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# CHARACTERIZATIONS OF HYPERPLANES IN EUCLIDEAN SPACE THROUGH HIGHER-ORDER ANISOTROPIC MEAN CURVATURES

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**Abstract**

Our aim in this article is to establish new characterizations of hyperplanes in Euclidean space via mild constraints on the higher-order anisotropic mean curvatures of an immersed complete hypersurface. Our approach is based on the study of the behavior of the height and angle functions naturally attached to a hypersurface, jointly with the use of appropriate maximum principles dealing with the notions of convergence at infinity and polynomial volume growth. In this context, we also obtain lower estimates for the minimum anisotropic relative null index of an immersed complete hypersurface, which can be regarded as a measure of how close such a hypersurface is to being flat.

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## 1. Introduction

Let $F : \mathbb{S}^n \to (0, +\infty)$ be a positive smooth function defined on the standard $n$-dimensional unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, satisfying the convexity condition:

$$(D^2F + FI)_q > 0 \quad \text{for all } q \in \mathbb{S}^n, \tag{1-1}$$

where $D^2F$ denotes the intrinsic Hessian of $F$ on $\mathbb{S}^n$, $I$ is the identity on $T_q\mathbb{S}^n$, and the inequality means the matrix is positive definite. We consider the map

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$$\begin{array}{rccl} \phi: & \mathbb{S}^n & \to & \mathbb{R}^{n+1} \\ & q & \mapsto & \phi(q) = F(q)q + (\nabla_{\mathbb{S}^n} F)_q, \end{array} \tag{1-2}$$

whose image $\mathcal{W}_F = \phi(\mathbb{S}^n)$ is a smooth convex hypersurface called the *Wulff shape* of $F$ (see, for example, [7, 15, 19]). When $F \equiv 1$, the Wulff shape $\mathcal{W}_F$ reduces to the unit sphere $\mathbb{S}^n$.

Now, let $x : M^n \looparrowright \mathbb{R}^{n+1}$ be a smooth immersion of an orientable Riemannian manifold $M^n$ into $\mathbb{R}^{n+1}$ and let $N : M^n \to \mathbb{S}^n$ denote its Gauss map, that is, $N$ is the unit inner normal vector field of $M^n$. In this setting, $\nu = \phi \circ N : M^n \to \mathcal{W}_F$ is the *anisotropic Gauss map* of $M^n$ in the Wulff shape $\mathcal{W}_F$ and $S_F = -d\nu = -A_F \circ dN$ is called the *F-Weingarten operator* of $x : M^n \looparrowright \mathbb{R}^{n+1}$, where $A_F = D^2F + FI$. The eigenvalues $\lambda_1^F, \ldots, \lambda_n^F$ of $S_F$ are called *anisotropic principal curvatures* of $x : M^n \looparrowright \mathbb{R}^{n+1}$. For $r \in \{1, \ldots, n\}$, one defines the *r th anisotropic mean curvature* (also called *higher-order anisotropic mean curvature*) $H_{r;F}$ of $x : M^n \looparrowright \mathbb{R}^{n+1}$ by

$$\binom{n}{r} H_{r;F} = \sum_{1 \le i_1 < \cdots < i_r \le n} \lambda_{i_1}^F \cdots \lambda_{i_r}^F, \tag{1-3}$$

which is a natural extension of the classic $r$th mean curvature $H_r$ of hypersurfaces immersed in $\mathbb{R}^{n+1}$ (here, it is obtained when $F \equiv 1$). In this context, $H_F = H_{1;F}$ is called the anisotropic mean curvature of $x : M^n \looparrowright \mathbb{R}^{n+1}$. For more details on the anisotropic Gauss map and the anisotropic mean curvatures, see Section 2.2.

We also recall that a hypersurface $x : M^n \looparrowright \mathbb{R}^{n+1}$ is said to be anisotropic minimal when $H_F = 0$, and anisotropic $r$-minimal when $H_{r+1;F} = 0$. In this setting, Wang and He [20] considered the set $W$ of points that are omitted by the hyperplanes tangent to $x(M)$. They showed that if an oriented hypersurface $x : M^n \looparrowright \mathbb{R}^{n+1}$ is anisotropic minimal, and the set $W$ is open and nonempty, then $x(M)$ is part of a hyperplane of $\mathbb{R}^{n+1}$. They also showed that if an oriented hypersurface $x : M^n \looparrowright \mathbb{R}^{n+1}$ is anisotropic $r$-minimal and its $r$ th anisotropic mean curvature $H_{r;F}$ is nonzero everywhere, and the set $W$ is open and nonempty, then $x : M^n \looparrowright \mathbb{R}^{n+1}$ has anisotropic relative nullity $n - r$. For the concept of anisotropic relative nullity, see Section 5.

In [8], the first and second authors named, jointly with da Silva studied the behavior of the angle function $f_\xi = \langle \xi, \nu \rangle_\nu$ defined on a complete hypersurface $x : M^n \looparrowright \mathbb{R}^{n+1}$, where $\nu$ is the anisotropic Gauss map of $x : M^n \looparrowright \mathbb{R}^{n+1}$, $\langle\ ,\ \rangle_\nu$ is the metric tensor in $\mathbb{R}^{n+1}$ defined in Section 2.1, which is associated to a Minkowski norm constructed through the function $F$ satisfying the convex condition (1-1), and $\xi \in \mathbb{R}^{n+1}$ is a fixed vector. In this context, they showed that if $f_\xi = \langle \xi, \nu \rangle_\nu$ and $H_F$ do not change sign, the height function $\ell_\xi = \langle x, \xi \rangle_\nu$ is bounded and $|\xi^\top|_\nu$ is Lebesgue integrable, then $x : M^n \looparrowright \mathbb{R}^{n+1}$ must be a hyperplane of $\mathbb{R}^{n+1}$. Here, $|\ |_\nu$ denotes the norm with respect to the metric $\langle\ ,\ \rangle_\nu$, and $\xi^\top$ is the tangent component of $\xi$ along $M^n$. Afterwards, they provided extensions of this result for the case of higher-order anisotropic mean curvatures $H_{r;F}$, $r \in \{2, \ldots, n\}$. Moreover, they inferred a lower estimate of the index of minimum anisotropic relative nullity of a complete noncompact hypersurface $x : M^n \looparrowright \mathbb{R}^{n+1}$.

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More recently, Li and Peng [16] considered characterizations of a closed orientable connected hypersurface $x : M^n \looparrowright \mathbb{R}^{n+1}$ in terms of anisotropic higher-order mean curvatures, improving a previous work of Onat [17] by removing the convexity condition there. Furthermore, they generalized the results concerning the rigidity of standard spheres to the anisotropic setting and showed that if the hypersurface $x : M^n \looparrowright \mathbb{R}^{n+1}$ satisfies some assumptions involving constant linear combinations of anisotropic mean curvatures or other relations, then it must be the Wulff shape, up to translations and homotheties.

Motivated by these previous works, our aim in this article is to continue studying the higher-order anisotropic mean curvatures $H_{r;F}$ defined in (1-3) and the angle function $f_\xi = \langle \xi, \nu \rangle_\nu$ naturally attached to a hypersurface $x : M^n \looparrowright \mathbb{R}^{n+1}$ immersed in Euclidean space $\mathbb{R}^{n+1}$ to obtain new characterizations for the hyperplanes of $\mathbb{R}^{n+1}$. Our approach is based on suitable formulas for the linearized operator $L_{r;F}$ of the higher-order anisotropic mean curvatures acting on the height function $\ell_\xi = \langle x, \xi \rangle_\nu$, the angle function $f_\xi = \langle \xi, \nu \rangle_\nu$, and the support function $u = \langle x, \nu \rangle_\nu$ (see Lemmas 2 and 3) jointly with the use of appropriated generalized maximum principles dealing with the notions of convergence at infinity and polynomial $\nu$-volume growth (see Lemmas 4 and 5). Characterizations for hyperplanes involving the first two anisotropic mean curvatures $H_F$ and $H_{2;F}$ are stated and proved in Section 3, while extensions of these results involving higher-order anisotropic mean curvatures $H_{r;F}$, with $r \in \{2, \ldots, n\}$, appear in Section 4. Finally, in Section 5, we establish lower estimates for the index of minimum anisotropic relative nullity of a complete noncompact hypersurface immersed in $\mathbb{R}^{n+1}$.

## 2. Background

In this section, we recall some basic facts related to the higher-order anisotropic mean curvatures of a hypersurface $x : M^n \looparrowright \mathbb{R}^{n+1}$ immersed in Euclidean space $\mathbb{R}^{n+1}$, which are constructed from a positive smooth function $F : \mathbb{S}^n \to (0, +\infty)$ satisfying the convexity condition (1-1).

**2.1. The Wulff shape and the metric tensor associated with a Minkowski norm.** Associated with the function $F : \mathbb{S}^n \to (0, +\infty)$ that satisfies the convex condition (1-1), we define the correspondence

$$\begin{array}{rccl} F^* : & \mathbb{R}^{n+1} & \to & \mathbb{R} \\ & y & \mapsto & F^*(y) = \sup\limits_{z \in \mathbb{S}^n} \dfrac{\langle y, z \rangle}{F(z)}, \end{array}$$

where $\langle\ ,\ \rangle$ stands for the canonical Euclidean metric. As observed in [14], we have that $F^*$ is a *Minkowski norm* on $\mathbb{R}^{n+1}$. Indeed, $F^* : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}$ is a smooth function and satisfies the properties listed in the following result (see [14, Proposition 2.5]).

