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Recursion by optimization: on the complexity of bidirectional optimality theory

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Abstract

The paper investigates the computational complexity of different versions of Optimality Theory (OT). The result of Frank and Satta (1998) is used as a starting point. These authors show that unidirectional optimization can be implemented by finite state techniques if only binary constraints are used. The consequences of (a) taking gradient constraints into account and (b) using bidirectional optimization in the sense of Blutner (2000) are explored. The central result of the paper is that the combination of gradient constraints and bidirectionality leads to a massive increase of computational complexity.

1 Introduction

Optimality Theory (OT) in the sense of Prince and Smolensky (1993) has found wide acceptance in all areas of theoretical linguistics from phonology to semantics and pragmatics, even though it was originally only designed for applications in phonology and morphology. In recent year there has been a growing interest in computational aspects of OT, both pertaining to its abstract complexity and to potential practical implementations.

In its standard form, OT is an abstract model of language production. Several researchers have argued though that the notion of optimization also plays a role in language comprehension, and that expressive and interpretative optimization actually work in tandem. Bidirectional versions of OT have been considered for the analysis of morphological and syntactic phenomena as well as in the realm of semantics and pragmatics (see Jäger (2002) for discussion and references). The recent work of Reinhard Blutner (see especially Blutner (2000)) represents the presumably best worked out version of bidirectional OT.

The computational complexity of OT is currently an area of active research, and several results have been obtained in the literature. Most importantly, (Frank and Satta 1998) show that under certain general restrictions, (unidirectional) optimization is a finite state technique. This means that an OT system can be implemented as a finite state transducer provided the generator can be modeled by means of

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finite state techniques and all constraints are regular languages. In Jäger (2002) it is shown that this result directly carries over to bidirectional OT.

These results are confined to OT systems that only use binary constraints. Many empirical works in OT assume gradient constraints though. Frank and Satta (1998) also prove that the relation of optimal input-output pairings may exceed the power of finite state techniques even if all components of the OT system in question are finite state implementable provided gradient constraints are involved. This negative result is countered by a positive result in Gerdemann and van Noord (2000). They present a finite state implementation of a certain OT system involving gradient constraints.

The present paper contains two main results. After reviewing the mentioned results from the literature, I will propose a generalization of Gerdemann and van Noord's construction for the unidirectional case, i.e. I will give general sufficient conditions when an OT system using gradient constraints can be implemented as a finite state automaton. The proof is constructive. It provides a recipe to construct finite state transducers that implement a certain OT analysis using gradient constraints provided regular expressions for the generator and the constraints are known. This result is thus of immediate relevance to computational phonology and morphology.

Furthermore, it will be shown that this result does not carry over to the bidirectional case. Two OT-systems will be presented that remain within the limits of finite state techniques if unidirectional optimization is used but which exceed this complexity class if bidirectional optimization is applied. Perhaps surprisingly, the second system even exceeds the power of pushdown automata. These results thus show that bidirectional OT is intrinsically more complex than unidirectional OT. Since recent work in theoretical linguistics indicates that bidirectionality plays a role in morphology (see, for instance, Aissen (2002)), this has repercussions for computational morphology too, since it identifies another potential source of complexity of OT systems, next to gradient constraints.

2 Basic concepts of finite state OT

Let me first make precise what is meant by the term "OT-system." In the general case, an OT-system consists of a relation **GEN** and a finite set of constraints that are linearly ordered by some constraint ranking. Constraints may be violated several times. So a constraint should be construed as a function from **GEN** into the natural numbers. Thus, an OT-system assigns every pair in **GEN** a finite sequence of natural numbers. The ordering of the elements of **GEN** that is induced by the OT-system is according to the lexicographic ordering of these sequences. This leads to the following definition:

Definition 1 (OT-System)

1. An OT-system is a pair $\mathcal{O} = \langle \mathbf{GEN}, C \rangle$, where **GEN** is a relation, and $C = \langle c_1, \ldots, c_p \rangle, p \in \mathbb{N}$ is a linearly ordered sequence of functions from **GEN** to $\mathbb{N}$.

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2. Let $a, b \in \mathbf{GEN}$. $a <_{\mathcal{O}} b$ iff there is an i with $1 \leq i \leq p$ such that $c_i(a) < c_i(b)$ and for all $j < i : c_j(a) = c_j(b)$.

Intuitively, an output o is an (unidirectionally) optimal output for some input i iff **GEN** relates i and o, and o is optimal among the possible outputs for i. This is expressed by the following definition:

Definition 2 (Unidirectional optimality)
Let $\mathcal{O} = \langle \mathbf{GEN}, C \rangle$ be an OT-system. Then $\langle i, o \rangle$ is unidirectionally optimal with respect to $\mathcal{O}$ iff

1. $\langle i, o \rangle \in \mathbf{GEN}$
2. there is no o' such that $\langle i, o' \rangle \in \mathbf{GEN}$ and $\langle i, o' \rangle <_{\mathcal{O}} \langle i, o \rangle$

It has frequently been observed that in realistic applications, candidate sets might be infinite, and there is hence no guarantee that such a brute force algorithm terminates. Thus the success of the OT research program crucially hinges on the issue whether there are computationally tractable evaluation algorithms.

It is obvious that the complexity of the task of finding the optimal candidates for a given OT-system depends upon the complexities of the generator and of the constraints. In the general case, these will provide a lower bound for the complexity of the OT-system as a whole. The crucial question is whether an OT-system as a whole may have a higher complexity than the most complex of its components. Furthermore, this issue may depend on the mode of evaluation that is chosen. For instance, unidirectional OT might be less complex than bidirectional OT.

While these issues are still open in the general case, the literature contains some promising results about the complexity of OT in cases where all components of the OT-system are finite state objects.

In this section I will discuss the most impressive piece of work on the complexity of OT, Frank and Satta's (1998) construction. This will pave the ground for the extrapolation of Frank and Satta's result to the bidirectional case that is to be presented later on. Before doing so, it is advisable to repeat some prerequisites.

