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Representing 3-manifolds by a universal branching set

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In this paper all 3-manifolds will be supposed to be *compact*, *connected*, *oriented* and *without 2-spheres in the boundary*.

Given a 3-manifold M we obtain a closed pseudomanifold $\hat{M}$ by capping off each boundary component of M with a cone. We prove that such an $\hat{M}$ is a covering of S^3 branched over a subcomplex G of S^3 which is independent of M, and such that $S^3 - G$ has free fundamental group on two generators. Hence $\hat{M}$ (and also M) can be represented by a transitive pair $\{\sigma, \tau\}$ of permutations in the symmetric group Σ_h on the set $\{1, 2, \ldots, h\}$, for some h. We show how to obtain $\{\sigma, \tau\}$ from a given Heegaard diagram of M.

Conversely, given $\{\sigma, \tau\}$ we reconstruct the corresponding manifold $M(\sigma, \tau)$ in two ways: by means of a polyhedral cell with pairs of faces identified and also from a Heegaard diagram of $M(\sigma, \tau)$. Thus we obtain a presentation for the fundamental group of $M(\sigma, \tau)$ directly from $\{\sigma, \tau\}$. Finally, given a finite presentation of a group, we give an algorithm which decides if the presentation comes from a Heegaard diagram of a 3-manifold with boundary.

1. *The universal branching set*

Let M be a closed 3-manifold and let B be a 1-dimensional subcomplex of M. In general, the coverings of M branched over B are not 3-manifolds because, though the link of any k-simplex, $k > 0$, is always a sphere, the link of a vertex might be a closed, connected orientable surface of non-zero genus. Thus we say that a compact connected three-dimensional polyhedron N is a *singular 3-manifold* if it admits a triangulation as an orientable closed pseudomanifold such that the link of each k-simplex, $k = 0, 1$, is connected. We will assume N *oriented* with a fixed orientation. It follows immediately that the link of any k-simplex, $k > 0$, is a sphere. Hence the most general singular 3-manifold is obtained by capping off each boundary component of a 3-manifold M with a cone. We call $\hat{M}$ such a singular 3-manifold and we say that the vertices of the cones with base a surface of non zero genus, are the *singular points* of $\hat{M}$. In this way, any 3-manifold can be represented by a singular 3-manifold $\hat{M}$. Conversely, a singular 3-manifold N determines (up to isomorphism) a unique 3-manifold $\check{N}$. And this uniqueness explains why we rule out 3-manifolds with 2-sphere boundary components.

The next theorem was proved by Ramírez (7) and generalizes another by Alexander (1). We state it for singular 3-manifolds since this does not introduce any real change in the proof, which we give for the benefit of the reader.

THEOREM 1 (*Ramírez*). *Every singular 3-manifold is a covering of* S^3 *branched over the 1-skeleton of a 3-simplex.*

Proof. We barycentrically subdivide a pseudomanifold triangulation of the singular

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3-manifold N. Then each new 3-simplex, having a natural order in the set of its vertices, gets a *natural orientation*. But each new 3-simplex has also an *inherited orientation* coming from the old simplex in which it lies. If the natural orientation of a new 3-simplex coincides with the inherited one we colour that simplex in black. Otherwise we colour it in white. The result is a chess-board coloration, because two new 3-simplices which touch in a common 2-face have coherent inherited orientations but the natural orientations are not coherent.

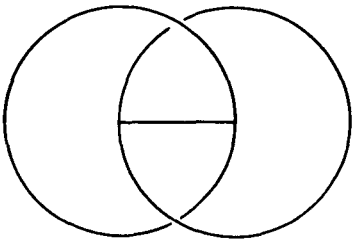


Fig. 1.

Let us form S^3 by pasting together two 3-simplices, one black and the other white, along their boundaries and give an order to the resulting four vertices. Define the simplicial map $f: N \to S^3$ which sends the black (white) 3-simplices onto the black (white) 3-simplex linearly, preserving the order of the vertices. Thus f is a branched covering (2) and the branching set lies in the 1-skeleton of a 3-simplex. |

Let G be the subset of S^3 depicted in Figure 1. We now have the following

COROLLARY 2. *Every singular 3-manifold is a covering of S^3 branched over G.*

Proof. By theorem 1, the singular 3-manifold N is a covering of S^3 branched over the 1-skeleton of a tetrahedron t, which we can assume regular and lying in $R^3 \subset R^3 + \infty$. Then the 120° rotation of $R^3 + \infty$ around an axis of three-fold symmetry of t, defines a covering of S^3 onto itself. The branching set of this covering together with the image of the 1-skeleton of t is G. |

It is well known (see (8) for instance) that h-fold coverings of S^3 branched over G are in one-to-one correspondence with natural equivalence classes of transitive representations of $\pi_1(S^3 - G)$ into the symmetric group Σ_h on the set $\{1, 2, \ldots, h\}$. Since the fundamental group of $S^3 - G$ is a free group on the generators $\{x, y\}$ depicted in Figure 2, it follows that given a pair of permutations $\{\sigma, \tau\}$ of Σ_h, we have a representation $\omega: \pi_1(S^3 - G) \to \Sigma_h$ sending x to σ and y to τ, which is transitive if and only if the subgroup of Σ_h generated by $\{\sigma, \tau\}$ acts transitively on $\{1, 2, \ldots, h\}$. Hence to each transitive pair of permutations $\{\sigma, \tau\}$ we can associate a singular 3-manifold $N(\sigma, \tau)$, namely the h-fold covering of S^3 branched over G, associated to the representation ω. Now we can restate Corollary 2 as follows.

COROLLARY 2′. *Every 3-manifold (with no 2-spheres in the boundary) can be represented as $\breve{N}(\sigma, \tau)$, for some transitive pair of permutations $\{\sigma, \tau\}$.*

The following proposition complements Corollary 2.

PROPOSITION 3. *We have the following orientation-preserving homeomorphisms:*

$$\text{(i)} \quad N(\sigma, \tau) \cong N(\sigma^{-1}, \tau^{-1})$$

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(ii) $N(\sigma,\tau) \cong N(\tau,\sigma)$

(iii) $N(\sigma,\tau) \cong -N(\sigma,\tau^{-1})$

where the minus sign denotes change of orientation.

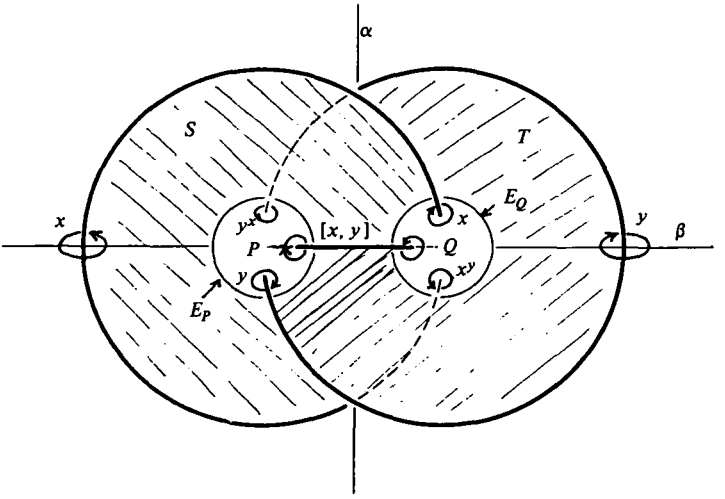


Fig. 2.

