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Boundary Classes of Planar Graphs

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We analyse classes of planar graphs with respect to various properties such as polynomial-time solvability of the DOMINATING SET problem or boundedness of the tree-width. A helpful tool to address this question is the notion of boundary classes. The main result of the paper is that for many important properties there are exactly two boundary classes of planar graphs.

1. Introduction

From an algorithmic point of view planar graphs form a difficult class, since many algorithmic problems are NP-hard in this class. Moreover, most of them remain intractable even when restricted to planar graphs of vertex degree at most 3. Planar graphs are also complicated from a structural point of view, because many width parameters, such as tree- or clique-width, are unbounded in this class. Thus, in many respects, planar graphs are difficult, but not all of them. For instance, outerplanar graphs enjoy many nice properties both from an algorithmic and a structural point of view. Obviously, the same is true for any subclass of outerplanar graphs, such as forests, *i.e.*, graphs without cycles. On the other hand, if a class of graphs X is 'hard' with respect to some property, then any class containing X is 'hard' too. This discussion leads to the following natural question: Is there any boundary separating 'hard' classes of graphs from 'simple' ones? An interesting tool for addressing this question was recently introduced under the name *boundary classes* [1]. In the present paper, we study this tool for classes of planar graphs with respect to various properties. Surprisingly enough, we show that for a large variety of properties there are exactly two boundary classes of planar graphs. Among those properties are classes of graphs of bounded tree- or clique-width, as well as classes with polynomial-time solvable DOMINATING SET or INDUCED MATCHING problems.

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All graphs in this paper are undirected, without loops and multiple edges. The set of vertices of a graph G is denoted $V(G)$ and the set of edges $E(G)$. The degree of a vertex $v \in V(G)$ is the number of edges incident to v. As usual, C_k denotes a chordless cycle with k vertices. We say that a numerical parameter is bounded in a class of graphs X if there is a constant k such that the parameter does not exceed k for any graph in X.

2. Hereditary properties and boundary classes

Let $\langle U, \prec \rangle$ be a partially ordered set, where U is a set of elements and $\prec$ a partial order on U. A *hereditary property* on $\langle U, \prec \rangle$ is a subset $P \subseteq U$ such that $x \in P$ implies $y \in P$ for any $y \prec x$. Two particular examples of hereditary properties are of special interest in this paper.

Example 1: *Hereditary classes* of graphs. If U is the set of all graphs and $\prec$ is the relation 'to be an induced subgraph', then we refer to a hereditary property on $\langle U, \prec \rangle$ as a *hereditary class* of graphs. Many classes of theoretical and practical importance are hereditary, which includes, among others, graphs of bounded vertex degree, planar, bipartite, interval, perfect, line graphs, forests.

Example 2: *Hereditary systems* of graph classes. Let U be the family of all hereditary classes of graphs and $\prec$ the relation 'to be a subclass'. We shall refer to a hereditary property on $\langle U, \prec \rangle$ as a *hereditary system.* The family of classes of graphs of bounded tree-width provides an example of a hereditary system. Another important example deals with graph classes on which an NP-hard graph problem admits a polynomial-time solution (under the assumption that P $\neq$ NP).

An important observation regarding hereditary classes of graphs is that these and only these classes admit a uniform description in terms of minimal forbidden induced subgraphs, *i.e.*, minimal graphs that are not in the class. More formally, if a graph G does not contain induced subgraphs isomorphic to graphs in a set Z, we say that G is Z-free. The set of all Z-free graphs will be denoted Free(Z). The following is well known.

Theorem 2.1. *A class of graphs X is hereditary if and only if $X = \mathrm{Free}(Z)$ for some set Z. The minimal set Z possessing this property is unique.*

An analogous result holds for any hereditary property P containing, for each element x, finitely many elements preceding x. However, this is not the case for hereditary systems of graph classes, since a class of graphs may contain infinitely many subclasses. To overcome this difficulty, we shall employ the notion of a boundary class of graphs, which was recently introduced in [1] with respect to the INDEPENDENT SET problem. Below we recall the definition of a boundary class and two important lemmas from [1].

Let X be a hereditary class of graphs and $\mathcal{H}$ a hereditary system. To simplify our discussion, we shall say that X is an $\mathcal{H}$-class if $X \in \mathcal{H}$ and non-$\mathcal{H}$ otherwise. We shall

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define the notion of a boundary class in two steps. First, let us define the notion of a *limit class.*

Definition 1. A class of graphs X is called a *limit class* for a hereditary system $\mathcal{H}$ ($\mathcal{H}$-limit for short) if there exists a sequence $X_1 \supseteq X_2 \supseteq \cdots$ of non-$\mathcal{H}$ classes such that $\bigcap_{n \geqslant 1} X_n = X$.