LEMMA 1. *With the notation considered above:*

(a) $F^*(y) > 0$ *for all* $y \in \mathbb{R}^{n+1} \setminus \{0\}$;
(b) $F^*(ty) = tF^*(y)$ *for any* $y \in \mathbb{R}^{n+1}$ *and all real numbers* $t \in (0, +\infty)$;

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(c) $F^*(y+z) \leq F^*(y) + F^*(z)$ *for every* $y, z \in \mathbb{R}^{n+1}$, *and the equality holds if and only if* $y = 0$, *or* $z = 0$, *or* $y = kz$ *for some constant* $k \in (0, +\infty)$;
(d) $\mathcal{W}_F = \{y \in \mathbb{R}^{n+1} : F^*(y) = 1\}$.

For $y \in \mathbb{R}^{n+1} \setminus \{0\}$, we define the components $\overline{g}_{\alpha\beta}(y)$ by putting

$$\overline{g}_{\alpha\beta}(y) = \frac{1}{2}\frac{\partial^2 (F^*)^2}{\partial y^\alpha \partial y^\beta}(y), \quad \alpha, \beta \in \{1, \ldots, n+1\}, \tag{2-1}$$

and we also define the symmetric bilinear form $\langle\ ,\ \rangle_y$ by

$$\langle X, Y\rangle_y = \sum_{\alpha,\beta=1}^{n+1} \overline{g}_{\alpha\beta}(y) X^\alpha Y^\beta \tag{2-2}$$

for any $X = (X^1, X^2, \ldots, X^{n+1})$, $Y = (Y^1, Y^2, \ldots, Y^{n+1}) \in T_y\mathbb{R}^{n+1} \simeq \mathbb{R}^{n+1}$.

From Euler's theorem for homogeneous functions jointly with item (b) of Lemma 1, for all $y \in \mathbb{R}^{n+1} \setminus \{0\}$,

$$\sum_{\beta=1}^{n+1} \frac{\partial F^*}{\partial y^\beta}(y) y^\beta = F^*(y) \quad \text{and} \quad \sum_{\beta=1}^{n+1} \frac{\partial^2 (F^*)}{\partial y^\alpha \partial y^\beta}(y) y^\beta = 0, \quad \alpha \in \{1, \ldots, n+1\}.$$

Thus, for any $y \in \mathbb{R}^{n+1} \setminus \{0\}$,

$$\langle y, y\rangle_y = (F^*(y))^2. \tag{2-3}$$

From (1-2), we have that the differential $d\phi : T_x\mathbb{S}^n \to T_{\phi(x)}\mathcal{W}_F$ is given by

$$d\phi(X) = (D^2 F + FI)_x(X)$$

for all $X \in T_x\mathbb{S}^n$. Then, from (1-1), we see that $T_{\phi(x)}\mathcal{W}_F$ is parallel to $T_x\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ and $d\phi$ is an isomorphism. Since $(F^* \circ \phi)(x) = 1$ for all $x \in \mathbb{S}^n$ (see item (d) of Lemma 1), from (1-2), we get for $x \in \mathbb{S}^n$ and $X \in T_x\mathbb{S}^n$ that

$$\langle \phi(x), X\rangle_{\phi(x)} = \sum_{\beta=1}^{n+1} \frac{\partial F^*}{\partial y^\beta}(\phi(x)) X^\beta = 0. \tag{2-4}$$

We note that (2-3) and (2-4) imply

$$\langle x, X\rangle = 0 \quad \text{if and only if} \quad \langle \phi(x), X\rangle_{\phi(x)} = 0 \tag{2-5}$$

for all $x \in \mathbb{S}^n$.

In what follows, for $v \in \mathbb{R}^{n+1}$ and $y \in \mathbb{R}^{n+1} \setminus \{0\}$, we write

$$|v|_y = \sqrt{\langle v, v\rangle_y}.$$

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We define the constants $\gamma, \Lambda, \mu$ by

$$\begin{cases} \lambda = \lambda(\mathbb{R}^{n+1}, F) = \inf\limits_{y,v\in\mathbb{R}^{n+1}\setminus\{0\}} \dfrac{|v|_y^2}{|v|^2}, \\ \Lambda = \Lambda(\mathbb{R}^{n+1}, F) = \sup\limits_{y,v\in\mathbb{R}^{n+1}\setminus\{0\}} \dfrac{|v|_y^2}{|v|^2}, \\ \mu = \mu(\mathbb{R}^{n+1}, F) = \dfrac{\Lambda}{\lambda}, \end{cases} \tag{2-6}$$

where $|\ |=\sqrt{\langle\ ,\ \rangle}$ denotes the Euclidean norm. Then, it is immediate to observe that $1 \le \mu < \infty$ and, in addition, $\mu = 1$ if and only if $F^*$ is a Euclidean norm.

**2.2. The anisotropic Gauss map and the higher-order anisotropic mean curvatures.** Let $x : M^n \hookrightarrow \mathbb{R}^{n+1}$ be a hypersurface, that is, a smooth immersion of an orientable Riemannian manifold $M^n$ in Euclidean space $\mathbb{R}^{n+1}$. If $N : M^n \to \mathbb{S}^n$ denotes the Gauss map of $x : M^n \hookrightarrow \mathbb{R}^{n+1}$ (namely, $N$ is the unit inner-normal vector field globally defined along $M^n$), then

$$\nu = \phi \circ N : M^n \to \mathcal{W}_F$$

is the generalized Gauss map of $M^n$ in the Wulff shape $\mathcal{W}_F$ of $F$, where $F$ is the positive smooth function on $\mathbb{S}^n$ defined in (1-1), which is called the *anisotropic Gauss map* of $x : M^n \hookrightarrow \mathbb{R}^{n+1}$. In what follows, we consider on $M^n$ the metric tensor $\langle\ ,\ \rangle_\nu$ defined according to (2-2). In this setting,

$$S_F = -d\nu = -A_F \circ dN$$

is called the *F-Weingarten operator* of $x : M^n \hookrightarrow \mathbb{R}^{n+1}$, where

$$A_F = D^2F + FI.$$

At each $p \in M^n$, $S_F$ restricts to a linear map $S_F(p) : T_pM \to T_pM$. The eigenvalues $\lambda_1^F, \ldots, \lambda_n^F$ of $S_F$ (obtained in this way) are called *anisotropic principal curvatures* of $x : M^n \hookrightarrow \mathbb{R}^{n+1}$. We know that the roots of the characteristic polynomial of $S_F$ are all real (see, for instance, [7, 12, 14]). Furthermore, for all $p \in M^n$, we observe from (2-3) and (2-4) that

$$\langle \nu(p), \nu(p)\rangle_{\nu(p)} = (F^*(\nu(p)))^2 = 1$$

and $\nu(p)$ is *$\nu$-orthogonal* to $T_pM$, namely,

$$\langle u, \nu(p)\rangle_{\nu(p)} = 0$$

for any $u \in T_pM$. For this reason, we call $\nu(p)$ an *anisotropic unit normal vector* at $p \in M^n$.

Let $\overline{\nabla}$ be the standard connection on the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. For vector fields $X, Y \in \mathfrak{X}(M)$, we define $II_\nu(X, Y) \in C^\infty(M)$ by

$$II_\nu(X, Y) = \langle \overline{\nabla}_X Y, \nu\rangle_\nu, \tag{2-7}$$

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where we decompose $\overline{\nabla}_X Y$ as the tangent part $\nabla_X Y$ and the anisotropic normal part $II(X,Y)\nu$ with respect to the metric tensor $\langle\ ,\ \rangle_\nu$. Hence, for $X, Y \in \mathfrak{X}(M)$, we can define $\nabla^\nu_X Y$ by putting

$$\nabla^\nu_X Y = \overline{\nabla}_X Y - II(X,Y)\nu. \tag{2-8}$$

Since $\langle \nu, \nu\rangle_\nu = 1$,

$$\langle \nabla^\nu_X Y, \nu\rangle_\nu = 0$$

and thus, from (2-5), we conclude that $\nabla^\nu_X Y$ is a smooth vector field tangent to $M^n$. Furthermore, since

$$\overline{\nabla}_X Y - \overline{\nabla}_Y X = [X,Y]$$

is tangent to $M^n$ and $\nu(p)$ is $\nu$-orthogonal to $T_pM$,

$$II_\nu(X,Y) - II_\nu(Y,X) = \langle [X,Y], \nu\rangle_\nu = 0.$$

Consequently, $II_\nu$ is symmetric, that is,

$$II_\nu(X,Y) = II_\nu(Y,X) \tag{2-9}$$

for all $X, Y \in \mathfrak{X}(M)$.