Proof. In Fig. 2 we have depicted two orthogonal discs S and T. The set G is the union of $S \cap T$ with the boundaries of S and T. The α-axis is at an angle $\pi/4$ to the discs S and T. To prove (i) consider the reflection in the β-axis which fixes G as a set and sends (x,y) to (x^{-1},y^{-1}). To prove (ii) note that the reflection in the α-axis fixes G as a set and sends (x,y) to (y^{-1},x^{-1}). Finally, to prove (iii) consider the orientation-reversing reflection in the disc T which fixes G as a set and sends (x,y) to (x,y^{-1}). ∎

COROLLARY 4. *$N(\sigma,\tau)$ is symmetric if $\sigma = \sigma^{-1}$ or $\tau = \tau^{-1}$.* ∎

PROPOSITION 5. *The branching set of $N(\sigma,\tau)$ can be reduced to two simply linked unknots if and only if $[\sigma,\tau] = 1$. In this case the covering is regular.* ∎

Example 1. If $\sigma = 1$, then $\tau = (12\ldots h)$ and $N(1,\tau) \cong S^3$.

Example 2. If $\sigma = \tau$, then $\sigma = \tau = (12\ldots h)$ and $N(\sigma,\sigma)$ is a cyclic covering branched over two simply linked unknots, and then $N(\sigma,\sigma)$ is the lens space $L(h,1)$. Also $N(\sigma,\sigma^{-1}) = -L(h,1)$.

Remark. The representation of Corollary 2′ gives a *universal ramification*, but note that the number of sheets can be arbitrarily high.

Let H be a subgroup of Σ_h. Then H acts on the set $\{1,2,\ldots,h\}$ dividing it into orbits. We denote the number of orbits of this action by $c(H)$. We use the notation $\mathcal{G}(S)$, where S in a non empty subset of Σ_h, to denote the subgroup of Σ_h generated by S. We write $c(\sigma)$ for $c(\mathcal{G}(\sigma))$, $\sigma \in \Sigma_h$. Thus $c(\sigma)$ is the number of cycles of the permutation σ. We use the notation $\sigma^\tau, \sigma,\tau \in \Sigma_h$, for $\tau\sigma\tau^{-1}$.

PROPOSITION 6. *The Euler characteristic $\chi(N(\sigma,\tau))$ of $N(\sigma,\tau)$ is*

$$h + c(\sigma,\sigma^\tau) + c(\tau,\tau^\sigma) - c(\sigma) - c(\tau) - c[\sigma,\tau].$$

Proof. We refer to Fig. 2. The two 2-discs S and T provide a cellular decomposition

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of S^3 in two 0-cells P and Q; three closed 1-cells ∂S, ∂T and $S \cap T$; two closed 2-cells S and T; and just one 3-cell. This cellular decomposition lifts to one for $N(\sigma,\tau)$ with h 3-cells, $2h$ 2-cells, $c(\sigma)+c(\tau)+c[\sigma,\tau]$ 1-cells and $c(\sigma,\sigma^\tau)+c(\tau,\tau^\sigma)$ 0-cells. To obtain the number of 0-cells we take two little 2-spheres E_P, E_Q with centres at P and Q,

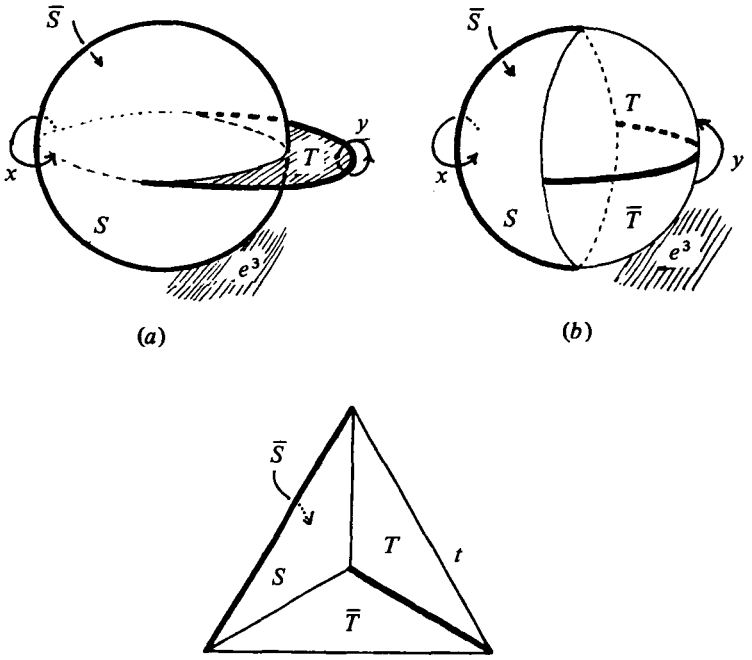


Fig. 3.

respectively, and we compute the number of components in the preimages of E_P and E_Q. The preimage of E_P is a covering of E_P branched over three points and corresponding to a representation $\pi_1(E_P - G) \to \Sigma_h$ whose image is

$$g\not{p}(\tau,[\sigma,\tau],\tau^\sigma) = g\not{p}(\tau,\tau^\sigma).$$

Hence the number of components in the preimage of E_P is $c(\tau,\tau^\sigma)$; similarly with E_Q. ∎

Since $\chi(N(\sigma,\tau))$ is the sum of the genera of the links of the singular points ((8); p. 216) of $N(\sigma,\tau)$ we have immediately

COROLLARY 7. *$N(\sigma,\tau)$ is a closed 3-manifold if and only if*

$$c(\sigma)+c(\tau)+c(\sigma,\tau] = h+c(\sigma,\sigma^\tau)+c(\tau,\tau^\sigma). \text{ ∎}$$

By the *exterior of a knot* we mean a 3-manifold E^3 whose boundary consists of a torus.

COROLLARY 8. *$\check{N}(\sigma,\tau)$ is the exterior of a knot if and only if*

$$c(\sigma)+c(\tau)+c[\sigma,\tau] = (h-1)+c(\sigma,\sigma^\tau)+c(\tau,\tau^\sigma). \text{ ∎}$$

Example 1. Let $\sigma = (123)$, $\tau = (124)$ and $h = 4$. Then

$$[\sigma,\tau] = (12)(34),\ c(\sigma,\sigma^\tau) = c((123),(243)) = 1,\ c(\tau,\tau^\sigma) = 1.$$

Thus $N(\sigma,\tau)$ is a 3-manifold.

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Example 2. Let $\sigma = (12345)$, $\tau = (123)(45)$ and $h = 5$. Then $[\sigma, \tau] = (24)(35)$, $c(\sigma, \sigma^\tau) = c(\tau, \tau^\sigma) = 1$. Hence $\check{N}(\sigma, \tau)$ is the exterior of a knot.