The classes of regular languages and of rational relations are subject to certain *closure properties* ($\overline{L}$ is the complement of the set L, $\mathbf{I}_L$ the identity relation on L, and $R^{\cup}$ the inverse of the relation R):

- Every finite language is regular.
- If L_1 and L_2 are regular languages, then $L_1 \cap L_2, L_1 \cup L_2, L_1 - L_2, L_1 L_2, \overline{L_1}$, and L_1^* are also regular languages.
- If R_1 and R_2 are rational relations, then $R_1 \cup R_2, R_1 \circ R_2, R_1^{\cup}, R_1 R_2$, and R_1^* are also rational relations.
- If R is a rational relation, then $Dom(R)$ and $Rg(R)$ (the domain $\{x | \exists y : xRy\}$ and the range $\{y | \exists x : xRy\}$ of R) are regular languages.
- If L_1 and L_2 are regular languages, then $L_1 \times L_2$ and $\mathbf{I}_{L_1}$ are rational relations.

Note that the rational relations are not closed under intersection and complement. Frank and Satta use these closure properties to show that for a significant class of OT-systems, unidirectional optimization is a rational relation provided all building blocks are rational.

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3 Frank and Satta (1998)

Now let us turn our attention to Frank and Satta's construction. They restrict the class of OT-systems in two ways. First, OT constraints in general "count," a given constraint may be violated arbitrarily many times. Frank and Satta restrict attention to binary constraints, i.e. constraints c with the property $Rg(c) = \{0, 1\}$. OT-systems which are not binary but have an upper limit for the number of constraint violations are implicitly covered; a constraint c that can be violated at most n times can be represented by n binary constraints of the form "Violate c less than i times" for $1 \leq i \leq n$. The ranking of these new constraints is inessential for the induced ordering relation.

Second, one may distinguish constraints that evaluate solely the output and constraints that properly evaluate an input-output pair. The former type of constraint is called *markedness constraints* in the literature, while the latter is covered under the term *faithfulness constraints*. Let us make this precise. We use the term "Output Markedness Constraint" since markedness constraints may also evaluate solely the input. Such input constraints have no effect for unidirectional OT, but they become important later when bidirectionality will be discussed.

Definition 3 ((Output) Markedness Constraint)
Let $\mathcal{O} = \langle \mathbf{GEN}, C \rangle$ be an OT-system. Constraint c_i is an *output markedness constraint* iff

$$\langle i, o \rangle \in \mathbf{GEN} \wedge \langle i', o \rangle \in \mathbf{GEN} \rightarrow c_i(\langle i, o \rangle) = c_i(\langle i', o \rangle))$$

Frank and Satta restrict attention to binary output markedness constraints. Obviously, these can be represented as languages over the output alphabet, namely as the set of outputs that obey them.

The central part of their construction is an operation called *conditional intersection* that combines a relation with a language. Karttunen (1998) calls this operation *lenient composition*, and I follow this terminology.

Definition 4 (Lenient composition)
Let R be a relation and $L \subseteq Rg(R)$. The *lenient composition* $R \upharpoonright L$ of R with L is defined as

$$R \upharpoonright L \doteq (R \circ \mathbf{I}_L) \cup \left(\mathbf{I}_{\overline{Dom(R \circ \mathbf{I}_L)}} \circ R\right)$$

By applying the definitions, it is easy to see that $\langle x, y \rangle \in R \upharpoonright L$ iff xRy and either $y \in L$ or there is no $z \in L$ such that xRz. In other words, $\{y | \langle x, y \rangle \in R \upharpoonright L\}$ is the set of ys that are related to x by R, and that are optimal with respect to the constraint L. Furthermore, it follows from the closure properties of regular sets that $R \upharpoonright L$ is a rational relation provided R is rational and L is a regular language.

Unidirectional optimality can now be implemented in a straightforward way, namely by successively leniently composing the (binary markedness) constraints of an OT-system with **GEN**.

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Theorem 1 (Frank and Satta)
Let $\mathcal{O} = \langle \mathbf{GEN}, C \rangle$ with $C = \langle c_1, \ldots, c_p \rangle$ be an OT-system such that C solely consists of binary output markedness constraints. Then $\langle i, o \rangle$ is unidirectionally optimal iff $\langle i, o \rangle \in \mathbf{GEN} \uparrow c_1 \cdots \uparrow c_p$.

The proof of this theorem is obvious from the definitions. Crucially, it follows that unidirectional optimality is a rational relation provided **GEN** is rational and all constraints are regular languages.

4 Gradient constraints in unidirectional OT

Despite the theoretical importance of Frank and Satta's result, the restriction to binary constraints limits its practical use in quite a significant way. This shortcoming is somewhat alleviated by the observation (also due to Frank and Satta) that their result readily extends to gradient constraints which impose an upper limit on the number of violations. Nevertheless, in the general case gradient constraints extend the generative power of an OT-system beyond the limits of finite state techniques (see Frank and Satta (1998) for a proof).

Gerdemann and van Noord (2000) show though that it is nevertheless possible to implement certain linguistically useful OT-systems using gradient (markedness) constraints. To bring this point home, they implement the constraint "Parse!" from Prince and Smolensky (1993) as a regular expression, even though "Parse!" can be violated arbitrarily many times. Their approach is based on the insight that the ordering on outputs that is induced by this constraint is itself a rational relation. In other words, there is a finite state transducer T such that every suboptimal candidate (with respect to "Parse!") can be obtained by applying T to some other candidate.

Even though the mentioned authors only consider this and closely related examples, their method can easily be generalized. Let us call a constraint c *rational* with respect to a generator relation R iff there is a rational relation S such that for all candidates x and y it holds that

$$\{\langle x, y \rangle | c(x) < c(y)\} \cap (R^{\cup} \circ R) = S \cap (R^{\cup} \circ R)$$

Note that two outputs x and y are competitors if they are both outputs of the same input. This means that x and y are competing iff $\langle x, y \rangle \in (R^{\cup} \circ R)$. A constraint c is thus rational with respect to R iff we can model the ranking that c induces among the competing outputs of R by means of some rational relation S.

In the sequel I write $rel_R(c)$ for the rational relation that represents a rational constraint c with respect to R. (I omit the subscript if no confusion is likely to arise.) Following the example of Gerdemann and van Noord, this relation can be used to construct a filter that eliminates all suboptimal candidates from a given set of competitors.