Remark. Notice that Definition 1 does not require the classes $X_1 \supseteq X_2 \supseteq \cdots$ to be distinct. Therefore, every non-$\mathcal{H}$ class is $\mathcal{H}$-limit. However, the converse statement is not generally true. Indeed, let $\mathcal{H}$ be the family of graph classes of bounded tree-width. Also, denote $X_k :=$
Free$(C_3, \ldots, C_k)$. It was shown in [10] that for each particular value $k \geqslant 3$, the class X_k does not belong to $\mathcal{H}$. It is not hard to see that $X_k \supseteq X_{k+1}$ for any k, and $\bigcap_{k \geqslant 3} X_k$ is the class of forests. Therefore, the class of forests is a limit class for the system in question. On the other hand, forests are exactly the graphs of tree-width 1. This example shows that an $\mathcal{H}$-limit class is not necessarily non-$\mathcal{H}$. However, for classes defined by finitely many forbidden induced subgraphs, the notions of $\mathcal{H}$-limit classes and non-$\mathcal{H}$ classes are equivalent, as the following lemma states.

Lemma 2.2 ([1]). *Let X be a class of graphs defined by finitely many forbidden induced subgraphs. Then X is an $\mathcal{H}$-limit class if and only if $X \notin \mathcal{H}$.*

One more important property of limit classes is given in the next lemma.

Lemma 2.3 ([1]). *If X is an $\mathcal{H}$-limit class and $X \subseteq Y$, then Y is also an $\mathcal{H}$-limit class.*

Proofs of Lemmas 2.2 and 2.3 can be found in the paper of Alekseev [1].

Remark. An important observation concerning the class of forests is that it is not a minimal limit class for the system of classes of bounded tree-width. In fact, the paper [10] shows that the tree-width is unbounded even for graphs with maximum vertex degree at most three in the class Free$(C_3, \ldots, C_k)$. Hence, the forests with vertex degree at most three constitute a limit class for the system in question. This observation motivates the following definition.

Definition 2. A minimal limit class for a system $\mathcal{H}$ is called a *boundary class* for $\mathcal{H}$ ($\mathcal{H}$-boundary for short).

In the next two sections, we study this notion with respect to several important hereditary systems of planar graphs.

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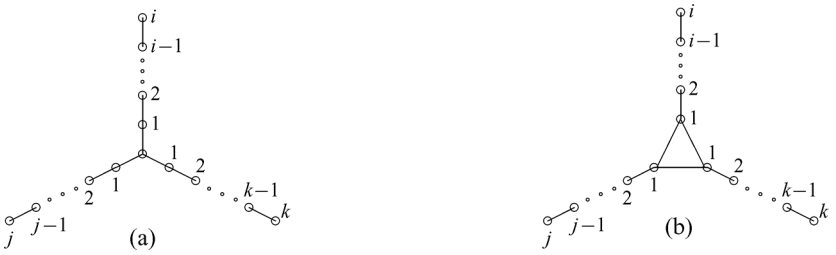


Figure 1. The graphs $S_{i,j,k}$ (a) and $T_{i,j,k}$ (b).

3. Hereditary $\mathcal{ST}$-systems

In this section, we define a special type of hereditary systems, which includes many systems of theoretical and practical importance. To this end, let us introduce two graph classes of particular interest. Denote by $S_{i,j,k}$ and $T_{i,j,k}$ the graphs represented in Figure 1 and let

$\mathcal{S}$ be the class of graphs every connected component of which is of the form $S_{i,j,k}$,
$\mathcal{T}$ be the class of graphs every connected component of which is of the form $T_{i,j,k}$.

Observe that $\mathcal{T}$ is simply the class of line graphs of graphs in $\mathcal{S}$.

Definition 3. A hereditary system $\mathcal{H}$ will be called an $\mathcal{ST}$-*system* if

(1) $\mathcal{S}$ and $\mathcal{T}$ are limit classes for $\mathcal{H}$,
(2) every hereditary class excluding a graph from $\mathcal{S}$ and a graph from $\mathcal{T}$ belongs to $\mathcal{H}$.