Thus Corollaries 7 and 8 give a *representation of closed 3-manifolds and exteriors of knots by pairs of permutations.*

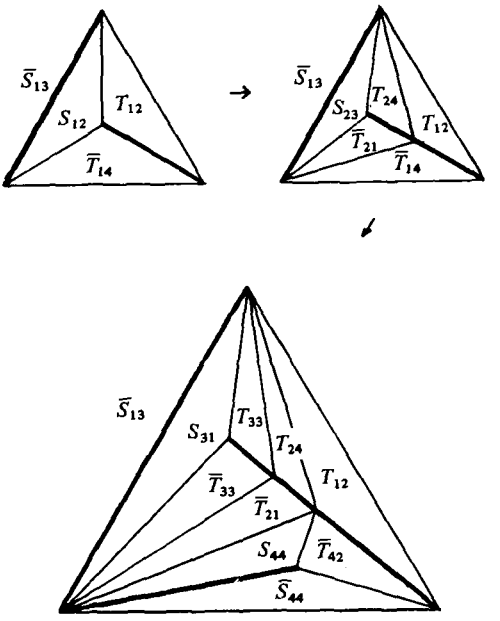


Fig. 4.

2. *Reconstructing the covering*

In this section, given the pair $\{\sigma, \tau\} \in \Sigma_h$ with $c(\sigma, \tau) = 1$ we are going to construct $N(\sigma, \tau)$ using the method of the 'splitting complex'(4).

A splitting complex for G is the set $S \cup T$ depicted in Fig. 2. The result of cutting S^3 open along $S \cup T$ is the tetrahedron t with two opposite edges marked, illustrated in Fig. 3*c*.

To reconstruct $N(\sigma, \tau)$ take h copies $\{t_1, \ldots, t_h\}$ of t; label the faces S, $\bar{S}$, T, $\bar{T}$ in the copy t_i as $S_{i\sigma(i)}$, $\bar{S}_{i\sigma^{-1}(i)}$, $T_{i\tau(i)}$, $\bar{T}_{i\tau^{-1}(i)}$ respectively; and identify S_{ij} with $\bar{S}_{ji}$ and T_{ij} with $\bar{T}_{ji}$ by an orientation-reversing linear homeomorphism respecting marked edges.

Example. Let $\sigma = (123), \tau = (124)$ and $h = 4$. Then $N(\sigma, \tau)$ is the result of performing, in the boundary of a 3-cell, the identifications shown in Fig. 4.

We call a presentation $(X; R)$ of a group *cubic*, following Neuwirth, if each generator x from X occurs exactly three times in the elements of the set R (compare ((3); page 146)). The following theorem was proved by Hoare for the case of closed 3-manifolds ((3); page 146).

THEOREM 9. *The fundamental group of any 3-manifold (possibly with boundary) admits a cubic presentation.*

Proof. It is easy to get a Heegaard diagram for $\check{N}(\sigma, \tau)$ as is indicated in Fig. 5, where Poincaré's notation (6) for a Heegaard diagram is used, i.e. there is a set of arcs properly embedded in a planar surface, bounded by circles $1, \bar{1}, 2, \bar{2}, \ldots$ with dots; the circles are

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to be identified in pairs to form one of the two systems of curves of the Heegaard diagram. This identification is determined by matching the dots together and also the endpoints of the arcs, so that these arcs constitute the second system of curves of the Heegaard diagram. From the Heegaard diagram one can obtain a presentation of

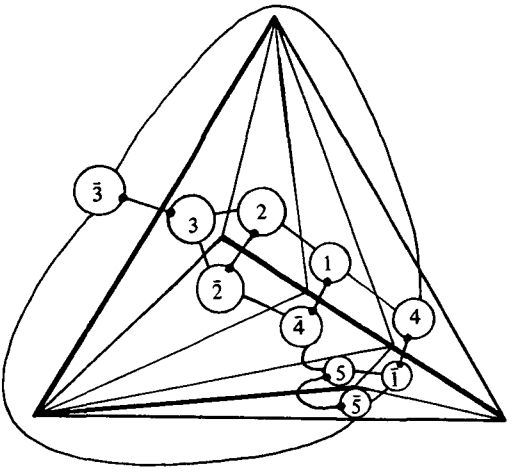


Fig. 5.

$\pi_1(\check{N}(\sigma,\tau))$, where the generators are the curves 1, 2, ... and the relations are obtained by recording the intersections (with signs) of the second system of curves with the circles $1, 2, \ldots$ (compare ((9);(4)). Since each circle of Fig. 5 meets exactly three arcs, the theorem follows. |

In the last example, the reduction of the Heegaard diagram of Figure 5 gives $N(\sigma,\tau) \cong S^3$.

Example. Let $\sigma = (12\ldots p)^q$, $\tau = (12\ldots p)$ and $h = p$, where p and q are coprime. Then $N(\sigma,\tau)$ is obtained by pasting p tetrahedra together as indicated in Fig. 6 ($p = 5$, $q = 2$). Thus $N(\sigma,\tau)$ is the lens space $L(p,q)$.

3. *The canonical Heegaard splitting of* $\check{N}(\sigma,\tau)$

Let S_g be a closed, oriented, surface of genus g. A *hollow handlebody* X_g of genus g is the 3-manifold obtained by adding (successively) a number of 2-handles and of 3-handles (in this order) to $S_g \times [0,1]$ along $S_g \times \{1\}$. We say that $S_g \times \{0\}$ is the *free boundary* of X_g.

A *generalized Heegaard splitting* of a 3-manifold M is a pair (M, S_g) such that M is the union of two hollow handlebodies X_g and Y_g whose intersection $X_g \cap Y_g$ equals S_g and is the free boundary of both X_g and Y_g. If at least one of X_g and Y_g has empty boundary we say that (M, S_g) is a *Heegaard splitting*.

Let F be a torus embedded in S^3 in such a way that the closures X and Y of the two components of $S^3 - F$ are regular neighbourhoods of ∂S and ∂T, respectively (Fig. 7).

THEOREM 10. *The preimage* $\tilde{F}$ *of* F *in* $N(\sigma,\tau)$ *is a closed, connected and oriented surface of genus* $1 + \frac{1}{2}(h - c[\sigma,\tau])$. *Moreover* $(\check{N}(\sigma,\tau), \tilde{F})$ *is a generalized Heegaard splitting.*

Proof. The preimage $\tilde{F}$ of F is the covering of F, branched over R, corresponding to

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the representation, $\pi_1(F-R) \cong \mathbb{Z} * \mathbb{Z} \to \Sigma_h$ which sends the generators (x, y) into (σ, τ). Since this is a transitive representation, $\tilde{F}$ is a connected (closed and orientable) surface whose Euler characteristic is $\chi(\tilde{F}) = -h + c[\sigma, \tau]$.