Now suppose R is a rational relation and c a rational constraint with respect to R. Then the *generalized lenient composition* of R with $rel(c)$ (written as "$R \restriction rel_R(c)$") relates an input i with an output o iff iRo and among the possible outputs of i under R, o violates c only minimally.

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Definition 5 (Generalized lenient composition)

$$R \upharpoonright S \doteq R \circ \mathbf{I}_{\overline{Rg(R \circ S)}}$$

Clearly, the generalized lenient composition of two rational relations is again a rational relation.

A word of caution is in order here, though. Consider the following toy OT-system. There are just two inputs, i_1 and i_2, and two outputs, o_1 and o_2. The generator relation **GEN** relates i_1 to both outputs while it maps i_2 just to o_2. There is one constraint c, which is violated by o_2, but not by o_1. So we have

$$\begin{aligned} \mathbf{GEN} &= \{\langle i_1, o_1\rangle, \langle i_1, o_2\rangle, \langle i_2, o_2\rangle\} \\ c(o_1) &= 0 \\ c(o_2) &= 1 \end{aligned}$$

It is easy to see that the optimal input-output pairing maps i_1 to o_1 and i_2 to o_2. When we consider the possible outputs of i_1, o_2 is blocked because it competes with o_1, and the latter is better according to c. When we consider the possible outputs of i_2 though, o_2 is optimal because there is no competitor that could block it. Now suppose we form the generalized lenient composition of **GEN** with $rel(c)$. The relation $rel(c)$ is simply $\{\langle o_1, o_2\rangle\}$. Composing **GEN** with $rel(c)$ and forming the range of the result gives us the singleton set $\{o_2\}$. So the filter that we construct from $rel(c)$ excludes o_2 in a global way, and we would wrongly predict that there is no optimal output for i_2 at all. The problem here is that in this scenario, competition and blocking between different outputs depends on the input. In other words, local optimality does not entail global optimality. Generalized lenient composition only works if this is not the case. To make this point precise, the filter construction is possible if optimality is global in the following sense: If some output o is optimal with respect to some input i, then o is also optimal with respect to the set of all possible outputs, regardless of the input. Formally, this reads as follows.

Definition 6

Let R and S be relations. **Optimality is global with respect to R and S** iff

$$\forall i, o((iRo \wedge \neg\exists o'(iRo' \wedge o'So)) \rightarrow \neg\exists o'(o' \in Rg(R) \wedge o'So))$$

In Gerdemann and van Noord's sample construction, optimality is certainly global with respect to **GEN** and R_P because it is always possible to recover a unique input from a given output candidate, just by erasing all brackets. Thus, competition only takes place between different bracketings over the same string, and thus local and global optimality coincide. Generally, if the inverse of **GEN** is functional, i.e. if the input is uniquely recoverable from the output, optimality is global with respect to all constraints. In many actual OT analyses, this is the case because the input is always contained in the output. Furthermore, generator relations not having this property can easily be redefined in a way that makes them inversely functional. Note though that inverse functionality of **GEN** is a sufficient condition for optimality

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to be global, but not a necessary one. In section 7 I discuss an example where optimality is global even though **GEN** is not inversely functional.

If the precondition that optimality be global is fulfilled, generalized lenient composition in fact provides the optimal outputs for each input:

Fact 1
Let R and S be relations such that optimality is global with respect to R and S. Then

$$\langle i, o\rangle \in R \upharpoonright S \text{ iff } iRo \wedge \neg\exists o'(iRo' \wedge o'So)$$

Proof
We start with the direction from left to right. Suppose $\langle i, o\rangle \in R \upharpoonright S$. By the definition of generalized lenient composition, this means that iRo and $o \in \overline{Rg(R \circ S)}$. So the first conjunct of the right hand side is fulfilled. Now suppose the second conjunct is false, i.e. there is an o' such that iRo' and $o'So$. Since $o \in \overline{Rg(R \circ S)}$, there is no o'' such that $o'' \in Rg(R)$ and $o''So$. $o' \in Rg(R)$ by assumption, thus $\neg o'So$. So we derive a contradiction.

We turn to the direction from right to left. Suppose iRo and there is no o' with iRo' and $o'So$. We furthermore assume that optimality is global with respect to R and S. Finally we assume that the left hand side of the biconditional is false, i.e. $\langle i, o\rangle \notin R \upharpoonright S$. By the definition of generalized lenient composition, this means that $o \notin \overline{Rg(R \circ S)}$. It follows that $o \in Rg(R \circ S)$. Thus there is an o' with $o' \in Rg(R)$ and $o'So$. Since optimality is global with respect to R and S, there is thus an o'' with iRo'' and $o''So$. This is in contradiction with the assumptions. □

Lenient composition with binary constraints is a special case of the more general notion of generalized lenient composition (provided optimality is global). Suppose the extension of some binary constraint c is the regular language $l(c)$. Then c is a rational constraint, and $rel(c) = l(c) \times \overline{l(c)}$. Then it follows from fact 1 that

$$\langle i, o\rangle \in R \upharpoonright (l(c) \times \overline{l(c)}) \text{ iff } iRo \wedge \neg\exists o'(iRo' \wedge o' \in l(c) \wedge o \notin l(c))$$

It furthermore follows from Frank and Satta's theorem that

$$\langle i, o\rangle \in R \uparrow l(c) \text{ iff } iRo \wedge \neg\exists o'(iRo' \wedge o' \in l(c) \wedge o \notin l(c))$$

because the right-hand side of this biconditional is just the definition of unidirectional optimality for the OT system $\mathcal{O} = \langle R, c\rangle$. From this one can infer the corollary that

$$R \uparrow l(c) = R \upharpoonright (l(c) \times \overline{l(c)})$$

Since neither Frank and Satta's proof of Theorem 1 nor the proof of Fact 1 requires the languages and relations involved to be regular/rational, this result holds regardless whether R and c are rational or not.