In Section 4, we will show that several important hereditary systems of planar graphs fall in the category of $\mathcal{ST}$-systems. In the rest of the present section, we prove that for any $\mathcal{ST}$-system $\mathcal{H}$, the classes $\mathcal{S}$ and $\mathcal{T}$ are $\mathcal{H}$-boundary, *i.e.*, they are minimal limit classes for $\mathcal{H}$.

Theorem 3.1. *For any $\mathcal{ST}$-system $\mathcal{H}$, the classes $\mathcal{S}$ and $\mathcal{T}$ are $\mathcal{H}$-boundary.*

Proof. We give a complete proof for the class $\mathcal{S}$. For $\mathcal{T}$, the proof is similar, so we will reduce it to a sketch.

Assume by contradiction that there exists an $\mathcal{H}$-limit class X which is a proper subclass of $\mathcal{S}$. Then there must exist a graph G belonging to $\mathcal{S} - X$. Therefore, $X \subseteq \mathcal{S} \cap \text{Free}(G)$. Since X is a hereditary class, we may assume without loss of generality that every connected component of G is of the form $S_{j,j,j}$ for some $j \geqslant 1$. Let M denote the set of graphs containing G as a spanning subgraph. Obviously G is the only graph in $\mathcal{S}$ belonging to M. We thus obtain the following inclusions:

$$X \subseteq \mathcal{S} \cap \text{Free}(G) \subseteq \text{Free}(M).$$

Since X is an $\mathcal{H}$-limit class, Free(M) is also an $\mathcal{H}$-limit class by Lemma 2.3, and since M is a finite set, Free(M) does not belong to $\mathcal{H}$ by Lemma 2.2.

On the other hand, Free(M) excludes a graph from $\mathcal{S}$ (the graph G) and a graph from $\mathcal{T}$ (a graph obtained from G by adding one edge to each connected component).

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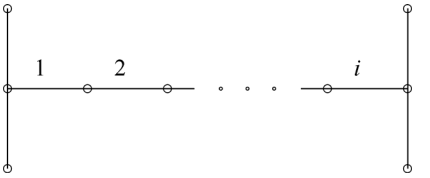


Figure 2. Graphs H_i.

Therefore, Free(M) belongs to $\mathcal{H}$ by Definition 3. This contradiction shows that $\mathcal{S}$ is a minimal $\mathcal{H}$-limit class.

To prove that $\mathcal{T}$ is minimal, we consider an $\mathcal{H}$-limit class $X \subset \mathcal{T}$ and a graph $G \in \mathcal{T} - X$. We assume that every connected component of G is of the form $T_{j,j,j+1}$ for some $j > 1$. It is not difficult to see that one can delete an edge from each connected component of G so that in the resulting graph, denoted G', every connected component has the form $S_{j,j,j}$, *i.e.*, G' belongs to $\mathcal{S}$. Now let M be the set of all graphs containing G' as a spanning subgraph. It is not hard to verify that $M \cap \mathcal{T} = \{G\}$, which implies $X \subseteq \mathcal{T} \cap \text{Free}(G) \subseteq \text{Free}(M)$. The rest of the proof is identical to that for the class $\mathcal{S}$. □

4. Particular $\mathcal{ST}$-systems

In this section, we study several important hereditary systems of planar graphs and show that all of them are $\mathcal{ST}$-systems. This includes:

(a) classes of planar graphs with polynomial-time solvable DOMINATING SET, INDEPENDENT DOMINATING SET, or INDUCED MATCHING problems (under the assumption that P ≠ NP);
(b) classes of planar graphs of bounded tree-, clique- or rank-width.

Section 4.1 shows that all these systems meet condition (1) of Definition 3, while Section 4.2 deals with condition (2).

4.1. $\mathcal{S}$ and $\mathcal{T}$ are limit classes

A typical way to show that $\mathcal{S}$ is a limit class for a hereditary system $\mathcal{H}$ is to prove that the class of graphs of vertex degree at most 3 does not belong to $\mathcal{H}$ and then to apply some 'stretching operations' to graphs in the class. For instance, double subdivisions of edges (which are known to increase the independence number by exactly one) have been used in [11] to prove that the INDEPENDENT SET problem is NP-hard for graphs without small cycles, *i.e.*, in the class Free($C_3, \ldots, C_k$) for any $k \geqslant 3$. In fact, the same operation allows us to exclude small induced graphs of the form H_i represented in Figure 2. As a result, we conclude that the INDEPENDENT SET problem is NP-hard for graphs of vertex degree at most three in the class Free($C_3, \ldots, C_k, H_1, \ldots, H_k$) for any particular value of k. It is not difficult to see that the intersection of all these classes is precisely the class $\mathcal{S}$, which means that $\mathcal{S}$ is a limit class for the INDEPENDENT SET problem. Some similar arguments have been used in [3, 7, 8, 10] to obtain results implying that $\mathcal{S}$ is a limit class for the hereditary systems listed in (a) and (b).

