the classical level.

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In the $\lambda\phi^4$ theory any two-soliton configuration must consist of a kink and an antikink, with their relative position determined by the fields at $x = \pm\infty$. For example, if $\phi(\pm\infty) = -v$, the kink (going from $-v$ to v) must always be to the left of the antikink (going from v to $-v$). If the two collide, they will recoil and reverse direction. The collision will not be completely elastic; some energy will be always be lost to radiation (i.e., small-amplitude oscillations of the scalar field). Throughout this process, the region between the solitons remains (approximately) in the vacuum at $\phi = v$. This should be contrasted with the sine-Gordon case, where the region between the solitons is in a different vacuum before and after the collision.

Behavior somewhat like the sine-Gordon case is possible in theories where $V(\phi)$ has more than two degenerate minima, not necessarily related by any symmetry. For example, consider a theory with degenerate vacua at $\phi = v_A$, v_B, and v_C, with $v_A < v_B < v_C$. There will be two types of kinks (A-to-B and B-to-C), and two types of antikinks. If a right-moving B-to-C kink collides with a left-moving C-to-B antikink, the emerging particles may be a right-moving A-to-B kink and a left-moving B-to-A antikink. The analogous behavior in the collision of domain walls separating different (not necessarily degenerate) vacua in four spacetime dimensions leads, in certain theories, to an interesting new possibility, termed classical transitions [22, 23], that are an alternative to the transitions via quantum mechanical bubble nucleation that will be discussed in Chap. 12.

There are no known examples of stable analogues of the sine-Gordon breather in other theories. However, it has been found that some theories support oscillating solutions that have a surprisingly long lifetime before their energy is eventually radiated away. These solutions, known as oscillons, can arise even in theories with a single vacuum, and hence no kink solutions [24–26]. For some recent work on these, see [27–29].

2.7 The sine-Gordon–massive Thirring model equivalence

In the previous section we saw that a weak-coupling analysis of the sine-Gordon theory reveals that the lowest breather state is in fact identical to the elementary boson of the theory. Even more remarkable results are suggested by an analysis of the spectrum in the strong-coupling limit.

A word of caution is in order. We might well expect that in the strong-coupling regime higher-order quantum corrections would be important, thus rendering both our expression for the kink mass and the WKB result for the breather masses meaningless. However, the results described below suggest that this may not be the case, and that the sine-Gordon theory is sufficiently special that the WKB formula for the mass might be exact, as indeed turns out to be the case [30, 31]. At the same time, though, it should be kept in mind that in the strong-coupling regime the kink Compton wavelength is greater than the classical soliton width and the direct significance of the classical field profile in the quantum state is uncertain.

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Equation (2.112) for the breather state masses gives

$$M_n = 2M_{\text{kink}} \sin\left[\frac{n\pi}{2}\,\frac{\lambda/(8\pi m^2)}{1-\lambda/(8\pi m^2)}\right]. \tag{2.116}$$

The requirement that

$$n < \frac{8\pi m^2}{\lambda} - 1 \tag{2.117}$$

implies that there are no breather states if $\lambda \geq 4\pi m^2$. Hence, the "elementary particle" of the theory, which we previously identified with the $n = 1$ breather state, is absent in this case. Indeed, already for $\lambda > 2\pi m^2$, the kink, and not the elementary ϕ, is the lightest particle in the theory.

To examine the spectrum for λ just below $4\pi m^2$, let us write

$$\frac{\lambda}{4\pi m^2} = \frac{1}{1+\delta/\pi}. \tag{2.118}$$

The energy of the lowest breather mode can then be written as

$$M_1 = M_{\text{kink}}\left[2 - \delta^2 + \frac{4\delta^3}{\pi} + O(\delta^4)\right]. \tag{2.119}$$

This suggests that for λ near the limiting value, the lowest (and only) breather state should perhaps be interpreted as a weakly bound state of a kink and an antikink.

Putting this result aside for a moment, let us consider the massive Thirring model [32]. This is a theory of a fermion field ψ in $(1+1)$-dimensional spacetime, with a Lagrangian density

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - M)\psi - \frac{g}{2}(\bar{\psi}\gamma^\mu\psi)^2. \tag{2.120}$$

The particle spectrum includes an elementary fermion and its antiparticle, each with mass M. In addition, if $g > 0$, so that the interaction is attractive, there are also fermion–antifermion bound states. The lowest of these (and the only one if g is sufficiently small), has a mass

$$M_{\text{bound}} = M\left[2 - g^2 + \frac{4g^3}{\pi} + O(g^4)\right]. \tag{2.121}$$

Note the resemblance between Eqs. (2.119) and (2.121). If we take this to be more than coincidence, we could identify δ and g, and write

$$\frac{\lambda}{4\pi m^2} = \frac{1}{1+g/\pi}. \tag{2.122}$$

The weak-coupling ($\lambda/m^2 \to 0$) limit of the sine-Gordon theory would then correspond to the strong-coupling ($g \to \infty$) limit of the massive Thirring model,