Generalized lenient composition can be used to model optimization with respect to a system of ranked constraints in a way completely analogous to Frank and Satta's construction with ordinary lenient composition:

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Theorem 2
Let $\mathcal{O} = \langle \mathbf{GEN}, C\rangle$ with $C = \langle c_1, \ldots, c_p\rangle$ be an OT-system such that C solely consists of rational output markedness constraints. Furthermore, let optimality be global with respect to $\mathbf{GEN} \upharpoonright rel(c_1) \cdots \upharpoonright rel(c_{i-1})$ and $rel(c_i)$ for all i with $1 \leq i \leq p$. Then $\langle i, o\rangle$ is unidirectionally optimal iff $\langle i, o\rangle \in \mathbf{GEN} \upharpoonright c_1 \cdots \upharpoonright c_p$.

Proof
Directly from Fact 1 and the definition of unidirectional optimality. □

5 Bidirectional OT

In the generative tradition of syntax, phonology and morphology, transformations are taken to be mappings from underlying abstract representations to concrete surface representations. OT researchers usually adopt this perspective too; competition takes place between different possible realizations of some underlying form. In other words, OT usually takes the generation perspective. It is a theory about the optimal realization of a given underlying form.

In the realm of semantics and pragmatics, interpretative optimization arguably plays a role too, as for instance exemplified by the Gricean Maxim of Manner. The Maxim of Quantity, on the other hand, involves productive optimization as well. So it seems that the mapping of linguistic forms to interpretations requires optimization both in the parsing and in the generation direction. This insight is not new, some form of bidirectional optimization has been assumed in the pragmatics literature for quite some time. In a series of recent publications, Reinhard Blutner has made the interplay between generation optimization and parsing optimization precise and integrated it into the overall framework of OT (Blutner 2000).

Taking both hearer perspective and speaker perspective into account, a grammatically licit form-meaning pair – or, more generally, a licit input-output pair – $\langle i, o\rangle$ may be blocked both by a more economical form alternative and a more economical meaning alternative. It should be added that a blocking expression should itself be optimal. So we arrive at the following definition of bidirectional optimality:

Definition 7 (Bidirectional optimality)
Let $\mathcal{O} = \langle \mathbf{GEN}, C\rangle$ be an OT-system. Then $\langle i, o\rangle$ is bidirectionally optimal iff

1. $\langle i, o\rangle \in \mathbf{GEN}$,
2. there is no bidirectionally optimal $\langle i', o\rangle \in \mathbf{GEN}$ such that $\langle i', o\rangle <_{\mathcal{O}} \langle i, o\rangle$, and
3. there is no bidirectionally optimal $\langle i, o'\rangle \in \mathbf{GEN}$ such that $\langle i, o'\rangle <_{\mathcal{O}} \langle i, o\rangle$.

Bidirectional optimality differs from the unidirectional case in that an input-output pair $\langle i, o\rangle$ can be blocked either by a better alternative output o' for i or a better alternative input i' for o. In Jäger (2002) it is shown that, despite its apparent circularity, bidirectional optimality is a well-defined notion.

6 Finite state bidirectional OT

In Jäger (2002) it is demonstrated that Frank and Satta's result carries over to the bidirectional case. To gain an intuitive understanding of the construction, let us

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consider another, slightly more complex example, which illustrates how bidirectional optimality is evaluated in case of a finite **GEN**. Suppose $\mathbf{GEN} = \{1,2,3\} \times \{1,2,3\}$, and we have two constraints which both say "Be small!". One of its instances applies to the input and one to the output. Thus, formally we have

- $\mathcal{O} = \langle \mathbf{GEN}, C \rangle$
- $\mathbf{GEN} = \{1,2,3\} \times \{1,2,3\}$
- $C = \langle c_1, c_2 \rangle$
- $c_1(\langle i, o \rangle) = i$
- $c_2(\langle i, o \rangle) = o$

It follows from the way constraints are evaluated that $\langle i_1, o_1 \rangle <_{\mathcal{O}} \langle i_2, o_2 \rangle$ iff $i_1 \leq i_2$, $o_1 \leq o_2$, and $\langle i_1, o_1 \rangle \neq \langle i_2, o_2 \rangle$. Now obviously $\langle 1,1 \rangle$ is bidirectionally optimal since both its input and its output obey the constraint in an optimal way. Accordingly, $\langle 1,2 \rangle, \langle 1,3 \rangle, \langle 2,1 \rangle$ and $\langle 3,1 \rangle$ are blocked, since they all share a component with a bidirectionally optimal candidate. Among the remaining candidates, $\langle 2,2 \rangle$ is certainly bidirectionally optimal because all of its competitors in either dimension are known to be blocked. This candidate in turn blocks $\langle 2,3 \rangle$ and $\langle 3,2 \rangle$. The only remaining candidate, $\langle 3,3 \rangle$, is again bidirectionally optimal since all its competitors are blocked. This example illustrates the general strategy for the finite case: Find the cheapest input-output pairs in the whole of **GEN** and mark them as bidirectionally optimal. Next mark all candidates that share either the input component or the output component (but not both) with one of these bidirectionally optimal candidates as blocked. If there are candidates left that are neither marked as bidirectionally optimal nor as blocked, repeat the procedure until **GEN** is exhausted.

Such an evaluation strategy is confined to OT-systems with a finite **GEN**. The finite state construction in Jäger (2002) follows a similar idea, but it is also applicable to systems with an infinite generator, provided all building blocks are finite state implementable.

The first step inside the evaluation strategy used above amounts to finding the globally optimal input-output pairs in the whole of **GEN**. As in Frank and Satta's construction, this global optimization is achieved by successively optimizing with respect to the constraints in **CON** in the order of their ranking. Jäger's (2002) construction is also restricted to binary markedness constraints. As a further complication, we have to distinguish between input constraints and output constraints (the former would not make sense in unidirectional OT). The definition of the notion of input markedness constraint is the mirror image of Definition 3:

Definition 8 (Input Markedness Constraint)
Let $\mathcal{O} = \langle \mathbf{GEN}, C \rangle$ be an OT-system. Constraint c_j is an *input markedness constraint* iff

$$\langle i, o \rangle \in \mathbf{GEN} \wedge \langle i, o' \rangle \in \mathbf{GEN} \rightarrow c_j(\langle i, o \rangle) = c_j(\langle i, o' \rangle))$$

If we want to leniently compose **GEN** with a binary input markedness constraint, we need a mirror image of Frank and Satta's lenient composition. Thus backward lenient composition is defined as

$$R \downarrow L \doteq (\mathbf{I}_L \circ R) \cup \left(R \circ \mathbf{I}_{\overline{Rg(\mathbf{I}_L \circ R)}}\right)$$

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Furthermore, for reasons that will become clear immediately, in bidirectional optimality it is not sufficient to consider the best outputs for a given input, but we have to look for the best input-output pairs in a global way. Thus, bidirectional lenient composition is defined in the following way.