One of the possible ways to show that $\mathcal{T}$ is a limit class for a hereditary system $\mathcal{H}$ is to prove that the class of line graphs does not belong to $\mathcal{H}$. This is not the case, for

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instance, if $\mathcal{H}$ is the system of classes with the polynomial-time solvable INDEPENDENT SET problem, as this problem, restricted to the class of line graphs, coincides with the MAXIMUM MATCHING problem and hence is polynomial-time solvable. However, this is the case for all hereditary systems listed in (a) and (b). For instance, with respect to the DOMINATING SET problem, this follows from the fact that when restricted to line graphs the problem coincides with EDGE DOMINATION, which is known to be NP-hard [13]. Moreover, in the class of line graphs, DOMINATING SET and INDEPENDENT DOMINATING SET problems are polynomially equivalent [13] (an independent proof of the fact that $\mathcal{T}$ is a limit class for the INDEPENDENT DOMINATING SET problem can be found in [3]). The NP-hardness of INDUCED MATCHING in the class of line graphs was proved in [6].

In order to prove that $\mathcal{T}$ is a limit class for the systems listed in (b), let us show that the tree-, clique- and rank-width is unbounded in the class of line graphs. Moreover, we will show that all these parameters are unbounded for line graphs of bounded vertex degree. This is a consequence of the fact that the tree-, clique- and rank-width is unbounded in the class of graphs of vertex degree at most 3 and the following theorem.

Theorem 4.1. *Let X be a class of graphs of bounded vertex degree and $L(X)$ the class of line graphs of graphs in X. Then the tree-, clique- and rank-width of graphs in $L(X)$ is bounded if and only if it is bounded for graphs in X.*

Proof. Observe that in the proof of this theorem all three parameters can be used interchangeably. Indeed, according to [12], clique-width is bounded if and only if rank-width is bounded, and according to [4], for graphs of bounded vertex degree, tree-width is bounded if and only if clique-width is bounded. We shall prove the theorem with the help of two operations: subdivision of an edge and local complementation (*i.e.*, complementing the subgraph induced by the neighbourhood of a vertex). The first of these operations is known to preserve the tree-width, while the second one preserves the rank-width.

Assume first that all three parameters are bounded for graphs in X. Let us subdivide each edge of a graph $G \in X$ by a new vertex and denote the graph obtained in this way by G'. We shall create the line graph $L(G)$ of G on the set of new vertices of G', as they correspond to the edges of G. For each vertex $v \in V(G)$, the set of edges incident to v induces a clique in $L(G)$. Such a clique can be created by local complementation in G' centred at v. The sequence of local complementations centred at each of the old vertices of G' creates a graph G'' such that $L(G)$ is induced by the set of new vertices of the graph. Since we applied only edge subdivisions and local complementations, all three parameters are bounded for G'', and hence for $L(G)$.

Conversely, suppose that all three parameters are bounded for graphs in $L(X)$. Let $L(G)$ be the line graph of a graph $G \in X$. For each vertex $v \in V(G)$, add a new vertex x_v to $L(G)$ and connect x_v to the vertices of $L(G)$ that represent the edges of G incident to v. Denote the new graph by $L^{(1)}(G)$. Notice that vertex degree is bounded both for $L(G)$ and $L^{(1)}(G)$, since it is bounded for G. In order to see that the tree-width is bounded for $L^{(1)}(G)$, observe that any triangulation of the subgraph of $L^{(1)}(G)$ induced by the vertices of $L(G)$ is also a triangulation of $L^{(1)}(G)$, since the neighbourhood of each new vertex x_v is a clique in $L^{(1)}(G)$.

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For each $v \in V(G)$, perform local complementation in $L^{(1)}(G)$ centred at x_v and denote the resulting graph by $L^{(2)}(G)$. We know that the rank-width of $L^{(2)}(G)$ is bounded. Obviously, all original vertices of $L^{(2)}(G)$ (*i.e.*, those that represent the edges of G) have degree 2. In other words, $L^{(2)}(G)$ can be viewed as a graph obtained from G by subdividing each edge once. Thus, the tree-width of G is bounded. □

4.2. Classes excluding graphs from $\mathcal{S}$ and $\mathcal{T}$ are $\mathcal{H}$-classes

In this section, we show that all hereditary systems listed in (a) and (b) meet condition (2) of Definition 3. In fact, we can restrict ourselves to classes of graphs of bounded tree-width. Indeed, according to [4], for planar graphs, tree-width is bounded if and only if clique-width is bounded. Also, as mentioned before, clique-width is bounded if and only if rank-width is bounded [12]. Finally, all problems listed in (a) are polynomially solvable for graphs of bounded tree-width. To reach the goal of the section, we need the following two results.