However, using once more Euler's theorem for homogeneous functions, from (2-1) and item (b) of Lemma 1, we obtain for all $y \in \mathbb{R}^{n+1}\backslash\{0\}$ that

$$\sum_{\beta=1}^{n+1} \frac{\partial \overline{g}_{\alpha\beta}}{\partial y^\gamma}(y) y^\beta = 0, \quad \alpha, \gamma \in \{1, 2, \ldots, n+1\}.$$

Consequently, for any smooth vector fields $X = (X^1, X^2, \ldots, X^{n+1}) \in \mathfrak{X}(M)$ and $Z = (Z^1, Z^2, \ldots, Z^{n+1}) \in \mathfrak{X}(\mathbb{R}^{n+1})$, defined along $M^n$,

$$X\langle Z, \nu\rangle_\nu = \sum_{\gamma,\alpha,\beta=1}^{n+1} X^\gamma \frac{\partial}{\partial y^\gamma}\{\overline{g}_{\alpha\beta}(\nu) Z^\alpha \nu^\beta\} = \sum_{\gamma,\alpha,\beta=1}^{n+1} X^\gamma\, \overline{g}_{\alpha\beta}(\nu)\Big\{\frac{\partial Z^\alpha}{\partial y^\gamma}\nu^\beta + Z^\alpha \frac{\partial \nu^\beta}{\partial y^\gamma}\Big\}.$$

That is, the following compatibility property of $\overline{\nabla}$ with $\langle\ ,\ \rangle_\nu$ holds:

$$X\langle Z, \nu\rangle_\nu = \langle \overline{\nabla}_X Z, \nu\rangle_\nu + \langle Z, \overline{\nabla}_X \nu\rangle_\nu. \tag{2-10}$$

Taking $Z = \nu$ in (2-10) and taking into account that $\langle \nu, \nu\rangle_\nu = 1$,

$$\langle \overline{\nabla}_X \nu, Y\rangle_\nu = -\langle \overline{\nabla}_X Y, \nu\rangle_\nu = -II_\nu(X,Y).$$

From (2-8), it is not difficult to verify that the correspondence

$$\nabla^\nu : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

defines a torsion-free connection on $M^n$. In turn, from (2-7) and (2-9),

$$II_\nu : \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$$

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is a symmetric second-order covariant tensor field on $M^n$, which we call from now on the *anisotropic second fundamental form* of $x : M^n \looparrowright \mathbb{R}^{n+1}$. We also have the expressions

$$\overline{\nabla}_X \nu = -S_F(X) \quad \text{and} \quad \langle S_F(X), Y\rangle_\nu = II_\nu(X, Y) \tag{2-11}$$

for all $X, Y \in \mathfrak{X}(M)$, which define anisotropic versions of the *Weingarten formula* of $x : M^n \looparrowright \mathbb{R}^{n+1}$.

Now, fix $r \in \{1, \dots, n\}$. At each $p \in M^n$, let $\mathcal{S}_r(p)$ denote the $r$th elementary symmetric function on the anisotropic principal curvatures $\lambda_1^F(p), \dots, \lambda_n^F(p)$ of $x : M^n \looparrowright \mathbb{R}^{n+1}$. Thus, we obtain $n$ smooth functions $\mathcal{S}_r^F : M^n \to \mathbb{R}$ such that

$$\det(tI - S_F) = \sum_{k=0}^{n} (-1)^k \mathcal{S}_k^F t^{n-k},$$

where $\mathcal{S}_0^F = 1$ by definition. One immediately sees that

$$\mathcal{S}_r^F = \sigma_r(\lambda_1^F, \dots, \lambda_n^F) = \sum_{1 \le i_1 < \cdots < i_r \le n} \lambda_{i_1}^F \cdots \lambda_{i_r}^F,$$

where $\sigma_r \in \mathbb{R}[X_1, \dots, X_n]$ is the $r$th elementary symmetric polynomial in the indeterminates $X_1, \dots, X_n$.

For $r \in \{1, \dots, n\}$, one defines the *r th anisotropic mean curvature* (or the *higher-order anisotropic mean curvature*) $H_{r;F}$ of $x : M^n \looparrowright \mathbb{R}^{n+1}$ by

$$\binom{n}{r} H_{r;F} = \mathcal{S}_r^F.$$

In this context, $H_F = H_{1;F}$ is called the anisotropic mean curvature. We note that when $F \equiv 1$, $H_{r;F}$ is just the standard $r$th mean curvature $H_r$ of $x : M^n \looparrowright \mathbb{R}^{n+1}$ (see, for instance, the seminal work of Barbosa and Colares [3]). Furthermore, after some algebraic calculations, we can obtain an important relationship between the $F$-Weingarten operator $S_F$ of $x : \Sigma^n \looparrowright \mathbb{R}^{n+1}$ and its first two anisotropic curvatures

$$H_F = \frac{1}{n} \sum_{i=1}^{n} \lambda_i^F$$

and

$$H_{2;F} = \frac{2}{n(n-1)} \sum_{i<j}^{n} \lambda_i^F \lambda_j^F,$$

which is given by

$$|S_F|_\nu^2 = n^2 H_F^2 - n(n-1) H_{2;F}, \tag{2-12}$$

where $|S_F|_\nu$ is the Hilbert–Schmidt norm of $S_F$.

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**2.3. The differential operators associated with the higher-order anisotropic mean curvatures.** Let $x : M^n \to \mathbb{R}^{n+1}$ be a hypersurface and let $S_F$ be its $F$-Weingarten operator, as in the previous section. We introduce the corresponding transformations $P_{r;F} : \mathfrak{X}(M) \to \mathfrak{X}(M)$, $r \in \{0, \dots, n\}$, which arise from the $F$-Weingarten operator $S_F$ (see, for instance, [12, 13]). According to our definition of higher-order anisotropic mean curvatures, such transformations are defined inductively from $S_F$ by

$$P_0 = I \quad \text{and} \quad P_{r;F} = \binom{n}{r} H_{r;F} I - P_{r-1;F} \circ S_F \quad \text{for } r = \{1, \dots, n\}, \tag{2-13}$$

where $I$ denotes the identity operator. Equivalently,

$$P_{r;F} = \sum_{j=0}^{r} (-1)^j \binom{n}{r-j} H_{r-j;F} S_F^j.$$

For each $p \in M^n$, we observe that $P_{r;F}(p) : T_pM \to T_pM$ is also a linear operator that commutes with $S_F(p) : T_pM \to T_pM$. Furthermore, about the operators $P_{r;F}$, we have the following properties:

$$\begin{cases} \operatorname{tr}(P_{r;F}) & = (n-r)\mathcal{S}_r^F = b_r H_{r;F}, \\ \operatorname{tr}(P_{r;F} \circ S_F) & = (r+1)\mathcal{S}_{r+1}^F = b_r H_{r+1;F}, \\ \operatorname{tr}(P_{r;F} \circ S_F^2) & = \mathcal{S}_1^F \mathcal{S}_{r+1}^F - (r+2)\mathcal{S}_{r+2}^F = \dfrac{b_r}{r+1} H_F H_{r+1;F} - b_{r+1} H_{r+2;F}, \end{cases} \tag{2-14}$$

where $b_r = (r+1)\binom{n}{r+1} = (n-r)\binom{n}{r}$.

Let $f \in C^\infty(M)$. As in [20], we define the gradient $\nabla^\nu f$ of $f$ on $M^n$ with respect to the metric tensor $\langle\,,\,\rangle_\nu$ by putting

$$\langle \nabla^\nu f, X \rangle_\nu = X(f),$$

where $X \in \mathfrak{X}(M)$ is any vector field.

Let $\{e_1, \dots, e_n\}$ be a local frame of $M^n$ and $\{\omega^1, \dots, \omega^n\}$ its dual frame (both considered with respect to $\langle\,,\,\rangle_\nu$). We define

$$dV_\nu = |e_1, \dots, e_n, \nu|\, \omega^1 \wedge \cdots \wedge \omega^n, \tag{2-15}$$

where $|e_1, \dots, e_n, \nu|$ is the determinant of the matrix $(e_1, \dots, e_n, \nu)$. Then, $dV_\nu$ is a volume element on $M^n$. Noting that $\nu = \phi \circ N = DF|_N + F(N)N$, we have $dV_\nu = F(N)\, dA$, where $dA$ is the area form of $M^n$ induced by the standard Euclidean metric of $\mathbb{R}^{n+1}$. Thus, the integral

$$\int_{M^n} dV_\nu \tag{2-16}$$

is just the anisotropic surface energy

$$\int_{M^n} F(N)\, dA,$$

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which has been studied by many authors (see, for instance, [6, 12, 15, 18, 21]). We also define the divergence $\operatorname{div}_\nu(X)$ of a vector field $X \in \mathfrak{X}(M)$ by

$$d(i(X)\,dV_\nu) = (\operatorname{div}_\nu X)\,dV_\nu,$$

where $i(X)\,dV_\nu$ is the interior product of $X$ with the volume element $dV_\nu$ given by

$$(i(X)\,dV_\nu)(Y_1,\dots,Y_{n-1}) = dV_\nu(X, Y_1\dots,Y_{n-1})$$

for all $Y_1,\dots,Y_{n-1} \in \mathfrak{X}(M)$.

Now, associated with each transformation $P_r$ given in (2-13), we can define the differential operator

$$\begin{array}{rccc} L_{r;F}: & C^\infty(M) & \to & C^\infty(M) \\ & f & \mapsto & L_{r;F}(f) = \operatorname{div}_\nu(P_{r;F}(\nabla^\nu f)). \end{array} \tag{2-17}$$

In particular, we observe that $L_{0;F} = \Delta$ is the standard Laplacian operator on $M^n$.

The next lemma gives us suitable formulas for the differential operator $L_{r;F}$ acting on two smooth functions naturally associated with a hypersurface $x : M^n \looparrowright \mathbb{R}^{n+1}$ with respect to a fixed vector of $\xi \in \mathbb{R}^{n+1}$ (see [8, Propositions 2 and 3]).