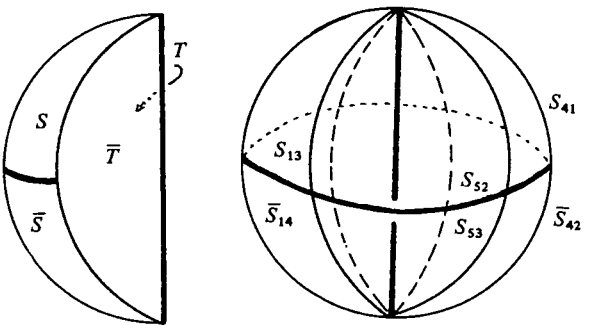


Fig. 6.

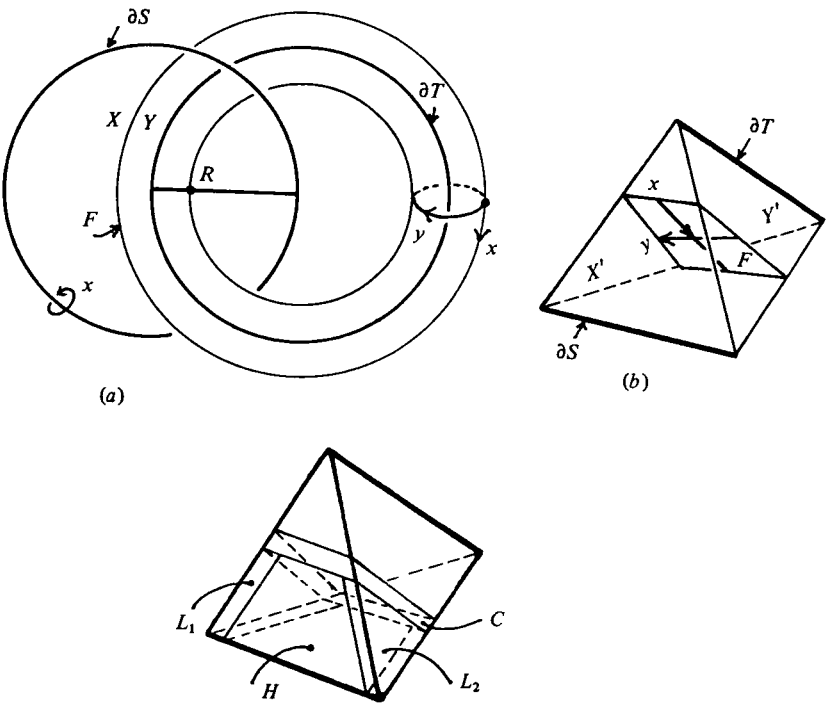


Fig. 7.

The closure of the two components, into which $\tilde{F}$ separates $N(\sigma, \tau)$, are the pre-images $\tilde{X}$ and $\tilde{Y}$ of X and Y respectively. To finish the proof we have to show that both $\tilde{X} \cap \check{N}(\sigma, \tau)$ and $\tilde{Y} \cap \check{N}(\sigma, \tau)$ are hollow handlebodies. In the rest of the proof we refer to Fig. 7. In section 2 we have seen that $\tilde{X}$ (and analogously $\tilde{Y}$) is obtained by pasting h copies of X' (Fig. 7*b*) together linearly. If from each one of the h copies of X' which constitute $\tilde{X}$ we subtract the slabs L_1 and L_2 of Fig. 7*c*, the effect in $\tilde{X}$ is to delete regular neighbourhoods of the vertices of $N(\sigma, \tau)$ that lie in its interior. The result,

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which we denote by $\tilde{X}'$, is a hollow handlebody whose free boundary is $\tilde{F}$. In fact, $\tilde{X}'$ is the addition of h copies of C (which produces a collar of $\tilde{F}$) and of h copies of H which define a collection of 2-handles added to $\partial C - \tilde{F}$. Finally, $\tilde{X} \cap \check{N}(\sigma,\tau)$ is the result of adding a number of 3-balls to $\tilde{X}'$, namely the stars of the vertices of $N(\sigma,\tau)$ which lie in $\tilde{X}$ and which are not singular points of $N(\sigma,\tau)$. |

PROPOSITION 11. *The sum of the genera of the links of the singular points in the preimage $\tilde{X}$ of X (resp. $\tilde{Y}$ of Y) is*

$\frac{1}{2}(h - c[\sigma,\tau]) - c(\sigma) + c(\sigma,\sigma^\tau)$ *(resp.* $\frac{1}{2}(h - c[\sigma,\tau]) - c(\tau) + c(\tau,\tau^\sigma)$*).*

Proof. By (8), the Euler characteristic of $2\tilde{X}$ is twice the sum of the genera of the links of the singular points in $\tilde{X}$. Now

$$\begin{aligned}\chi(2\tilde{X}) &= 2\chi(\tilde{X}) - \chi(\tilde{F}) = 2\chi(\text{preimage of } \partial S) - (2 - 2\,\text{genus } \tilde{F}) \\ &= 2(c(\sigma,\sigma^\tau) - c(\sigma)) + h - c[\sigma,\tau]. \;|\end{aligned}$$

Thus we have:

ADDENDUM TO THEOREM 10. *$(\check{N}(\sigma,\tau),\tilde{F})$ is a Heegaard splitting if and only if* $\frac{1}{2}(h - c[\sigma,\tau]) - c(\sigma) + c(\sigma,\sigma^\tau) = 0$ *or* $\frac{1}{2}(h - c[\sigma,\tau]) - c(\tau) + c(\tau,\tau^\sigma) = 0$. |

We also have (compare Corollaries 7 and 8):

COROLLARY 12. (i) *The condition $c(\sigma) + c(\tau) + c[\sigma,\tau] = h + c(\sigma,\sigma^\tau) + c(\tau,\tau^\sigma)$ for $N(\sigma,\tau)$ to be a closed 3-manifold is equivalent to the conditions*

$$\tfrac{1}{2}(h - c[\sigma,\tau]) = c(\sigma) - c(\sigma,\sigma^\tau) = c(\tau) - c(\tau,\tau^\sigma).$$

(ii) *The condition $c(\sigma) + c(\tau) + c[\sigma,\tau] = (h-1) + c(\sigma,\sigma^\tau) + c(\tau,\tau^\sigma)$ for $\check{N}(\sigma,\tau)$ to be the exterior of a knot is equivalent to the conditions*

$$\tfrac{1}{2}(h - c[\sigma,\tau]) = c(\sigma) - c(\sigma,\sigma^\tau) = 1 + c(\tau) - c(\tau,\tau^\sigma),$$

modulo changing order in $\{\sigma,\tau\}$. |

4. *The canonical Heegaard diagram for $\check{N}(\sigma,\tau)$*

Let (M, S_g) be a generalized Heegaard splitting. It is then possible to reconstruct M by adding to $S_g \times [-1,1]$ a number of 2-handles and after that capping off the 2-spheres of the resulting boundary. Thus (M, S_g) is determined by a *generalized Heegaard diagram* $(S_g; v, w)$ where $v = \{v_1, \ldots, v_s\}$ is the system of attaching spheres of the 2-handles added to $S_g \times \{1\}$ and $w = \{w_1, \ldots, w_t\}$ is the system for the 2-handles attached to $S_g \times \{-1\}$. If (M, S_g) is a Heegaard splitting we say that $(S; v, w)$ is simply a *Heegaard diagram*, i.e. $S_g - v$ (say) must be planar.