Definition 9 (Bidirectional Lenient Composition)
Let $\mathcal{O} = \langle \mathbf{GEN}, C \rangle$ be an OT-system with $C = \langle c_1, \ldots, c_p \rangle$, and c_i be a binary markedness constraint.

$$R \Uparrow c_i \doteq \begin{cases} R \circ \mathbf{I}_{Rg((\{\varepsilon\} \times Rg(R)) \uparrow c_i)} \\ \text{if } c_i \text{ is an output markedness constraint} \\ \\ \mathbf{I}_{Dom((Dom(R) \times \{\varepsilon\}) \downarrow c_i)} \circ R \\ \text{if } c_i \text{ is an input markedness constraint} \end{cases}$$

Let us look at this construction in detail. Suppose c_i is an output markedness constraint. $\{\varepsilon\} \times Rg(R)$ is a relation that relates the empty string to any possible output of R. Leniently composing this relation with c_i leads to a relation that relates the empty string to those possible outputs of R that are optimal with respect to c_i. So if c_i is fulfilled by some output of R, this relation is just $\{\varepsilon\} \times (Rg(R) \cap c_i)$. If no output of R obeys c_i, the relation is just $\{\varepsilon\} \times Rg(R)$. In either way, $Rg((\{\varepsilon\} \times Rg(R)) \uparrow c_i)$ is the set of outputs of R that are optimal with respect to c_i. Since c_i only evaluates outputs, $R \Uparrow c_i$ is thus the set of $\langle i, o \rangle \in R$ that are optimal with respect to c_i. The same holds *ceteris paribus* if c_i is an input markedness constraint.

Note that a certain input-output pair may be evaluated as sub-optimal according to this construction even if it neither shares the input component nor the output component with any better candidate. So while Frank and Satta's lenient composition operates pointwise for each input, bidirectional lenient composition is global.

Fact 2
Let $\mathcal{O} = \langle \mathbf{GEN}, C \rangle$ be an OT-system (with binary markedness constraints only), where $C = \langle c_1, \ldots, c_p \rangle$, and let $R \subseteq \mathbf{GEN}$. Then

$$\langle i, o \rangle \in R \Uparrow c_1 \cdots \Uparrow c_p$$

iff $\langle i, o \rangle \in R$, and there are no i', o' with $\langle i', o' \rangle \in R$ and $\langle i', o' \rangle < \langle i, o \rangle$.

The proof of this fact – a straightforward induction over the number of constraints – can be found in Jäger (2002).

For simplicity, the notation R^C is used as shorthand for $\{x \in R | \neg \exists y \in R : y <_{\mathcal{O}} x\}$, where $\mathcal{O}$ is some OT-system with C as its constraint component. Intuitively, this operation picks out the globally optimal set of input-output pairs from R. The above fact states that $R^C = R \Uparrow c_1 \cdots \Uparrow c_p$ (where $C = \langle c_1, \ldots, c_p \rangle$). Note that R^C is a rational relation if R is rational and all constraints in C are regular languages.

After finding the globally optimal input-output pairs from **GEN**, we have to reevaluate those pairs from **GEN** that neither share the input component nor the output component with any globally optimal input-output pair. These pairs form the set

$$\mathbf{I}_{\overline{Dom(\mathbf{GEN}^C)}} \circ \mathbf{GEN} \circ \mathbf{I}_{\overline{Rg(\mathbf{GEN}^C)}}$$

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Optimization has to be applied to this set as well, and this procedure has to be repeated until all elements of **GEN** are either evaluated as optimal or as blocked. This leads to the following recursive definition of bidirectional optimality:

$$OPT_0 = \emptyset$$
$$OPT_{n+1} = OPT_n \cup \left(\mathbf{I}_{\overline{Dom(OPT_n)}} \circ \mathbf{GEN} \circ \mathbf{I}_{\overline{Rg(OPT_n)}}\right)^C$$

Fact 3
Let $\mathcal{O} = \langle \mathbf{GEN}, C \rangle$ be an OT-system such that all c_i in C are binary markedness constraints. If **GEN** is a rational relation and all constraints can be represented as regular languages, OPT_n is a rational relation for each n.

Proof
Immediately from the construction of OPT and $(\cdot)^C$ and the closure conditions on regular languages/rational relations. □

Even if **GEN** is infinite, there are only finitely many possible patterns of constraint violations if we are dealing with OT-systems comprising only binary constraints. This guarantees that bidirectional optimization with binary markedness constraints remains within the reach of finite state techniques.

Lemma 1
Let $\mathcal{O} = \langle \mathbf{GEN}, C \rangle$ be an OT-system with $C = c_1, \ldots, c_p$, where all c_i are binary markedness constraints. Then $\langle i, o \rangle$ is bidirectionally optimal iff $\langle i, o \rangle \in OPT_{2^p}$.

Proof
The proof proceeds in three stages. First it has to be shown that all elements of OPT_n are optimal for arbitrary n. This can be done by induction over n. Second it has to be established that each bidirectionally optimal candidate is element of OPT_n for some n. This can be shown by induction over the the "better-than" relation that is induced by $\mathcal{O}$. Induction is possible here since this relation is always well-founded. Finally one can observe that there are only 2^p different equivalence classes of candidates, which are defined by their pattern of constraint violations. This entails that 2^p recursion steps are sufficient. The details of this proof can be found in Jäger (2002).

This leads directly to the central result of Jäger (2002), namely the bidirectional counterpart of Frank and Satta's (1998) result.

Theorem 3
Let $\mathcal{O} = \langle \mathbf{GEN}, C \rangle$ be an OT-system with $C = \langle c_1, \ldots, c_p \rangle$, where all c_i are binary markedness constraints. Furthermore, let **GEN** be a rational relation and let all c_i be regular languages. Then the set of bidirectionally optimal elements of **GEN** is a rational relation.