LEMMA 2. *Let $F : \mathbb{S}^n \to (0,+\infty)$ be a smooth function satisfying (1-1) and let $x : M^n \looparrowright \mathbb{R}^{n+1}$ be a hypersurface. We consider on $M^n$ the metric tensor $\langle\,,\,\rangle_\nu$ defined according to (2-2). Given $\xi \in \mathbb{R}^{n+1}$, let us consider on $M^n$ the height function*

$$\begin{array}{rccc} \ell_\xi: & M^n & \to & \mathbb{R} \\ & p & \mapsto & \ell_\xi(p) = \langle x(p), \xi\rangle_\nu \end{array} \tag{2-18}$$

*and the angle function*

$$\begin{array}{rccc} f_\xi: & M^n & \to & \mathbb{R} \\ & p & \mapsto & f_\xi(p) = \langle \nu(p), \xi\rangle_\nu, \end{array} \tag{2-19}$$

*where $\nu$ is the anisotropic Gauss map of $x : M^n \looparrowright \mathbb{R}^{n+1}$. Then:*

(a) $\nabla^\nu(\ell_\xi) = \xi^\top$;
(b) $\nabla^\nu(f_\xi) = -S_F(\xi^\top)$;
(c) $L_{r;F}(\ell_\xi) = b_r H_{r+1;F} f_\xi$, *with* $r \in \{0,\dots,n-1\}$;
(d) $L_{r;F}(f_\xi) = -b_r/(r+1)\langle \nabla H_{r+1;F}, \xi\rangle_\nu - (n(b_r/r+1)H_F H_{r+1;F} - b_{r+1}H_{r+2;F})f_\xi$, *with* $r \in \{0,\dots,n-2\}$,

*where $S_F$ and $H_{j;F}$ are respectively the $F$-Weingarten operator and the $j$ th anisotropic mean curvature of $x : M^n \looparrowright \mathbb{R}^{n+1}$, with $j \in \{1, r+1, r+2\}$ and $b_r = (r+1)\binom{n}{r+1} = (n-r)\binom{n}{r}$.*

In turn, we collect the formula of $L_{r;F}$ for a suitable support function associated with a hypersurface $x : M^n \looparrowright \mathbb{R}^{n+1}$ in the last lemma of this section (see [20, Lemma 20]).

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LEMMA 3. *Let $F : \mathbb{S}^n \to (0, +\infty)$ be a smooth function satisfying (1-1) and let $x : M^n \looparrowright \mathbb{R}^{n+1}$ be a hypersurface. We consider on $M^n$ the metric tensor $\langle\,,\,\rangle_\nu$ defined according to (2-2). If*

$$\begin{array}{rccl} u: & M^n & \to & \mathbb{R} \\ & p & \mapsto & u(p) = \langle x(p), \nu(p)\rangle_\nu, \end{array} \tag{2-20}$$

*then:*

(a) $\nabla^\nu(u) = -S_F(x^\top)$;
(b) $L_{r;F}(u) = -b_r/(r+1)\langle \nabla H_{r+1;F}, \xi\rangle_\nu - b_r H_{r+1;F}$
$-(n(b_r/r+1)H_F H_{r+1;F} - b_{r+1}H_{r+2;F})u$, *with* $r \in \{0, \ldots, n-2\}$,

*where $S_F$ and $H_{j;F}$ are respectively the F-Weingarten operator and the j th anisotropic mean curvature of $x : M^n \looparrowright \mathbb{R}^{n+1}$, with $j \in \{1, r+1, r+2\}$ and $b_r = (r+1)\binom{n}{r+1} = (n-r)\binom{n}{r}$.*

## 3. Characterizing hyperplanes using the first two anisotropic mean curvatures

In this section, our main objective is to obtain rigidity results with respect to complete hypersurfaces immersed in Euclidean space $\mathbb{R}^{n+1}$ in terms of its first two anisotropic mean curvatures. To do this, we need to describe some suitable maximal principles on Riemannian manifolds. We do this through the use of maximum principles on Riemannian manifolds presented in [1, 2].

Regarding the notion of completeness of hypersurfaces $x : M^n \looparrowright \mathbb{R}^{n+1}$, we can make the following remark. From the first two equalities of (2-6),

$$\lambda|v|^2 \le |v|_\nu^2 \le \Lambda|v|^2 \quad \text{for all } v \in \mathbb{R}^{n+1}, \tag{3-1}$$

where $|\ | = \sqrt{\langle\,,\,\rangle}$ is the Euclidean norm and $|\ |_\nu = \sqrt{\langle\,,\,\rangle_\nu}$ denotes the norm induced by the metric tensor $\langle\,,\,\rangle_\nu$, which is defined according to (2-2). Consequently, if $x : M^n \looparrowright \mathbb{R}^{n+1}$ is a complete hypersurface with respect to $\langle\,,\,\rangle$, then $x : M^n \looparrowright \mathbb{R}^{n+1}$ is also complete with respect to the tensor metric $\langle\,,\,\rangle_\nu$. Moreover, the result is valid when we change the order of the metric tensors in the last sentence.

**3.1. Via convergence at infinity.** Let $x : M^n \looparrowright \mathbb{R}^{n+1}$ be a smooth immersion of an oriented, connected, complete noncompact Riemannian manifold $M^n$ in Euclidean space $\mathbb{R}^{n+1}$ and consider on $M^n$ the metric tensor $\langle\,,\,\rangle_\nu$ defined according to (2-2). As set out in [1], consider $d_\nu(\cdot, \mathbf{o}) : M^n \to [0, +\infty)$ to represent the Riemannian distance of $M^n$ with respect to $\langle\,,\,\rangle_\nu$, measured from a fixed point $\mathbf{o} \in M^n$. Thus, if $f \in C^\infty(M)$ satisfies

$$\lim_{d_\nu(p,\mathbf{o})\to+\infty} f(p) = 0,$$

we simply say that *f converges to zero at infinity*.

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REMARK 1. We observe that the existence of nonnegative nonvanishing smooth functions that converge to zero at infinity is related to the existence of cut-off functions, which always occur in a complete noncompact Riemannian manifold.

In this context, we have the following maximum principle, which is a version of the one presented by Alias *et al.* in [1, Theorem 2.2] for the metric tensor $\langle\ ,\ \rangle_\nu$.

LEMMA 4. *Let $F:\mathbb{S}^n\to(0,+\infty)$ be a smooth function satisfying (1-1), let $x:M^n\looparrowright\mathbb{R}^{n+1}$ be a smooth immersion of an oriented, connected, complete noncompact Riemannian manifold $M^n$ into Euclidean space $\mathbb{R}^{n+1}$, and let $X\in\mathfrak{X}(M)$ be a vector field on $M^n$. We consider on $M^n$ the metric tensor $\langle\ ,\ \rangle_\nu$ defined according to (2-2). Suppose there is a nonidentically null and nonnegative function $f\in C^\infty(M)$, converging to zero at infinity. If $\langle\nabla^\nu f,X\rangle_\nu\geq 0$ and $\operatorname{div}_\nu X\geq 0$ on $M^n$, then:*

(a) $\langle\nabla^\nu f,X\rangle_\nu\equiv 0$ *on* $M^n$;
(b) $\operatorname{div}_\nu X\equiv 0$ *on* $M^n\setminus f^{-1}(0)$;
(c) $\operatorname{div}_\nu X\equiv 0$ *on* $M^n$, *provided that* $f^{-1}(0)$ *has zero Lebesgue measure.*

As established in [1], given the unit fields $N$ and $\xi$ defined along the immersion $x:M^n\looparrowright\mathbb{R}^{n+1}$, let $\theta\colon M^n\to\mathbb{R}$ be the function that associates to each point the angle between $\nu$ and $\xi$ where $|\xi|_\nu=1$, given by

$$\cos\theta=\langle\nu,\xi\rangle_\nu. \tag{3-2}$$

If $\theta$ vanishes identically, then $\nu$ is identical to $\xi$. If $\theta$ does not vanish identically, we want suitable conditions that still force $x:M^n\looparrowright\mathbb{R}^{n+1}$ to be *totally geodesic* (that is, the $F$-Weingarten operator $S_F$ of $x:M^n\looparrowright\mathbb{R}^{n+1}$ vanishes identically). With that objective, if $x:M^n\looparrowright\mathbb{R}^{n+1}$ is complete and noncompact, we say that $\nu$ *converges to* $\xi$ *at infinity*, provided that $\theta$ converges to zero at infinity.

Now, we can state our first result, which establishes a certain rigidity for hypersurfaces $x:M^n\looparrowright\mathbb{R}^{n+1}$ that admit constant anisotropic mean curvature.

THEOREM 1. *Let $F:\mathbb{S}^n\to(0,+\infty)$ be a smooth function satisfying (1-1) and let $x:M^n\looparrowright\mathbb{R}^{n+1}$ be a complete noncompact hypersurface with constant anisotropic mean curvature $H_F$. Suppose that there exists $\xi\in\mathbb{R}^{n+1}$ with $|\xi|_\nu=1$ and such that the angle function $f_\xi$ defined in (2-19) is positive on $M^n$. If $\nu$ converges to $\xi$ at infinity, then $x(M^n)$ is a hyperplane of $\mathbb{R}^{n+1}$.*

PROOF. We consider the function $f=1-f_\xi\in C^\infty(M)$. Since $|\xi|_\nu=1$, from (3-2), we infer that $f$ is a nonnegative function on $M^n$. We also consider the vector field $X=-\nabla^\nu f_\xi\in\mathfrak{X}(M)$.