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while the $g \to 0$ weak-coupling limit of the latter would correspond to the limit $\lambda/m^2 \to 4\pi$. This would then suggest the correspondences

$$\begin{aligned} \text{kink} &\longleftrightarrow \text{elementary } \psi \\ \text{antikink} &\longleftrightarrow \text{elementary } \bar{\psi} \\ \text{elementary } \phi &\longleftrightarrow \bar{\psi}\text{-}\psi \text{ bound state} \\ \text{topological charge} &\longleftrightarrow \text{fermion number} \end{aligned}$$

between the two theories.[6]

The above remarks are suggestive, but hardly conclusive. They are made rigorous by an analysis of Coleman [33]. Adopting Eq. (2.122) and making the identifications

$$\begin{aligned} \frac{\sqrt{\lambda}}{2\pi m}\,\epsilon^{\mu\nu}\,\partial_\nu \phi &= -\bar{\psi}\gamma^\mu \psi, \\ \frac{m^4}{\lambda}\cos\left(\frac{\sqrt{\lambda}}{m}\phi\right) &= -M\bar{\psi}\psi \end{aligned} \tag{2.123}$$

(with appropriate renormalization and normal ordering of the composite operators), he showed that the matrix elements of the two theories agree to all orders of perturbation theory.

These identifications relate expressions involving the bosonic field ϕ with fermion bilinears. Mandelstam [34] showed that one could go further and construct the fermionic field from the bosonic field. In a basis where the gamma matrices are $\gamma^0 = \sigma_1$ and $\gamma^1 = i\sigma_2$, the upper and lower components of $\psi(x)$ are given by

$$\begin{aligned} \psi_1(x) &= C : \exp\left[-\frac{i\beta}{2}\phi(x) - \frac{2\pi i}{\beta}\int_{-\infty}^{x} dz\,\dot{\phi}(z)\right] :, \\ \psi_2(x) &= -iC : \exp\left[\frac{i\beta}{2}\phi(x) - \frac{2\pi i}{\beta}\int_{-\infty}^{x} dz\,\dot{\phi}(z)\right] :, \end{aligned} \tag{2.124}$$

where colons indicate normal ordering and C is a constant whose value need not concern us here. By using the formula

$$e^A e^B = e^{[A,B]} e^B e^A, \tag{2.125}$$

which holds when $[A, B]$ is a c-number, one can easily verify that $\psi(x)$ and $\psi(y)$ anticommute at equal times for $x \neq y$. Verifying that the correct anticommutator is obtained for $x = y$ is more subtle, but this can be done by an appropriate choice of the constant C and careful handling of the singularities.

[6] The correspondence between the bosonic kink and the fermionic ψ may seem particularly strange. Recall, however, that there is no spin in two-dimensional spacetime, so the distinction between fermions and bosons is based only on statistics. A change in statistics can be compensated by an interaction; e.g., the Pauli exclusion principle for fermions can be replaced by an appropriate repulsive interaction between bosons.

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Thus, the sine-Gordon and massive Thirring model are not two different theories, but rather two "dual" formulations that describe precisely the same physics. To state this differently, an experimenter in two-dimensional spacetime would not be able to distinguish between the two. The fact that the duality relates the strong-coupling regime of one formulation to the weak-coupling regime of the other provides a useful calculational tool, since quantities that are not perturbatively accessible in one formulation may be calculable in the other.

The duality also teaches us important conceptual lessons. Elementary introductions to quantum field theory typically begin with free field theory, where there is a one-to-one correspondence between quantum fields and "elementary" particles. Even after going on to treat the interacting theory, there is often a bias towards viewing these particles as the fundamental ones, with other objects, whether found as bound states or from semiclassical quantization of a soliton, seen as rather different entities. The sine-Gordon–massive Thirring duality shows that this is misguided, since a particle that is solitonic in one formulation is elementary in another. A solitonic particle may be no more and no less fundamental than one corresponding to a quantum of a weakly coupled field. The various descriptions—elementary particle, soliton, bound state—should be viewed as characterizations that are useful within the context of a particular formulation in certain regions of parameter space. In other regions of parameter space (e.g., when no formulation is in a weak-coupling regime) it may be that none of these characterizations are either useful or appropriate.

A final lesson concerns the existence of degenerate vacua. The sine-Gordon theory appears to have an infinite number of vacuum states, whereas the massive Thirring model has a unique vacuum. It might therefore seem strange that the two theories could be equivalent. However, there is no difficulty here, because the sine-Gordon vacua are all physically equivalent and cannot be distinguished by any local measurement. Indeed, if one views $\phi(x)$ as a periodic variable, with ϕ and $\phi + v$ identified, there is just a single sine-Gordon vacuum.

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