Proof
Directly from Fact 3, Lemma 1, and the closure properties of regular languages and rational relations. □

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7 Gradient constraints and bidirectionality

The results summarized in the previous section suggest that bidirectional OT is not more complex than unidirectional OT in an automata theoretic sense, despite its considerable conceptual complexity. This is not true though; the construction from section 4 does not carry over to the bidirectional case. To see why, consider the following OT-system:

- $\mathcal{O} = \langle \mathbf{GEN}, C \rangle$
- $\mathbf{GEN} = \{\langle a^i b^j, a^k b^l \rangle | i = k \vee j = l\}$
- $C = \langle c_1, c_2 \rangle$
- $c_1(\langle i, o \rangle) = \#_a(i)$
- $c_2(\langle i, o \rangle) = \#_b(o)$

GEN is a rational relation, since it can be constructed in the following way, using only regular building blocks and finite state operations:

$$(a^* \times a^*)(b \times b)^* \cup (a \times a)^*(b \times b)^*$$

We assume two constraints: c_1: "No a!" applies to the input, and c_2: "No b!" applies to the output. So the number of violations of c_1 equals the number of as in the string that is evaluated, and likewise for c_2. c_1 is ranked higher than c_2 (but this plays no role in the sequel).

Both constraints are rational in the sense defined in section 4. Recall that a constraint is rational iff there is a rational relation that holds between two candidates α and β iff β violates the constraint more severely than α. The domain of **GEN** consists only of elements of the language a^*b^*. Thus we can transform any input candidate which violates c_1 n times into any other input candidates which violates c_1 m times for $m > n$ by adding a number of as at the beginning, and optionally adding or erasing some bs at the end. There is thus a rational relation corresponding to this constraint, namely

$$rel(c_1) : (a \times aa^*)^*(b^* \times b^*)$$

The same holds *ceteris paribus* for the output constraint c_2 which counts the number of bs in a string.

Finally, optimality is global with respect to **GEN** and both $rel(c_1)$ and $rel(c_2)$.[1] This follows from the fact that any two outputs of **GEN** are competitors, i.e. they share at least one input. For the same kind of reasoning, interpretive optimization is global as well.

If the construction from the previous section could be generalized to rational gradient constraints in a way similar to Gerdemann and van Noord's generalization of Frank and Satta's result, we would expect the set of bidirectionally optimal input-output pairs for the OT-system to form a rational relation. This is not so, however.

[1] In the bidirectional case, the definition of global optimality has to be adjusted so that both competition between inputs and competition between outputs is taken into account.

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To prove this fact, another recursive definition of bidirectional optimality is used that is somewhat more general than the one used in the last section. It is also taken from Jäger (2002).

Definition 10
Let $\mathcal{O} = \langle \mathbf{GEN}, C \rangle$ be an OT-system using only markedness constraints.

$$\begin{aligned} OPT_0 &= \emptyset \\ OPT_{\alpha+1} &= OPT_\alpha \cup \left(\mathbf{I}_{\overline{Dom(OPT_\alpha)}} \circ \mathbf{GEN} \circ \mathbf{I}_{\overline{Rg(OPT_\alpha)}}\right)^C \\ OPT_\alpha &= \bigcup_{\beta<\alpha} OPT_\beta \qquad \text{if } \alpha \text{ is a limit ordinal} \\ OPT &= \bigcup_{\alpha} OPT_\alpha \end{aligned}$$

This definition generalizes that used in the previous section to the transfinite case. This is necessary if we deal with gradient constraints since there may be infinitely many different patterns of constraint violations. This cumulative definition coincides with the standard definition of bidirectional optimality:

Fact 4
Let $\mathcal{O} = \langle \mathbf{GEN}, C \rangle$ be an OT-system using only markedness constraints. $\langle i, o \rangle$ is bidirectionally optimal iff $\langle i, o \rangle \in OPT$.

Proof
See Jäger (2002). It is analogous to the proof of Lemma 1, except that one has to use transfinite induction over the ordinal numbers.

We can use this construction to determine the set of bidirectionally optimal candidates from the OT-System given above. The crucial step is the observation that for all finite n:

$$OPT_n = \{\langle a^i b^y, a^z b^i \rangle | i < n \wedge (y = i \vee z = i)\} \tag{1}$$

We prove this by complete induction over n. The base case ($n = 0$) is obvious since $OPT_0 = \emptyset$ by definition. Now suppose the claim holds for n. Then, by the definition given above,

$$OPT_{n+1} = OPT_n \cup \left(\mathbf{I}_{\overline{Dom(OPT_n)}} \circ \mathbf{GEN} \circ \mathbf{I}_{\overline{Rg(OPT_n)}}\right)^C$$

Now $Dom(OPT_n) = \{a^i b^k | i < n\}$, thus $\overline{Dom(OPT_n)} = \{a^i b^k | i \geq n\}$. Likewise, $\overline{Rg(OPT_n)} = \{a^k b^i | i \geq n\}$. We will use the abbreviation

$$R_0 = \mathbf{I}_{\overline{Dom(OPT_n)}} \circ \mathbf{GEN} \circ \mathbf{I}_{\overline{Rg(OPT_n)}}$$

We thus have

$$\begin{aligned} OPT_{n+1} &= OPT_n \cup R_0^C \\ R_0 &= \{\langle a^x b^y, a^z b^w \rangle | x, w \geq n \wedge (x = z \vee y = w)\} \end{aligned}$$

To determine R_0^C, we have to bidirectionally optimize R_0 with respect to c_1 and c_2. The effect of c_1 is to minimize the number of as in the input. So the globally

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optimal elements of R_0 with respect to c_1 are those with the minimal number of as in the input, i.e.

$$\{\langle a^n b^y, a^z b^w\rangle | (n = z \vee y = w) \wedge w \geq n\}$$

Optimizing this relation with respect to c_2 minimizes the number of bs in the output, so we get

$$R_0^C = \{\langle a^n b^y, a^z b^n\rangle | n = z \vee n = y\}$$

Together with the induction hypothesis, we now have

$$OPT_{n+1} = \{\langle a^i b^y, a^z b^i\rangle | i \leq n \wedge (y = i \vee z = i)\}$$