Initially, if $f$ is identically zero on $M^n$, then $\xi\equiv\nu$ on $M^n$ and, from (2-11), we get $S_F(Y)=-\overline{\nabla}_Y\nu=0$ for any $Y\in\mathfrak{X}(M)$, where $\overline{\nabla}$ is the standard connection of $\mathbb{R}^{n+1}$, and it follows that $x:M^n\looparrowright\mathbb{R}^{n+1}$ is totally geodesic; since $x:M^n\looparrowright\mathbb{R}^{n+1}$ is complete and noncompact, we must have that $x(M^n)$ is a hyperplane of $\mathbb{R}^{n+1}$.

Now, let us assume that $f$ is not identically zero on $M^n$. So, on $M^n$,

$$\langle\nabla^\nu f,X\rangle_\nu=\langle\nabla^\nu f_\xi,\nabla^\nu f_\xi\rangle_\nu=|\nabla^\nu f_\xi|_\nu^2\geq 0. \tag{3-3}$$

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Furthermore, taking into account that $H_F$ is constant and the positivity of $f_\xi$, by making $r = 0$ in item (d) of Lemma 2, from (2-12), we can get

$$\operatorname{div}_\nu X = -\Delta(f_\xi) = |S_F|^2_\nu f_\xi \geq 0 \tag{3-4}$$

on $M^n$, where $S_F$ is the $F$-Weingarten operator of $x : M^n \looparrowright \mathbb{R}^{n+1}$.

Thus, since $\nu$ converges to $\xi$ at infinity, from (3-3), (3-4), and Lemma 4, we get $|\nabla^\nu f_\xi|_\nu \equiv 0$ on $M^n$. Next, $f_\xi$ is constant on $M^n$ and, consequently, $\Delta(f_\xi) = 0$ on $M^n$. So, since $f_\xi$ is positive on $M^n$, by returning to (3-4), we get that $|S_F|^2_\nu = 0$, which leads us to affirm that $S_F$ vanishes identically. Therefore, $x : M^n \looparrowright \mathbb{R}^{n+1}$ is totally geodesic and, consequently, $x(M^n)$ is a hyperplane of $\mathbb{R}^{n+1}$. □

We can also characterize hyperplanes of $\mathbb{R}^{n+1}$ by requiring the $F$-Weingarten operator of the hypersurfaces to be negative semi-definite, as established by the following result.

THEOREM 2. *Let $F : \mathbb{S}^n \to (0, +\infty)$ be a smooth function satisfying (1-1) and let $x : M^n \looparrowright \mathbb{R}^{n+1}$ be a complete noncompact hypersurface with $F$-Weingarten operator $S_F$ being negative semi-definite. Suppose that there is $\xi \in \mathbb{R}^{n+1}$ with $|\xi|_\nu = 1$ and such that the angle function $f_\xi$ defined in (2-19) is positive on $M^n$. If $\nu$ converges to $\xi$ at infinity, then $x(M^n)$ is a hyperplane of $\mathbb{R}^{n+1}$.*

PROOF. Initially, noting that the anisotropic mean curvature $H_F$ of $x : M^n \looparrowright \mathbb{R}^{n+1}$ is given by $nH_F = \operatorname{tr}(S_F)$, since $S_F$ is negative semi-definite, we must have $H_F \leq 0$ on $M^n$.

Consider the nonnegative function $f = 1 - f_\xi \in C^\infty(M)$ (see the relation given in (3-2)) and the vector field $X = -\xi^\top \in \mathfrak{X}(M)$. If $f$ is identically zero on $M^n$, then $\xi \equiv \nu$ on $M^n$ and, as we know in this case, $x(M^n)$ is a hyperplane of $\mathbb{R}^{n+1}$. Because of this, let us assume that $f$ is not identically zero on $M^n$.

From items (a), (b), and (c) (in the latter, by making $r = 0$) of Lemma 2,

$$\langle \nabla^\nu f, X\rangle_\nu = \langle \nabla^\nu f_\xi, \xi^\top\rangle_\nu = -\langle S_F(\xi^\top), \xi^\top\rangle_\nu \geq 0 \tag{3-5}$$

and

$$\operatorname{div}_\nu X = -\operatorname{div}_\nu(\xi^\top) = -\operatorname{div}_\nu(\nabla^\nu \ell_\xi) = -\Delta(\ell_\xi) = -nH_F f_\xi \geq 0 \tag{3-6}$$

on $M^n$, with $\ell_\xi$ being the height function of $x : M^n \looparrowright \mathbb{R}^{n+1}$ defined in (2-18), where in the inequalities, we make use of our restrictions regarding $S_F$, $H_f$, and $f_\xi$.

Thus, since $\nu$ converges to $\xi$ at infinity, by (3-5), (3-6), and Lemma 4, we conclude that $\langle S_F(\xi^\top), \xi^\top\rangle_\nu \equiv 0$ on $M^n$ and $0 \equiv \Delta(\ell_\xi) = nH_F f_\xi$ on $M^n \setminus f^{-1}(0)$. Since $f_\xi > 0$ and $H_F \leq 0$ on all $M^n$, we must have $H_F \equiv 0$ on $M^n \setminus f^{-1}(0)$.

However, by observing that

$$f^{-1}(0) = \{p \in M^n \,:\, \nu(p) = \xi(p)\},$$

if $p$ is an interior point of $f^{-1}(0)$, then $\nu \equiv \xi$ in a neighborhood $\mathcal{U}$ of $p$; and, consequently, from (2-11), we obtain $S_F(W) = -\overline{\nabla}_W \nu = 0$ for any smooth vector field $W$ defined on $\mathcal{U}$, where $\overline{\nabla}$ is the standard connection of $\mathbb{R}^{n+1}$. This guaranties $S_F \equiv 0$

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on $\mathcal{U}$, which in turn implies that $nH_F = \mathrm{tr}(S_F) \equiv 0$ on $\mathcal{U}$. Thus, $H_F \equiv 0$ in the interior of $f^{-1}(0)$ and, since it already vanishes in $M^n \setminus f^{-1}(0)$, we conclude that $H_F \equiv 0$ on all $M^n$.

For what follows, let us consider the vector field

$$Y = \frac{1}{2}\nabla^\nu(\ell_\xi^2) = \ell_\xi \nabla^\nu(\ell_\xi) \in \mathfrak{X}(M).$$

Given that we already have $\langle S_F(\xi^\top), \xi^\top\rangle_\nu \equiv 0$ and $H_F \equiv 0$ on $M^n$ then, by making use of items (a), (b), and (c) of Lemma 2 again,

$$\langle \nabla^\nu f, Y\rangle_\nu = -\ell_\xi \langle \nabla^\nu f_\xi, \nabla^\nu(\ell_\xi)\rangle_\nu = \ell_\xi \langle S_F(\xi^\top), \xi^\top\rangle_\nu = 0$$

and

$$\mathrm{div}_\nu Y = \mathrm{div}_\nu(\ell_\xi \nabla^\nu(\ell_\xi)) = \ell_\xi \Delta(\ell_\xi) + |\nabla^\nu \ell_\xi|_\nu^2 = nH_F f_\xi \ell_\xi + |\nabla^\nu \ell_\xi|_\nu^2 = |\xi^\top|_\nu^2 \geq 0$$

on $M^n$. Thus, Lemma 4 ensures that $\xi^\top \equiv 0$ on $M^n \setminus f^{-1}(0)$. As, by the definition of the set $f^{-1}(0)$, we already have $\xi^\top \equiv 0$ in the interior of $f^{-1}(0)$, we can state that $\xi^\top \equiv 0$ throughout $M^n$.

Next, from $\nabla^\nu(f_\xi) = -S_F(\xi^\top)$, we obtain that the angle function $f_\xi = \langle \nu, \xi\rangle_\nu$ is constant on $M^n$; but as $\nu$ converges to $\xi$ at infinity, then we must have that this constant is equal to 1 and, therefore, $\nu$ is identical to $\xi$ on $M^n$, from which we can conclude that $x(M^n)$ is a hyperplane of $\mathbb{R}^{n+1}$. □

The following result shows that the hypothesis on the $F$-Weingarten operator $S_F$ in Theorem 2 can be replaced by an upper bound on the height function $\ell_\xi$. Geometrically, requiring $\ell_\xi$ to be bounded above is equivalent to the hypersurface lying in a half-space of $\mathbb{R}^{n+1}$, a common assumption in rigidity theorems.

THEOREM 3. *Let $F : \mathbb{S}^n \to (0, +\infty)$ be a smooth function satisfying (1-1) and let $x : M^n \looparrowright \mathbb{R}^{n+1}$ be a complete noncompact hypersurface. Suppose that there is a vector $\xi \in \mathbb{R}^{n+1}$ such that $\sup_M(\ell_\xi) \leq c$ for some constant $c \in \mathbb{R}$, where $\ell_\xi$ is the height function of $x : M^n \looparrowright \mathbb{R}^{n+1}$ defined in (2-18). If the angle function $f_\xi$ defined in (2-19) and the anisotropic mean curvature $H_F$, both of $x : M^n \looparrowright \mathbb{R}^{n+1}$, have opposite signs, and the function $f = c - \ell_\xi$ converges to zero at infinity, then $x(M^n)$ is a hyperplane of $\mathbb{R}^{n+1}$.*

PROOF. Since $\ell_\xi \leq \sup_M(\ell_\xi) \leq c$ on $M^n$, we get that $f = c - \ell_\xi$ is a nonnegative function on $M^n$. Initially, we observe that if $f$ is identically zero on $M^n$, then the height function $\ell_\xi = \langle x, \xi\rangle_\nu$ is the constant $c$ throughout in $M^n$ and, consequently, $x(M^n)$ is a hyperplane of $\mathbb{R}^{n+1}$ orthogonal to $\xi$.