Theorem 4.2. *For any positive integer k and any two graphs $H_1 \in \mathcal{S}$ and $H_2 \in \mathcal{T}$, there is a number N such that every graph of vertex degree at most k with no induced subgraphs isomorphic to H_1 or H_2 has tree-width at most N.*

Theorem 4.3. *Let X be a subclass of planar graphs and Y a class of graphs obtained from graphs in X by edge contractions and vertex deletions. If the tree-width of graphs of bounded vertex degree in Y is bounded, then the tree-width of graphs in X is bounded.*

Theorem 4.2 was proved in [9], while Theorem 4.3 is a direct corollary of the following technical lemma proved in [5], where an apex graph is any graph which becomes planar after deleting one vertex.

Lemma 4.4. *Let H be an apex graph, let G be H-minor-free graph, let $r = 14|V(H)| - 22$, and let $m > 2r$ be the largest integer such that $\mathrm{tw}(G) \geqslant m^{4|V(H)|^2(m+2)}$. Then G be can be contracted into an augmented grid R, i.e., an $(m - 2r) \times (m - 2r)$ grid augmented with additional edges (and no additional vertices) such that each vertex $v \in V(R)$ is adjacent to less than $(r + 1)^6$ non-boundary vertices of the grid.*

Now all we have to do is to show that, for any $H_1 \in \mathcal{S}$ and $H_2 \in \mathcal{T}$, the class of (H_1, H_2)-free graphs is closed under edge contractions (closedness under vertex deletion is obvious). For a graph G, let us denote by nG the disjoint union of n copies of G. Clearly, every graph $H_1 \in \mathcal{S}$ is an induced subgraph of $nS_{k,k,k}$ for some n and k, and every graph $H_2 \in \mathcal{T}$ is an induced subgraph of $nT_{k,k,k}$ for some n and k. Therefore, it is enough to prove that the class of $(nS_{k,k,k}, nT_{k,k,k})$-free graphs is closed under edge contractions. Unfortunately, this is not true even for $n = 1$. However, we can show that by means of edge contractions a graph from this class cannot be transformed 'too much'.

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Lemma 4.5. *Let G be an $(nS_{k,k,k}, nT_{k,k,k})$-free graph, and let G' be a graph obtained from G by a sequence of edge contractions. Then, G' is $((2n-1)S_{k+1,k+1,k+1}, (2n-1)T_{k+1,k+1,k+1})$-free.*

Proof. Assume by contradiction that G' contains an induced subgraph H isomorphic either to $(2n-1)S_{k+1,k+1,k+1}$ or to $(2n-1)T_{k+1,k+1,k+1}$.

Assume first that $n = 1$. Denote by U the vertices of degree 3 in H and by R the remaining vertices of H. Also, let U_G, R_G be the vertices of G 'contracted' to U and R, respectively. It is not difficult to see that, without loss of generality, we may assume that $R_G = R$. Finally, let a, b, c denote the vertices of R that have a neighbour of degree 3 in U. Obviously, b must be connected to c in G by a path P whose every internal vertex (other than b and c) belongs to U_G. Similarly, there must exist a path Q connecting a to P with all internal vertices belonging to U_G. If P and Q are chordless, the reader can easily find in G an induced $S_{k,k,k}$ or $T_{k,k,k}$, contradicting the assumption.

For $n > 1$, we apply the above arguments to each connected component of H and conclude by analogy that each of them corresponds to an induced copy of $S_{k,k,k}$ or $T_{k,k,k}$ in G. Since the number of components is $2n-1$, we conclude that G contains either $nS_{k,k,k}$ or $nT_{k,k,k}$ as an induced subgraph. □

This lemma together with Theorems 4.2 and 4.3 implies the following conclusion.

Theorem 4.6. *For any two graphs $H_1 \in \mathcal{S}$ and $H_2 \in \mathcal{T}$, there is a number N such that every planar graph with no induced subgraphs isomorphic to H_1 or H_2 has tree-width at most N.*

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