Let $(\check{N}(\sigma,\tau),\tilde{F})$ be the generalized Heegaard splitting of section 3. From the proof of Theorem 10, we deduce that the preimages $v = \{v_1, \ldots, v_{c(\sigma)}\}$ and $w = \{w_1, \ldots, w_{c(\tau)}\}$ of the curves x and y of Fig. 7*a*, respectively, define a generalized Heegaard diagram $(\tilde{F}; v, w)$. To see this, note that x and y appear in Fig. 7*b* as depicted, and thus their copies in the h tetrahedra $t_1, \ldots, t_h$ constitute the attaching spheres of the 2-handles that are mentioned in the proof of Theorem 10. Thus we have:

THEOREM 13. *$(\tilde{F}; v, w)$, where $v = \{v_1, \ldots, v_{c(\sigma)}\}$, $w = \{w_1, \ldots, w_{c(\tau)}\}$ are the preimages of x and y* (Fig. 7), *is a generalized Heegaard diagram for $\check{N}(\sigma,\tau)$.* |

Remark. $(\tilde{F}; v, w)$ can be constructed directly, given $\{\sigma,\tau\}$. For take h copies $C_1, \ldots, C_h$

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of the square C of Fig. 8 and label the edges S, $\bar{S}$, T, $\bar{T}$ in C_i by $S_{i\sigma(i)}$, $\bar{S}_{i\sigma^{-1}(i)}$, $T_{i\tau(i)}$, $\bar{T}_{i\tau^{-1}(i)}$, respectively. Now identify S_{ij} with $\bar{S}_{ji}$ and T_{ij} with $\bar{T}_{ji}$. Then we get $\tilde{F}$, and the union of the copies of the curves x (resp. y) compose the system v (resp. w).

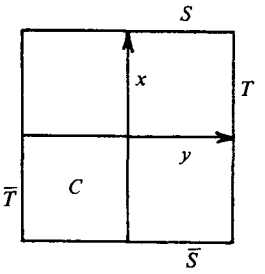


Fig. 8.

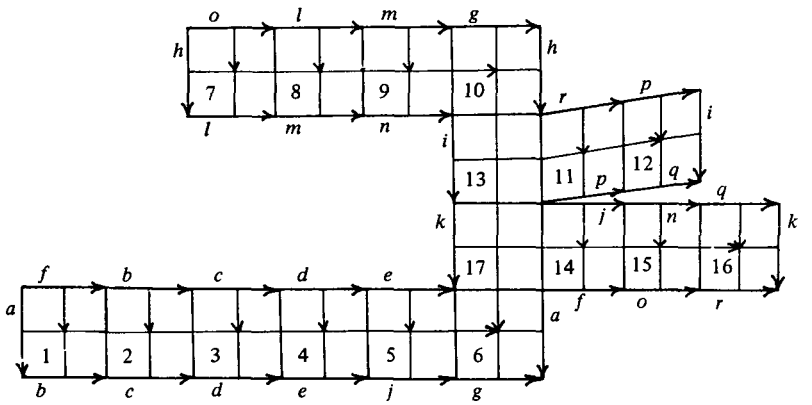


Fig. 9.

Example. Let $\sigma = (1\,2\,3\,4\,5\,6)(7\,8\,9,10)(11,12,13)(14,15,16,17)$, $\tau = (1\,2\,3\,4\,5,14)(7\,8\,9,15)(11,12,16)(6,10,13,17)$ and $h = 17$. The Heegaard diagram $(\tilde{F}; v, w)$ for the closed 3-manifold $N(\sigma,\tau)$ (Poincaré manifold) is shown in Fig. 9.

We call a generalized Heegaard diagram $(S_g; v, w)$ *positive* if it is possible to orient v and w in such a way that, for every $z \in v \cap w$, the intersection number of v and w in z is $+1$.

From the last remark and example we easily deduce:

ADDENDUM TO THEOREM 13. *The diagram $(\tilde{F}; v, w)$ is positive.* |

5. *Determining $\{\sigma,\tau\}$ from a Heegaard diagram.*

THEOREM 14. *Let $(S_g; v, w)$ be a Heegaard diagram. Then, there is an algorithm which determines $h \geqslant 1$ and permutations $\sigma,\tau \in \Sigma_h$, with $c(\sigma,\tau) = 1$, in such a way that $\check{N}(\sigma,\tau)$ is the 3-manifold given by the Heegaard diagram.*

Proof. If $g = 0$, then $M \cong S^3$. For $g > 0$ we describe the algorithm in several steps.

Step 1. Delete curves from v to make $S_g - v$ planar and connected.

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Step 2. If necessary add to $(S_g; v, w)$ a cancelling pair of handles to make $w \neq \phi$.

Step 3. *Make* $w_i \cap v \neq \phi$ *for every* $w_i \in w$. To do this, cut S_g along v and also along the components of w cutting v. Thus we get a number of connected planar surfaces

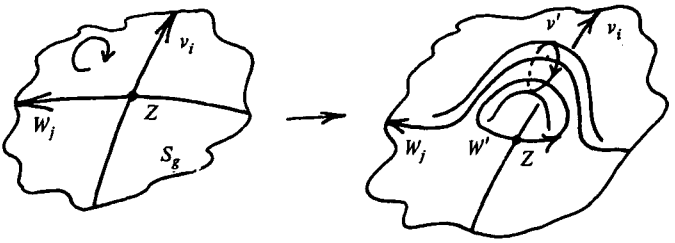


Fig. 10.

$C_1, \ldots, C_r$. Assume there is w_i such that $w_i \cap v = \phi$. Then $w_i \subset C_j$, for some j. Among all components of w lying in the interior of C_j select one, say w_k, such that by pushing part of w_k through C_j to cut $\partial C_j \cap v$ we do not disturb $w - w_k$. In this way we can reduce the components of w which do not cut v.

Step 4. *Make each component of* $S_g - v \cup w$ *simply connected*. To do this, cut S_g along $v \cup w$ to get a number of connected planar surfaces $C_1, \ldots, C_r$. By steps 1 and 2 $\partial C_i \cap w \neq \phi$, and by step 3 $\partial C_i \cap v \neq \phi$, $1 \leqslant i \leqslant r$. If C_i has more than one boundary component we can push part of w through C_i to cut v in a different boundary component. This reduces the number of C_i's with no connected boundary. Hence we can assume ∂C_i connected, for $1 \leqslant i \leqslant r$.

Step 5. *Make the diagram positive*. Orient S_g, v and w arbitrarily. If in $z \in v \cap w$ the intersection number of v and w is -1 we modify $(S_g; v, w)$ near z according to Fig. 10

Step 6. If h is the number of intersection points of $v \cap w$, define a bijective map φ from $\{1, 2, \ldots, h\}$ to the set of points of $v \cap w$.