From now on, we consider the case where $f$ is not identically zero. We consider the vector field $X = -\nabla^\nu \ell_\xi \in \mathfrak{X}(M)$. So, from items (a) and (c) (in the latter, by making $r = 0$) of Lemma 2,

$$\langle \nabla^\nu f, X\rangle_\nu = \langle \nabla^\nu \ell_\xi, \nabla^\nu \ell_\xi\rangle_\nu = |\xi^\top|_\nu^2 \geq 0 \tag{3-7}$$

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and

$$\operatorname{div}_\nu X = -\Delta(\ell_\xi) = -nH_F f_\xi \geq 0, \tag{3-8}$$

on $M^n$, where in the last inequality, we use our restrictions on $H_F$ and $f_\xi$.

Thus, since $f$ converges to zero at infinity, from (3-8), (3-7), and Lemma 4, we can conclude that $\xi^\top \equiv 0$ throughout $M^n$. So, reasoning as in the last part of the proof of Theorem 2, we infer that $x(M^n)$ is a hyperplane of $\mathbb{R}^{n+1}$. □

Observing the proof of Theorem 3, we can see that the result is also true if we consider the following sufficient conditions.

THEOREM 4. *Let $F : \mathbb{S}^n \to (0, +\infty)$ be a smooth function satisfying (1-1) and let $x : M^n \looparrowright \mathbb{R}^{n+1}$ be a complete noncompact hypersurface. Suppose that there is a vector $\xi \in \mathbb{R}^{n+1}$ such that $\inf_M(\ell_\xi) \geq -c$ for some constant $c \in [0, +\infty)$, where $\ell_\xi$ is the height function of $x : M^n \looparrowright \mathbb{R}^{n+1}$ defined in (2-18). If the angle function $f_\xi$ defined in (2-19) and the anisotropic mean curvature $H_F$, both of $x : M^n \looparrowright \mathbb{R}^{n+1}$, have the same signs, and the function $f = c + \ell_\xi$ converges to zero at infinity, then $x(M^n)$ is a hyperplane of $\mathbb{R}^{n+1}$.*

PROOF. Since $-c \leq \inf_M(\ell_\xi) \leq \ell_\xi$ on $M^n$, $f = c + \ell_\xi$ is a nonnegative function on $M^n$. From now on, the result is obtained in a similar way to the proof of Theorem 3, but this time by considering the vector field $X = \nabla^\nu \ell_\xi \in \mathfrak{X}(M)$. □

Proceeding, we need to consider the *modified F-Weingarten operator* $\widetilde{S}_F$ of a hypersurface $x : M^n \looparrowright \mathbb{R}^{n+1}$, which is defined at each $p \in M^n$ by putting

$$\widetilde{S}_F(p) = (H_F)(p)(S_F)(p) : T_pM \to T_pM, \tag{3-9}$$

where $H_F$ and $S_F$ are respectively the anisotropic mean curvature and the $F$-Weingarten operator of $x : M^n \looparrowright \mathbb{R}^{n+1}$.

Now, compared with the sufficient conditions of Theorems 3 and 4, in the next theorem, we no longer require a limitation on the height function $\ell_\xi$ and a behavior on the anisotropic mean curvature $H_F$ of a hypersurface $x : M^n \looparrowright \mathbb{R}^{n+1}$ and, in compensation, we place restrictions on the second anisotropic mean curvature $H_{2;F}$ and the modified $F$-Weingarten operator $\widetilde{S}_F$.

THEOREM 5. *Let $F : \mathbb{S}^n \to (0, +\infty)$ be a smooth function satisfying (1-1) and let $x : M^n \looparrowright \mathbb{R}^{n+1}$ be a complete noncompact hypersurface. Suppose that there is a vector $\xi \in \mathbb{R}^{n+1}$ with $|\xi|_\nu = 1$ such that the angle function $f_\xi$ defined in (2-19) satisfies the inequality $H_{2;F} f_\xi \leq 0$, where $H_{2;F}$ is the second anisotropic mean curvature of $x : M^n \looparrowright \mathbb{R}^{n+1}$. If the modified F-Weingarten operator $\widetilde{S}_F$ of $x : M^n \looparrowright \mathbb{R}^{n+1}$ given in (3-9) is negative definite in direction $\xi^\top$ and $\nu$ converges to $\xi$ at infinity, then $x(M^n)$ is a hyperplane of $\mathbb{R}^{n+1}$.*

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PROOF. Let us consider the nonnegative function $f = 1 - f_\xi \in C^\infty(M)$ (see the relation given in (3-2)). If $f$ is identically zero on $M^n$, then $\xi \equiv \nu$ on $M^n$ and, as already observed on some occasions previously, $x(M^n)$ is a hyperplane of $\mathbb{R}^{n+1}$. Because of this, let us assume that $f$ is positive on $M^n$.

We also consider the vector field $X = -\nabla^\nu f_\xi - nH_F\xi^\top \in \mathfrak{X}(M)$. From items (a), (b), (c), and (d) of Lemma 2,

$$\langle \nabla^\nu f, X\rangle_\nu = |\nabla^\nu f_\xi|_\nu^2 + nH_F\langle \nabla^\nu f_\xi, \xi^\top\rangle_\nu = |S_F(\xi^\top)|_\nu^2 - n\langle \widetilde{S}_F(\xi^\top), \xi^\top\rangle_\nu \geq 0 \tag{3-10}$$

and

$$\begin{aligned} \operatorname{div}_\nu X &= \operatorname{div}(-\nabla^\nu f_\xi) - n\langle \nabla^\nu H_F, \xi^\top\rangle_\nu - nH_F\operatorname{div}(\xi^\top) \\ &= -\Delta(f_\xi) - n\langle \nabla^\nu H_F, \xi\rangle_\nu - nH_F\Delta(\ell_\xi) \\ &= n\langle \nabla^\nu H_F, \xi\rangle_\nu + (n^2H_F^2 - n(n-1)H_{2;F})f_\xi - n\langle \nabla^\nu H_F, \xi^\top\rangle_\nu - n^2H_F^2 f_\xi \\ &= -n(n-1)H_{2;F}f_\xi \geq 0 \end{aligned} \tag{3-11}$$

on $M^n$, where $\ell_\xi$ is the height function of $x : M^n \looparrowright \mathbb{R}^{n+1}$ defined in (2-18). Here, in the last inequalities, we use our restrictions on $\widetilde{S}_F$, $H_{2;F}$ and $f_\xi$.

Thus, since $\nu$ converges to $\xi$ at infinity, from (3-10), (3-11), and Lemma 4, we can conclude that $\langle \widetilde{S}_F(\xi^\top), \xi^\top\rangle_\nu \equiv 0$ on $M^n$. However, since $\widetilde{S}_F$ is negative definite in the direction $\xi^\top$, we must have $\xi^\top \equiv 0$ throughout $M^n$. Finally, reasoning as at the end of the proof of Theorem 2, we obtain the result. □

Next, we apply the formula of $L_{r;F}$ for a suitable support function associated with a hypersurface $x : M^n \looparrowright \mathbb{R}^{n+1}$, given in Lemma 3.

THEOREM 6. *Let $F : \mathbb{S}^n \to (0, +\infty)$ be a smooth function satisfying (1-1) and let $x : M^n \looparrowright \mathbb{R}^{n+1}$ be a complete, connected noncompact hypersurface. Assume that $H_F$ is a nonnegative constant and $0 \leq u \leq \delta$ for some positive constant $\delta$, where $u$ is the function defined in (2-20). If $u$ converges to $\delta$ at infinity, then $x(M^n)$ is a hyperplane of $\mathbb{R}^{n+1}$.*

PROOF. Consider the nonnegative function $f = \delta - u$ and the vector field $X = \nabla^\nu(-u)$, where $u$ is the function defined in (2-20). By item (b) of Lemma 3,

$$\operatorname{div}(X) = -\Delta u = nH_F + |II_\nu|^2 u \geq 0. \tag{3-12}$$

Moreover,

$$\langle \nabla^\nu f, X\rangle_\nu = \langle \nabla^\nu(-u), X\rangle_\nu = |\nabla^\nu u|_\nu^2 \geq 0. \tag{3-13}$$

Thus, since $u$ converges to $\delta$ at infinity, from (3-12), (3-13), and Lemma 4, we can conclude that $\langle \nabla^\nu f, X\rangle_\nu \equiv 0$ on $M^n$ and $-\Delta u = 0$ on $M^n \setminus f^{-1}(0)$. From (3-13), we get $|\nabla^\nu u|_\nu \equiv 0$, so $u$ is constant on $M^n$; since $u$ converges to $\delta$ at infinity, we must have $u \equiv \delta$ on $M^n$.

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From (3-12),

$$nH_F + |II_\nu|^2\delta = 0.$$

Since $H_F \geq 0$ and $\delta > 0$, we must have $|II_\nu|^2 = 0$.

The condition $|II_\nu|^2 = 0$ implies that the anisotropic second fundamental form vanishes identically, which means that the hypersurface is totally geodesic with respect to the anisotropic metric. Since $M^n$ is complete and connected, and the immersion is totally geodesic, we conclude that $x(M^n)$ must be a hyperplane in $\mathbb{R}^{n+1}$. □

Assuming that $H_F$ is a nonpositive constant and $-\delta \leq u \leq 0$ for some positive constant $\delta$, consider the nonnegative function $f = \delta + u$ and the field $X = \nabla^\nu u$. We have a new version of Theorem 3.