This completes the proof of (1). □

From this we can infer that

$$OPT_\omega = \{\langle a^n b^y, a^z b^n\rangle | y = n \vee z = n\}$$

We will now demonstrate that $OPT = OPT_\omega$. By the recursive definition of OPT_α, it holds that

$$OPT_{\omega+1} = OPT_\omega \cup \left(\mathbf{I}_{\overline{Dom(OPT_\omega)}} \circ \mathbf{GEN} \circ \mathbf{I}_{\overline{Rg(OPT_\omega)}}\right)^C$$

It obviously holds that $Dom(OPT_\omega) = Dom(\mathbf{GEN})$ and $Rg(OPT_\omega) = Rg(\mathbf{GEN})$. Therefore, $\overline{Dom(OPT_\omega)} = \overline{Rg(OPT_\omega)} = \emptyset$. Furthermore, $\emptyset^C = \emptyset$. We therefore conclude that

$$OPT_{\omega+1} = OPT_\omega$$

The same argument can be applied to every transfinite ordinal number α. So if $\alpha \geq \omega$, it holds that

$$OPT_\alpha = OPT_\omega$$

Hence

$$OPT = OPT_\omega = \{\langle a^n b^y, a^z b^n\rangle | y = n \vee z = n\}$$

Now suppose this relation were rational. Then the result of composing it with a rational relation would again be a rational relation. The relation $\{\langle a^n, a^n\rangle | n > 0\}$ ($=\mathbf{I}_{aa^*}$) is rational. Thus $\mathbf{I}_{aa^*} \circ OPT$ should be rational as well. This is the relation $\langle a^n, a^n b^n\rangle | n > 0\}$. The range of this relation is known to be a non-regular language, thus it cannot be a rational relation, and hence OPT cannot be rational either.

This simple example demonstrates that a finite state implementation of gradient constraints along the lines of Gerdemann and van Noord (2000) is not possible in the context of bidirectional OT. In other words, the optimization strategy of bidirectional OT is intrinsically more complex than its unidirectional counterpart.

The complexity of bidirectional optimization can be pushed even further. While the previous example can still be modeled by a pushdown automaton, it is possible to transcend even this complexity class. To see why, consider the following OT-system:

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- $\mathcal{O} = \langle \mathbf{GEN}, C \rangle$
- $\mathbf{GEN} = \{\langle a^i b^j c^k d^l, a^x b^y c^z d^w \rangle | (i = x \wedge j = y) \vee (k = z \wedge l = w)\}$
- $C = \langle c_1, c_2, c_3, c_4 \rangle$
- $c_1(\langle i, o \rangle) = \#_a(i)$
- $c_2(\langle i, o \rangle) = \#_b(i)$
- $c_3(\langle i, o \rangle) = \#_c(o)$
- $c_4(\langle i, o \rangle) = \#_d(o)$

GEN is the relation that takes elements of the language $a^* b^* c^* d^*$ as input and either arbitrarily changes the number of *a*s and *b*s while leaving the *c*s and *d*s unchanged, or else keeps the *a*s and *b*s untouched while arbitrarily changing the number of *c*s and *d*s. It is again a rational relation, because it can be represented as given below using only finite state building blocks and regular operations:

$$(a^* \times a^*)(b \times b)^*(c^* \times c^*)(d \times d)^* \cup (a \times a)^*(b^* \times b^*)(c \times c)^*(d^* \times d^*)$$

The four constraints penalize occurrences of *a*s, *b*s, *c*s and *d*s respectively, where the first two constraints apply to the input and the other two to the output. Analogously to the previous example, these constraints can be represented by rational relations. Finally, local and global optimality coincide in this OT-system because every output competes with every other output, no matter what the input is.

The relation that is defined by this OT-system if we apply bidirectional optimization is

$$\{\langle a^i b^j c^k d^l, a^x b^y c^i d^j \rangle | (i = k \wedge j = l) \vee (i = x \wedge j = y)\}$$

The proof of this fact is similar to the analogous proof in the previous example, except that we have to make use of transfinite induction now. The crucial lemma is

$$\begin{aligned} \forall m, n \in \omega : OPT_{(\omega \cdot m)+n} = \{&\langle a^i b^j c^k d^l, a^x b^y c^i d^j \rangle | \\ &((i = x \wedge j = y) \vee (i = k \wedge j = l)) \wedge \\ &(\omega \cdot i) + j < (\omega \cdot m) + n\} \end{aligned} \tag{2}$$

We proof this fact by transfinite induction over $\alpha = (\omega \cdot m) + n$. So let $\alpha \in \omega^2$, and let us suppose that it holds for all $\beta < \alpha$ that

$$\begin{aligned} OPT_\beta = \{&\langle a^i b^j c^k d^l, a^x b^y c^i d^j \rangle | \\ &((i = x \wedge j = y) \vee (i = k \wedge j = l)) \wedge \\ &(\omega \cdot i) + j < \beta\} \end{aligned}$$

We have to show that it follows from this assumption that

$$\begin{aligned} OPT_\alpha = \{&\langle a^i b^j c^k d^l, a^x b^y c^i d^j \rangle | \\ &((i = x \wedge j = y) \vee (i = k \wedge j = l)) \wedge \\ &(\omega \cdot i) + j < \alpha\} \end{aligned}$$

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where $\alpha = (\omega \cdot m) + n$. This is certainly the case for $\alpha = 0$ because $OPT_0 = \emptyset$ by definition. Now suppose that α is a successor ordinal. This means that $n > 0$. It thus holds that

$$\begin{aligned} OPT_{\alpha-1} = \{\langle a^i b^j c^k d^l, a^x b^y c^i d^j\rangle | \\ ((i = x \wedge j = y) \vee (i = k \wedge j = l)) \wedge \\ (\omega \cdot i) + j < \alpha - 1\} \end{aligned}$$

By definition, in this case it holds that

$$\begin{aligned} OPT_\alpha = OPT_{(\omega\cdot m)+(n-1)} \cup \big(\mathbf{I}_{\overline{Dom(OPT_{(\omega\cdot m)+(n-1)})}} \circ \\ \mathbf{GEN} \circ \mathbf{I}_{\overline{Rg(OPT_{(\omega\cdot m)+(n-1)})}}\big)^C \end{aligned}$$