Then the *oriented* curves $v_i \in v$ (resp. $w_i \in w$) define cycles $c(v_i) \in \Sigma_h$. (resp. $c(w_i)$). Define $\sigma = c(v_1) \ldots c(v_g)$, $\tau = c(w_1) \ldots c(w_t)$.

CLAIM. $\check{N}(\sigma, \tau)$ *is the 3-manifold represented by the original Heegaard diagram* $(S_g; v, w)$.

Proof of the claim. The changes made in steps 1–5 give a new Heegaard diagram $(S'_g; v', w')$ which defines the same 3-manifold as $(S_g; v, w)$. Now $S'_g - (v' \cup w')$ is composed of simply connected components. We take the cellular decomposition of S_g which is dual to the one defined by $v \cup w$. The 2-cells of this dual decomposition are four-sided polygons each one containing a point $z \in v \cap w$. Give the label $C_{\varphi(z)}$ to the polygon containing z. Then S_g is composed of those polygons identified exactly in the same way as in the remark after Theorem 13. |

Example. Fig. 11 shows how to get $\{\sigma, \tau\}$ giving $N(\sigma, \tau) \cong S^1 \times S^2$: $\sigma = (1\,2\,3)(4\,5\,6)$, $\tau = (1\,3\,6\,4)(2\,5)$.

COROLLARY 15. (i) *Every 3-manifold is homeomorphic to* $\check{N}(\sigma, \tau)$ *where* $c(\sigma, \sigma^\tau) = 1$ *and* $c(\sigma) = 1 + \frac{1}{2}h - c[\sigma, \tau])$.

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(ii) *Every closed 3-manifold is homeomorphic to $N(\sigma,\tau)$ where*

$$c(\sigma) = c(\tau), \quad c(\sigma,\sigma^\tau) = c(\tau,\tau^\sigma) = 1.$$

(iii) *Every exterior of a knot is homeomorphic to $\check{N}(\sigma,\tau)$ where*

$$c(\sigma) = c(\tau)+1, \quad c(\sigma,\sigma^\tau) = c(\tau,\tau^\sigma) = 1.$$

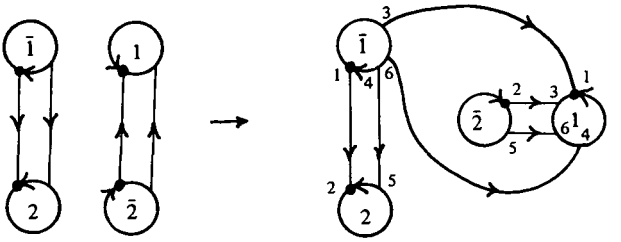


Fig. 11.

Proof. (i) Since every 3-manifold M has a Heegaard diagram, Theorem 14 gives $\{\sigma,\tau\}$, representing M, such that genus $\tilde{F} = c(\sigma)$ and $\tilde{X}$ is a handlebody. Then by Theorem 10, we have $c(\sigma) = 1+\frac{1}{2}(h-c[\sigma,\tau])$ and by Proposition 11, we have

$$\tfrac{1}{2}(h-c[\sigma,\tau]) = c(\sigma)-c(\sigma,\sigma^\tau).$$

(ii) Since every closed 3-manifold M has a Heegaard diagram $(S_g; v, w)$ where g equals the number of curves in v and in w, Theorem 14 gives $\{\sigma,\tau\}$ representing M with $c(\sigma) = c(\tau)$. By (i) $c(\sigma,\sigma^\tau) = c(\tau,\tau^\sigma) = 1$.

(iii) Every knot exterior has a Heegaard diagram $(S_g; v, w)$ where v has g curves and w has $g-1$. Theorem 14 gives $\{\sigma,\tau\}$ where $c(\sigma) = c(\tau)+1$. Part (i) gives $c(\sigma,\sigma^\tau) = 1$ and Proposition 11 gives $\frac{1}{2}(h-c[\sigma,\tau]) = 1+c(\tau)-c(\tau,\tau^\sigma)$. Hence $c(\tau,\tau^\sigma) = 1$. ∎

6. *Presentation for $\pi_1(\check{N}(\sigma,\tau))$.*

Let $\sigma = s_1 s_2 \dots s_{c(\sigma)}$, $\tau = t_1 t_2 \dots t_{c(\tau)}$ be elements of Σ_h written as a product of independent cycles and such that $c(\sigma,\tau) = 1$. Then the orbits $A_1,\dots,A_c$, of the action of $\mathcal{G}(\sigma,\sigma^\tau)$ on $\{1,2,\dots,h\}$ are union of cycles of σ, where $c = c(\sigma,\sigma^\tau)$. We can construct a graph with vertices $A_1,\dots,A_c$, where $c = c(\sigma,\sigma^\tau)$, as follows. For each cycle s_i in σ take the orbit A_k containing s_i^7 and join the orbit A_l, containing s_i, with A_k by an edge marked s_i. We get a graph R with $c(\sigma)$ edges and $c(\sigma,\sigma^\tau)$ vertices which can be identified with the preimage of ∂S in $N(\sigma,\tau)$. Consider a maximal tree $T \subset R$. Then the edges in T are marked with cycles $s_{i_1}, s_{i_2},\dots,s_{i_e}$, where $e = c(\sigma,\sigma^\tau)-1$, which we call a *superfluous system of cycles of* $\{\sigma,\tau\}$.

THEOREM 16. *Let $\{\sigma,\tau\}$ be such that $c(\sigma,\tau) = 1$ and $\frac{1}{2}(h-c[\sigma,\tau]) = c(\sigma)-c(\sigma,\sigma^\tau)$ and write $\sigma = s_1 s_2 \dots s_{c(\sigma)}$, $\tau = t_1 t_2 \dots t_{c(\tau)}$ as products of independent cycles. Then*

$$\{s_1, s_2,\dots,s_{c(\sigma)}\colon t_1 = t_2 = \dots = t_{c(\tau)} = 1,\ s_{i_1} = s_{i_2} = \dots = s_{i_e} = 1\}$$

is a presentation for $\pi_1(\check{N}(\sigma,\tau))$, where $s_{i_1},\dots,s_{i_e}$ is a superfluous system of cycles of $\{\sigma,\tau\}$, e is $c(\sigma,\sigma^\tau)-1$, and t_j is the word in $\{s_1,\dots,s_{c(\sigma)}\}$ obtained by replacing each number in the cycle t_j by the cycle s_k to which that number belongs.

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Remark. By Corollary 12, the condition $\frac{1}{2}(h-c[\sigma,\tau]) = c(\sigma)-c(\sigma,\sigma^\tau)$ follows automatically when $N(\sigma,\tau)$ is a closed 3-manifold or, up to a change of order in $\{\sigma,\tau\}$, when $\check{N}(\sigma,\tau)$ is a knot exterior. Note also that every M can be represented by $\check{N}(\sigma,\tau)$ satisfying that condition (Corollary 15(i)).