THEOREM 7. *Let $F : \mathbb{S}^n \to (0, +\infty)$ be a smooth function satisfying (1-1) and let $x : M^n \looparrowright \mathbb{R}^{n+1}$ be a complete, connected noncompact hypersurface. Suppose that $H_F$ is a nonpositive constant, u, defined in (2-20), and satisfies $-\delta \leq u \leq 0$ for some positive constant $\delta$. If u converges to $-\delta$ at infinity, then $x(M^n)$ is a hyperplane of $\mathbb{R}^{n+1}$.*

**3.2. Via polynomial $\nu$-volume growth.** Another analytic result that we use is a maximum principle due to Alías, Caminha, and Nascimento given in [2, Theorem 2.1]. To state it, we denote by $B(p, t)$ the geodesic ball centered at $p \in M^n$ and with radius $t \in (0, +\infty)$. Given a polynomial function

$$\sigma : (0, +\infty) \to (0, +\infty),$$

we say that $x : M^n \looparrowright \mathbb{R}^{n+1}$ has *polynomial $\nu$-volume growth* like $\sigma(t)$ if there exists $p \in M^n$ such that

$$\mathrm{vol}_\nu(B(p, t)) = O(\sigma(t)), \quad \text{as } t \to +\infty,$$

where $\mathrm{vol}_\nu(\cdot)$ denotes the volume calculated according to the volume element $dV_\nu$ given in (2-15). As was already observed in the beginning of [2, Section 2], if $p, q \in M^n$ are at distance $d$ from each other, we can verify that

$$\frac{\mathrm{vol}_\nu(B(p, t))}{\sigma(t)} \geq \left\{\frac{\mathrm{vol}_\nu(B(q, t-d))}{\sigma(t-d)}\right\}\left\{\frac{\sigma(t-d)}{\sigma(t)}\right\}.$$

Thus, the choice of $p \in M^n$ in the notion of the $\nu$-volume growth is immaterial. For this reason, we just say that $x : M^n \looparrowright \mathbb{R}^{n+1}$ has *polynomial $\nu$-volume growth*.

LEMMA 5. *Let $F : \mathbb{S}^n \to (0, +\infty)$ be a smooth function satisfying (1-1) and let $x : M^n \looparrowright \mathbb{R}^{n+1}$ be a smooth immersion of an oriented, connected, complete noncompact Riemannian manifold $M^n$ into Euclidean space $\mathbb{R}^{n+1}$. We consider on $M^n$ the metric tensor $\langle\,,\,\rangle_\nu$ defined according to (2-2). Assume that there exists a vector field $X \in \mathfrak{X}(M)$ such that $|X|_\nu \leq c \ll +\infty$ for some constant $c > 0$, and a function $f \in C^\infty(M)$ such that $\langle \nabla^\nu f, X\rangle_\nu \geq 0$ and $\mathrm{div}_\nu X \geq af$ on $M^n$ for some constant $a > 0$. If $x : M^n \looparrowright \mathbb{R}^{n+1}$ has polynomial $\nu$-volume growth, then $f \leq 0$ on $M^n$.*

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In the first result of this subsection, when a hypersurface $x : M^n \looparrowright \mathbb{R}^{n+1}$ with $\nu$-polynomial volume growth admits nonidentically zero anisontropic mean curvature $H_F$, we can establish the following rigidity result.

THEOREM 8. *Let $F : \mathbb{S}^n \to (0, +\infty)$ be a smooth function satisfying (1-1), and let $x : M^n \looparrowright \mathbb{R}^{n+1}$ be a complete hypersurface with polynomial $\nu$-volume growth and with $F$-Weingarten operator $S_F$ being positive semi-definite. If there exists a vector $\xi \in \mathbb{R}^{n+1}$ such that the angle function $f_\xi$ defined in (2-19) satisfies*

$$f_\xi \geq \frac{|\xi|_\nu}{1 + H_F} \tag{3-14}$$

*on $M^n$, where $H_F$ is the anisotropic mean curvature of $x : M^n \looparrowright \mathbb{R}^{n+1}$, then $x(M^n)$ is a hyperplane of $\mathbb{R}^{n+1}$.*

PROOF. Initially, we note that $H_F \geq 0$ on $M^n$, because $nH_F = \operatorname{tr}(S_F)$ and $S_F$ is positive semi-definite. So, consider the vector field $X = \xi^\top \in \mathfrak{X}(M)$ and the function $f = |\xi|_\nu - f_\xi \in C^\infty(M)$. From Schwarz's inequality, we obtain that $f$ is a nonnegative function on $M^n$ and, since $\xi \in \mathbb{R}^{n+1}$ is fixed, there exists a constant $c > 0$ such that

$$|X|_\nu = |\xi^\top|_\nu \leq |\xi|_\nu = c \ll +\infty$$

on $M^n$. Furthermore, since $S_F$ is positive semi-definite, from item (b) of Lemma 2,

$$\langle \nabla^\nu f, X\rangle_\nu = \langle -\nabla^\nu f_\xi, \xi^\top\rangle_\nu = \langle S_F(\xi^\top), \xi^\top\rangle_\nu \geq 0 \tag{3-15}$$

on $M^n$, and, if $\ell_\xi$ is the height function of $x : M^n \looparrowright \mathbb{R}^{n+1}$ defined in (2-18), by making $r = 0$ in item (c) of Lemma 2,

$$\operatorname{div}_\nu(X) = \operatorname{div}_\nu(\nabla^\nu \ell_\xi) = \Delta(\ell_\xi) = nH_F f_\xi \geq n(|\xi|_\nu - f_\xi) = nf \tag{3-16}$$

on $M^n$, where in the last inequality, we use (3-14).

Suppose that $M^n$ is a noncompact hypersurface, since $x : M^n \looparrowright \mathbb{R}^{n+1}$ has polynomial $\nu$-volume growth, by (3-15), (3-16), and by Lemma 5, we have $f \equiv 0$ on $M^n$, which tells us that $f_\xi = |\xi|_\nu = c$ on $M^n$. Then, using Schwarz's inequality, we verify that $\xi$ and $\nu$ are collinear on $M^n$. So, $\xi = c\nu$ on $M^n$ and, from (2-11),

$$S_F(Y) = -\overline{\nabla}_Y \nu = -\frac{1}{c}\overline{\nabla}_Y \xi = 0$$

for all $Y \in \mathfrak{X}(M)$, where $\overline{\nabla}$ is the standard connection of $\mathbb{R}^{n+1}$. Therefore, $x : M^n \looparrowright \mathbb{R}^{n+1}$ must be totally geodesic and, for completeness, we conclude that $x(M^n)$ is a hyperplane of $\mathbb{R}^{n+1}$.

However, if $M^n$ is compact, from (2-16) and applying the divergence theorem in (3-16),

$$0 = \int_{M^n} \operatorname{div}_\nu(X)\, dV_\nu \geq n \int_{M^n} f\, dV_\nu \geq 0.$$

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So, this implies that the function $f$ is identically zero on $M^n$ or, alternatively, that $x(M^n)$ is a hyperplane of $\mathbb{R}^{n+1}$. This generates a contradiction with the compactness of $M^n$. Therefore, there are no compact sets in this configuration. □

REMARK 2. In Theorem 8, (3-14) may not seem a mild hypothesis since, by Schwarz's inequality, $f_\xi \leq |\xi|_\nu$. However, since $H_F \geq 0 > -1$, the condition (3-14) tells us that the graph of the angle function $f_\xi$ is contained in the range determined by the length of the vector $\xi \in \mathbb{R}^{n+1}$ and by the quotient $|\xi|_\nu/(1 + H_F)$.

Based on the proof of Theorem 8, we can see that if the $F$-Weingarten operator $S_F$ of the hypersurface $x : M^n \looparrowright \mathbb{R}^{n+1}$ has an opposite behavior; by considering the vector field $X = -\xi^\top \in \mathfrak{X}(M)$, we obtain an analogous result.

THEOREM 9. *Let $F : \mathbb{S}^n \to (0, +\infty)$ be a smooth function satisfying (1-1), and let $x : M^n \looparrowright \mathbb{R}^{n+1}$ be a complete hypersurface with polynomial $\nu$-volume growth and with $F$-Weingarten operator $S_F$ being negative semi-definite. If there exists a vector $\xi \in \mathbb{R}^{n+1}$ such that the angle function $f_\xi$ defined in (2-19) satisfies*

$$f_\xi \geq \frac{|\xi|_\nu}{1 - H_F}$$

*on $M^n$, where $H_F$ is the anisotropic mean curvature of $x : M^n \looparrowright \mathbb{R}^{n+1}$, then $x(M^n)$ is a hyperplane of $\mathbb{R}^{n+1}$.*

Still in the context of polynomial $\nu$-volume growth, to obtain the next result, the main assumptions are about the function $f_\xi$, defined in (2-19), and the second mean anisotropic curvature $H_{2;F}$.