It follows directly from the definitions that

$$\overline{Dom\left(OPT_{(\omega\cdot m)+(n-1)}\right)} = \{\langle a^i b^j c^k d^l\rangle | (\omega \cdot i) + j \geq \alpha - 1\}$$

and

$$\overline{Rg\left(OPT_{(\omega\cdot m)+(n-1)}\right)} = \{\langle a^x b^y c^z d^w\rangle | (\omega \cdot z) + w \geq \alpha - 1\}$$

We can therefore conclude that

$$\mathbf{I}_{\overline{Dom(OPT_{(\omega\cdot m)+(n-1)})}} \circ \mathbf{GEN} \circ \mathbf{I}_{\overline{Rg(OPT_{(\omega\cdot m)+(n-1)})}}$$

$$=$$

$$\begin{aligned} \{\langle a^i b^j c^k d^l, a^x b^y c^z d^w\rangle | ((i = x \wedge j = y) \vee (k = z \wedge l = w)) \wedge \\ (\omega \cdot i) + j \geq \alpha - 1 \wedge (\omega \cdot z) + w \geq \alpha - 1\} \end{aligned}$$

Let us call this relation R. Applying bidirectional lenient composition of this relation with c_1 means minimizing the number of as in the input. This amounts to fixing i to m. Applying c_2 minimizes j, so j becomes $n-1$. Likewise, applying c_3 and c_4 fixes z and w to m and $n-1$, respectively. We thus get

$$R^C = \{\langle a^m b^{n-1} c^k d^l, a^x b^y c^m d^{n-1}\rangle | ((m = x \wedge n - 1 = y) \vee (k = m \wedge l = n - 1))\}$$

Therefore, we have

$$\begin{aligned} OPT_\alpha = \{\langle a^i b^j c^k d^l, a^x b^y c^i d^j\rangle | \\ ((i = x \wedge j = y) \vee (i = k \wedge j = l)) \\ (\omega \cdot i) + j < \alpha\} \end{aligned}$$

So the induction step goes through for successor ordinals, and it remains to be shown that the same holds for limit ordinals. Suppose α is a limit ordinal. Then by definition,

$$OPT_\alpha = \bigcup_{\beta<\alpha} OPT_\beta$$

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Therefore

$$\begin{aligned} OPT_\alpha = \bigcup_{\beta<\alpha} \{\langle a^i b^j c^k d^l, a^x b^y c^i d^j\rangle | \\ ((i = x \wedge j = y) \vee (i = k \wedge j = l)) \wedge \\ (\omega \cdot i) + j < \beta\} \\ = \{\langle a^i b^j c^k d^l, a^x b^y c^i d^j\rangle | \\ ((i = x \wedge j = y) \vee (i = k \wedge j = l)) \wedge \\ (\omega \cdot i) + j < \alpha\} \end{aligned}$$

This completes the proof (2). □

From this it follows straightforwardly that

$$OPT_{\omega^2} = \{\langle a^i b^j c^k d^l, a^x b^y c^i d^j\rangle | (i = x \wedge j = y) \vee (i = k \wedge j = l)\}$$

Now the domain and the range of this relation coincide with the domain and range of **GEN**. Hence $OPT_\alpha = OPT_{\omega^2}$ for all $\alpha > \omega^2$. Therefore

$$OPT = \{\langle a^i b^j c^k d^l, a^x b^y c^i d^j\rangle | (i = x \wedge j = y) \vee (i = k \wedge j = l)\}$$

The same argument that served to prove that the relation defined by the previous OT-system is not rational can be applied to demonstrate that this relation cannot be implemented by a pushdown automaton. Suppose it were otherwise. Restricting the input of a relation that is defined by a pushdown automaton to a regular language leads to another context free relation. (This can be shown by the same argument that proves the closure of the context free languages under intersection with regular languages.) Thus, restricting the domain of OPT to the regular language a^+b^+ should lead to another context free relation. However, the result of this operation is the relation

$$\{\langle a^i b^j, a^i b^j c^i d^j\rangle | i, j > 0\}$$

If there were a pushdown automaton that implements this relation, its range would be a context free language, but it is well-known that $a^i b^j c^i d^j$ is not context free. So we can conclude that bidirectional optimization exceeds the complexity of pushdown automata even if all building blocks can be implemented by finite state automata.

8 Conclusion

In this paper some issues pertaining to the computational complexity of optimality were discussed, both in its standard (i.e. unidirectional) and its bidirectional variety (where bidirectional OT is interpreted in the sense of Blutner (2000)). Two noteworthy results on unidirectional OT from the literature were reviewed. Frank and Satta (1998) propose a general way to implement unidirectional optimization by means of finite state techniques. Their construction is confined to OT systems that only employ binary markedness constraints. Gerdemann and van Noord (2000) give a finite state implementation of an OT system using gradient constraints. The

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present paper generalized this construction and showed that it is always applicable (a) if the OT system in question only uses markedness constraints, and (b) the notions of global and local optimality defined by this system coincide.

The present paper further reviewed the main result from Jäger (2002), which gives a finite state construction for bidirectional optimality that is based on Frank and Satta's construction, and is applicable under the same restrictions.

Finally, it was shown that a similar extrapolation of Gerdemann and van Noord's construction to bidirectionality is not possible. Even if both the generator and the constraints of an OT-system are confined to rational relations, the bidirectionally optimal relation may have a higher complexity than what can be implemented by pushdown automata, let alone finite state automata. The fact that bidirectional optimization goes beyond the power of finite state techniques follows from the fact that bidirectionality adds an element of recursion that need not be present in the building blocks of the OT-system in question. Using several constraints, it is even possible to implement simultaneous recursion at several positions in a string, and there are no restrictions on the linear positions of the split points where pumping occurs. It is thus possible to implement both crossing and nesting dependencies, while only the former can be modeled by pushdown automata.

The paper only presented two example constructions that illustrate recursion by bidirectional optimization. At present time, it is not known how these results can be generalized. My conjecture is that actually all context free languages can be implemented by finite state OT-systems under bidirectional interpretation, next to some mildly context-sensitive languages like $a^i b^j c^i d^j$. A precise characterization of the automata theoretic power of bidirectionality has to be left to further research though.

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