Proof. By Proposition 11, $\tilde{X}$ is a handlebody of genus

$$1+\tfrac{1}{2}(h-c[\sigma,\tau]) = c(\sigma)-(c(\sigma,\sigma^\tau)-1).$$

By the proof of Theorem 10, there are $c(\sigma)$ 2-handles in $\tilde{X}$, corresponding to the cycles $s_1,\dots,s_{c(\sigma)}$, such that the cocore of the 2-handle corresponding to s_i is part of the edge marked s_i in the graph R constructed above (which was identified with the preimage of ∂S in $N(\sigma,\tau)$). Then the cores of the 2-handles corresponding to the set

$$\{s_1,\dots,s_{c(\sigma)}\}-\{s_{i_1},\dots,s_{i_e}\},$$

where $e = c(\sigma,\sigma^\tau)-1$, are a complete system of meridians for $\tilde{X}$, which we can interpret as a system of generators for $\pi_1(\check{N}(\sigma,\tau))$. The relations are obtained from the attaching spheres of the $c(\tau)$ 2-handles which define $\tilde{Y}$. Now, compare this with the canonical Heegaard diagram of section 4 to understand the last part of the Theorem. ∎

Example. Let $\sigma = (1\,2\,3\,4)$, $\tau = (1\,2\,3)(4\,5\,6)$. Note that $c(\sigma,\tau) = 1$ and that the orbits of the action of $\mathcal{G}\!\!\!/(\sigma,\sigma^\tau)$ are $A_1 = (1\,2\,3\,4\,5)$, $A_2 = (6)$. Then for $s_2 = (5)$ we have $s_2^\tau = (6) = A_2$. Hence s_2 is superfluous. Thus we have

$$\pi_1(\check{N}(\sigma,\tau)) = \{s_1,s_2,s_3,: s_1^3 = s_1s_2s_3 = 1, s_2 = 1\} \cong \mathbb{Z}_3.$$

A presentation of a group is *positive* when the letters in the relators appear always with positive exponents.

ADDENDUM TO THEOREM 16. *The presentation given in theorem* 16 *is positive.*

Proof. This is an immediate consequence of the fact that the canonical Heegaard diagram for $\check{N}(\sigma,\tau)$ is positive.

Remark. When $c(\sigma,\sigma^\tau) = 1$ there are no superfluous cycles and the presentation for $\pi_1(\check{N}(\sigma,\tau))$ can be read directly. This occurs in the case (dual to the previous example) when $\sigma = (1\,2\,3)(4\,5\,6)$, $\tau = (1\,2\,3\,4)(5)(6)$ in which $c(\sigma,\sigma^\tau) = 1$. However, even when $N(\sigma,\tau)$ is a closed 3-manifold one can have both $c(\sigma,\sigma^\tau)$ and $c(\tau,\tau^\sigma)$ bigger than 1.

Example (pointed out by E. Bernal and J. Sebastián).

Let $\sigma = (1\,2\,3\,4)(5\,6\,7\,8)$, $\tau = (1\,5\,3\,7)(2\,8\,4\,6)$. Here $c(\sigma,\sigma^\tau) = c(\tau,\tau^\sigma) = 2$, and $N(\sigma,\tau)$ is a closed 3-manifold whose fundamental group is $\mathbb{Z}_2$.

7. *An algorithm*

Assume we are given a finite presentation P of a group. What 3-manifolds, if any, have a Heegaard diagram whose associated presentation for the fundamental group is P? Neuwirth(5) has solved this problem for closed 3-manifolds and balanced presentations (same number of generators and relations). We are going to obtain results for 3-manifolds with boundary.

Let $(S_g; v, w)$ be a Heegaard diagram such that v has g curves. Cutting S_g open along v we get a connected, planar surface S_v. After collapsing each boundary component of S_v to a point, the parts of w lying in S_v plus the collapsed boundary give rise to a graph that we call the *Whitehead graph of the Heegaard diagram* $(S_g; v, w)$. The name is given

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because that graph is also the Whitehead graph of the group presentation coming from the Heegaard diagram (10).

We note that if the Whitehead graph of the Heegaard diagram $(S_g; v, w)$ is not connected, the manifold it represents is a connected sum (interior or in the boundary) of manifolds with simpler Heegaard diagrams. Thus we do not lose much in *restricting our study to presentations and diagrams with connected Whitehead graphs*. It is clear that the Whitehead graph of $(S_g; v, w)$ is connected if and only if the components of $S_g - v \cup w$ are simply connected.

Description of the algorithm. The starting point is a connected, finite presentation $P = \{s_1, \dots, s_a : t_1, \dots, t_b\}$. Let $\ell(t_i)$ be the length of the word t_i. Let $n(s_i)$ be the number of times that the letter s_i appears in the relators of P.

Step 1. *Make P positive.* If it appears the relation $t_j = (\dots s_i^{-1} \dots)$ we replace P by $P' = \{s_1, \dots, s_a, s : t_1, \dots, t_j = (\dots s \dots), \dots, t_b, s_i s\}$. The resulting positive presentation we call P again.

Step 2. Take the permutation

$$\tau = (1, 2, \dots, \ell(t_1))\,(\ell(t_1)+1, \dots, \ell(t_1)+\ell(t_2)) \overset{b}{\dots} (\dots, h)$$

of Σ_h, where $h = \ell(t_1) + \dots + \ell(t_b)$. From the $(n(s_1)-1)!\,(n(s_2)-1)! \dots (n(s_a)-1)!$ permutations σ of Σ_h which act on $\{1, 2, \dots, h\}$ with orbits $d_1 \dots d_a$ defined as follows. The number $j \in \{1, 2, \dots, h\}$ belongs to d_i if and only if there is t_k whose $(j - \ell(t_{k-1}))^{th}$ letter is s_i.

Step 3. Select the permutations σ (obtained in Step 2) such that $c(\sigma, \sigma^\tau) = 1$ and $c[\sigma, \tau] = h - 2a + 2$. Thus we get a set $\Sigma \subset \Sigma_h$ of permutations.

We say that *the algorithm ends* in Σ if $\Sigma \neq \phi$.

THEOREM 17 (i) *Let $P = \{s_1, \dots, s_a : t_1, \dots, t_b\}$ be a presentation with connected Whitehead graph. Then P corresponds to a Heegaard diagram*

$$(S_a; v = \{v_1, \dots, v_a\}, w = \{w_1, \dots, w_b\})$$

with connected Whitehead graph, if and only if the above algorithm ends.

(ii) *If the algorithm ends in Σ, and $\sigma \in \Sigma$, then the sum of the genera of the boundary components of $\check{N}(\sigma, \tau)$ is $\frac{1}{2}(h - c[\sigma, \tau]) + c(\tau, \tau^\sigma) - b$.*

(iii) *If the algorithm ends in Σ, and $\sigma \in \Sigma$, then P is a presentation for $\pi_1(\check{N}(\sigma, \tau))$.*

Remark. Step 1 does not increase the number $(n(s_1)-1)! \dots (n(s_a)-1)!$ because it introduces relations of length two. In a sense, Step 1 does not complicate the algorithm.

Proof. Note that in Step 1, P satisfies parts (i) and (iii) of the theorem if and only if P' does. This is an immediate consequence of Step 5 of theorem 14, which realizes the algebraic move sending P to P'.

If P is a positive presentation coming from the Heegaard diagram $(S_g; v, w)$ whose Whitehead graph is connected, then the components of $S_g - v \cup w$ are simply connected. Then Theorem 14 (starting in Step 6) provides a permutation σ such that the natural Heegaard diagram for $\check{N}(\sigma, \tau)$ is $(S_g; v, w)$. Also (compare part (i) of Corollary 15) $c(\sigma, \sigma^\tau) = 1$ and $a = c(\sigma) = 1 + \frac{1}{2}(h - c[\sigma, \tau])$. Hence $\Sigma \neq \phi$, and the algorithm must end. Since $\tilde{X}$ is a handlebody, Proposition 11 implies that the sum of the genera of the boundary components of $\check{N}(\sigma, \tau)$ is $\frac{1}{2}(h - c[\sigma, \tau]) - c(\tau) + c(\tau, \tau^\sigma)$, and $c(\tau) = b$.

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Conversely, assume the algorithm ends in Σ, starting with a positive presentation P. Let $\sigma \in \Sigma$. Then, by section 4, $\check{N}(\sigma, \tau)$ admits a generalized Heegaard diagram $(\tilde{F}; v, w)$ where $v = \{v_1, \ldots, v_{c(\sigma)}\}$ and $w = \{w_1, \ldots, w_{c(\tau)}\}$, and $c(\sigma) = a$, $c(\tau) = b$. Now the conditions $c[\sigma, \tau] = h - 2a + 2$, $c(\sigma, \sigma^\tau) = 1$ imply $\frac{1}{2}(h - c[\sigma, \tau]) - c(\sigma) + c(\sigma, \sigma^\tau) = 0$. Hence $\tilde{X}$ is a handlebody and $(\tilde{F}; v, w)$ is a Heegaard diagram. Theorem 16 gives a presentation for $\check{N}(\sigma, \tau)$ coming from $(\tilde{F}; v, w)$. But since $c(\sigma, \sigma^\tau) = 1$, there are no superfluous cycles, and that presentation is precisely P. |

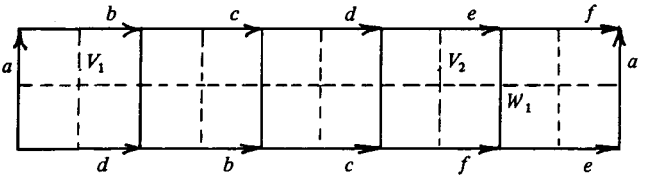


Fig. 12.

Example 1. Let $P = \{A, B, C, Z: A^5Z = B^3Z = C^2Z = ABCZ = 1\}$ the group of the Poincaré manifold. We apply the algorithm. Step 1 is not necessary, since P is positive. We have $\tau = (1\,2\,3\,4\,5\,6)(7\,8\,9, 10)(11, 12, 13)(14, 15, 16, 17)$. The orbits $d_1, \ldots, d_4$ for σ are $\{(1\,2\,3\,4\,5, 14), (7\,8\,9, 15), (11, 12, 16), (6, 10, 13, 17)\}$. We have to check $5!\,3!\,2!\,3! = 8640$ cases. But $\sigma = (1\,2\,3\,4\,5, 14)(7\,8\,9, 15)(11, 12, 16)(6\,10, 13, 17)$ ends the algorithm, because $c(\sigma, \sigma^\tau) = 1$ and $c[\sigma, \tau] = 11$. Thus P is a presentation coming from a Heegaard diagram of genus 4 representing a 3-manifold P^3 compact and oriented in which the sum of the genera of the boundary components is

$$\tfrac{1}{2}(h - c[\sigma, \tau]) - c(\sigma) + c(\tau, \tau^\sigma) = 0.$$

Hence P^3 is closed. The natural Heegaard diagram was constructed in the example of section 4.

Example 2. Let $P = |A, B: A^3 = B^2| = |A, C: A^3C^2 = 1|$. Here $\tau = (1\,2\,3\,4\,5)$ and $d_1 = (1\,2\,3)$, $d_2 = (4\,5)$. There are two cases to check and both $\sigma_1 = (1\,2\,3)(4\,5)$ and $\sigma_2 = (1\,3\,2)(4\,5)$ end the algorithm. Since $(\sigma_1, \tau) = (\sigma_2^{-1}, \tau)$, then $\check{N}(\sigma_1, \tau) = -\check{N}(\sigma_2, \tau)$ by Proposition 3. In both cases $\frac{1}{2}(h - c[\sigma, \tau]) - c(\sigma) + c(\tau, \tau^\sigma) = 1$. Hence P corresponds to the exterior of a knot (trefoil knot). The natural Heegaard diagram for $\{\sigma_1, \tau\}$ is shown in Fig. 12.

8. *Invertible presentations*

Let $\sigma = s_1 \ldots s_a$ and $\tau = t_1 \ldots t_b$ be elements of Σ_h written as products of independent cycles and call $P(\sigma, \tau)$ the presentation $\{s_1, \ldots, s_a: t_1, \ldots, t_b\}$ where t_j is the word in $\{s_1, \ldots, s_a\}$ whose ith letter is s_k if and only if the ith place in the cycle t_j is a number which belongs to the cycle s_k.

Let $P = \{s_1, \ldots, s_a: t_1, \ldots, t_a\}$ be a positive presentation with the same number of generators and relators. We say P is *invertible* if there are permutations $\{\sigma, \tau\} \in \Sigma_h$ such that (i) P coincides with $P(\sigma, \tau)$. (ii) $P(\tau, \sigma)$ presents the same group as P.

PROBLEM. *Characterize the positive and invertible presentations with connected Whitehead graph.*

One would think that they are exactly the ones coming from Heegaard diagrams, with connected Whitehead graph, representing closed 3-manifolds.

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THEOREM 18. *If* $P = \{s_1, \ldots, s_a : t_1, \ldots, t_a\}$ *is positive with Whitehead graph connected, and the algorithm of section* 7 *ends in* $\sigma \in \Sigma$, *such that* $\frac{1}{2}(h - c[\sigma, \tau]) = a - c(\tau, \tau^\sigma)$, *then* P *is invertible.* |

Is the converse of Theorem 18 true?

Example 1. The presentations $P(\sigma, \tau)$ and $P(\tau, \sigma)$ for example 1 of section 7 are equal.

Example 2. Let us find $P(\sigma, \tau)$ and $P(\tau, \sigma)$ for example 1 of section 6. We have $\sigma = (1\,3\,6\,4)(2\,5)$, $\tau = (1\,2\,3)(4\,5\,6)$. Hence

$$P(\sigma, \tau) = |s_1, s_2 : s_1 s_2 s_1 = s_1 s_2 s_1 = 1| \cong \mathbb{Z}$$
$$P(\tau, \sigma) = |t_1, t_2 : t_1 t_1 t_2 t_2 = t_1 t_2 = 1| \cong \mathbb{Z}.$$

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