THEOREM 10. *Let $F : \mathbb{S}^n \to (0, +\infty)$ be a smooth function satisfying (1-1) and let $x : M^n \looparrowright \mathbb{R}^{n+1}$ be a complete, connected hypersurface with polynomial $\nu$-volume growth. Suppose $|S_F|_\nu$ is bounded and $H_{2;F} < 1$ on $M^n$. If there is a vector $\xi \in \mathbb{R}^{n+1}$ such that the linear map $\tilde{S}_F$ defined in (3-9) is negative semidefinite in direction $\xi^\top$ and the function $f_\xi$, defined in (2-19), satisfies*

$$f_\xi \geq \frac{|\xi|_\nu}{1 - H_{2,F}},$$

*then $x(M^n)$ is a hyperplane of $\mathbb{R}^{n+1}$.*

PROOF. Indeed, for $r = 0$, from (2-17), and items (c) and (d) of Lemma 2, respectively,

$$\Delta(\ell_\xi) = nH_F f_\xi \quad \text{and} \quad \Delta(f_\xi) = -(n^2H_F^2 - n(n-1)H_{2;F})f_\xi - n\langle \nabla^\nu H_F, \xi\rangle_\nu. \tag{3-17}$$

Thus, considering the smooth vector field $X = -\nabla^\nu f_\xi - nH_F\xi^\top$, from item (a) of Lemma 2 and (3-17),

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$$\begin{aligned}\operatorname{div}_\nu(X) &= \operatorname{div}_\nu(-\nabla^\nu f_\xi) - n\langle \nabla^\nu H_F, \xi^\top\rangle_\nu - nH_F \operatorname{div}_\nu(\xi^\top)\\ &= -\Delta f_\xi - n\langle \nabla^\nu H_F, \xi\rangle_\nu - n^2 H_F^2 f_\xi\\ &= (n^2H_F^2 - n(n-1)H_{2;F})f_\xi + n\langle \nabla^\nu H_F, \xi\rangle_\nu - n\langle \nabla^\nu H_F, \xi^\top\rangle_\nu - n^2H_F^2 f_\xi\\ &= -n(n-1)H_{2;F} f_\xi. \end{aligned} \tag{3-18}$$

Furthermore, since $H_{2;F} < 1$, using (3-18), we see that $f_\xi \geq |\xi|_\nu/(1 - H_{2;F})$ if, and only if,

$$\operatorname{div}_\nu(X) \geq n(n-1)f, \tag{3-19}$$

where $f = |\xi|_\nu - f_\xi$. However, notice that, by the Schwarz inequality, the function $f$ is nonnegative and since, by hypothesis, the linear map $\tilde{S}_F$ defined in (3-9) is negative semidefinite in direction $\xi^\top$, by item (b) of Lemma 2,

$$\begin{aligned}\langle \nabla^\nu f, X\rangle_\nu &= \langle -\nabla^\nu f_\xi, -\nabla^\nu f_\xi - nH_F\xi^\top\rangle_\nu = |\nabla^\nu f_\xi|_\nu^2 + n\langle H_F \nabla^\nu f_\xi, \xi^\top\rangle_\nu\\ &= |\nabla^\nu f_\xi|_\nu^2 - n\langle \tilde{S}_F(\xi^\top), \xi^\top\rangle_\nu \geq 0. \end{aligned} \tag{3-20}$$

Now, note that bounded $S_F$ implies bounded $H_F$; thus, $|X|_\nu$ is bounded on $M^n$, since $|\xi^\top|_\nu \leq |\xi|_\nu \ll +\infty$ and, by item (b) of Lemma 2, $|\nabla^\nu f_\xi| = |S_F(\xi^\top)| \ll +\infty$; hence,

$$|X|_\nu = |-\nabla^\nu f_\xi - nH_F\xi^\top| \leq |S_F(\xi^\top)| + n|H_F||\xi|_\nu \ll +\infty.$$

Thus, suppose that $M^n$ is a noncompact hypersurface; since $M^n$ has polynomial $\nu$-volume growth, by (3-19), (3-20), and by Lemma 5, we have $f \equiv 0$ on $M^n$, that is, $f_\xi = |\xi|_\nu$. Then, using the Schwarz inequality, it is verified that $\xi$ and $\nu$ are collinear on $M^n$. Therefore, $\xi = f_\xi \nu$ on $M^n$; using (3-16) and arguing as at the end of the proof of Theorem 8, we conclude that $x(M^n)$ is a hyperplane of $\mathbb{R}^{n+1}$.

However, if $M^n$ is compact, from (2-16) and applying the divergence theorem in (3-19),

$$0 = \int_{M^n} \operatorname{div}_\nu(X)\, dV_\nu \geq n(n-1)\int_{M^n} f\, dV_\nu \geq 0,$$

implying that the function $f$ is identically zero on $M^n$ or, alternatively, that $x(M^n)$ is a hyperplane of $\mathbb{R}^{n+1}$. This generates a contradiction with the compactness of $M^n$. Therefore, there are no compact sets in this configuration. □

We finish this section by applying the formula of $L_{r;F}$ for a suitable support function associated with a hypersurface $x : M^n \looparrowright \mathbb{R}^{n+1}$, given in Lemma 3. In this context, let us consider again the hypothesis of polynomial $\nu$-volume growth. For $r \in \{0, \ldots, n-1\}$, we remember that $x : M^n \looparrowright \mathbb{R}^{n+1}$ is called *anisotropic r-minimal* if its $(r+1)$-anisotropic mean curvature $H_{r+1;F}$ vanishes identically. Here, to make the notion clearer in the simplest case, when $H_F \equiv 0$, we say that $x : M^n \looparrowright \mathbb{R}^{n+1}$ is *anisotropic minimal*.

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THEOREM 11. *Let $F : \mathbb{S}^n \to (0, +\infty)$ be a smooth function satisfying (1-1) and let $x : M^n \hookrightarrow \mathbb{R}^{n+1}$ be a complete, connected hypersurface with polynomial $\nu$-volume growth. Assume that $H_F$ is a nonnegative constant and $0 \le u \le \delta$ for some positive constant $\delta$, where $u$ is the function defined in (2-20). If $|S_F(x^\top)|_\nu$ is bounded, then $M^n$ is anisotropic minimal. If, in addition,*

$$u \ge \frac{\varepsilon\delta}{\varepsilon - n(n-1)H_{2;F}} \tag{3-21}$$

*for some constant $\varepsilon > 0$, then $x(M^n)$ is a hyperplane of $\mathbb{R}^{n+1}$.*

PROOF. Consider the nonnegative function $f = \delta - u$ and the vector field $X = \nabla^\nu(-u)$. By item (b) of Lemma 3,

$$\operatorname{div}(X) = -\Delta u = nH_F + |II_\nu|^2 u \ge nH_F - \frac{nH_F}{\delta}u = \frac{nH_F}{\delta}(\delta - u) = \frac{nH_F}{\delta}f. \tag{3-22}$$

Moreover,

$$\langle \nabla^\nu f, X\rangle_\nu = \langle \nabla^\nu(-u), X\rangle_\nu = |\nabla^\nu u|_\nu^2 \ge 0. \tag{3-23}$$

However, since, by item (a) of Lemma 3, $\nabla^\nu u = -S_F(x^\top)$ and, by hypothesis, $|S_F(x^\top)|_\nu$ is bounded, we have that $|X|_\nu$ is bounded. Thus, since $x : M^n \to \mathbb{R}^{n+1}$ has polynomial $\nu$-volume growth, if $H_F \neq 0$, then, by (3-22), (3-23), and from Lemma 5, we conclude that $f \equiv 0$ on $M^n$, that is, $u$ is constant. Hence, $-\Delta u = 0$ and $-nH_F = |II_\nu|^2 u \ge 0$. Thus, $H_F$ is nonpositive, which is a contradiction. Therefore, $nH_F = 0$ on $M^n$, that is, $M^n$ is anisotropic minimal.

Note that, since $H_F = 0$, from (2-12) and (3-21),

$$\operatorname{div}(X) = -\Delta u = |II_\nu|^2 u = -n(n-1)H_{2:F}\, u \ge \varepsilon\, f. \tag{3-24}$$

Then, suppose that $M^n$ is a noncompact hypersurface; from (3-23), (3-24), and Lemma 5, again we conclude that $f \equiv 0$ on $M^n$, that is, $u$ is constant. Hence, $0 = -\Delta u = |II_\nu|^2 u$. Thus, $|II_\nu|^2 = 0$ and because it is complete, we conclude that $x(M^n)$ is a hyperplane of $\mathbb{R}^{n+1}$.

However, if $M^n$ is compact, from (2-16) and applying the divergence theorem in (3-24),

$$0 = \int_{M^n} \operatorname{div}_\nu(X)\, dV_\nu \ge \epsilon \int_{M^n} f\, dV_\nu \ge 0,$$

which tells us that the function $f$ is identically zero on $M^n$ or, alternatively, that $x(M^n)$ is a hyperplane of $\mathbb{R}^{n+1}$. This generates a contradiction with the compactness of $M^n$. Therefore, there are no compact sets in this configuration. □

From the proof of Theorem 11, we realize that the hypotheses of $u$, defined in (2-20), and $H_F$ being nonnegative with constant $H_F$ and $u$ bounded above can be replaced by constant $H_F$ and bounded $u$ with the same sign as $H_F$. Indeed, suppose that $H_F$

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is a nonpositive constant and $-\delta \leq u \leq 0$ for some positive constant $\delta$. Consider the nonnegative function $f = \delta + u$ and the field $X = \nabla^{\nu} u$.

THEOREM 12. *Let $F : \mathbb{S}^n \to (0, +\infty)$ be a smooth function satisfying (1-1) and let $x : M^n \looparrowright \mathbb{R}^{n+1}$ be a complete, connected hypersurface with polynomial $\nu$-volume growth. Suppose that $H_F$ is constant, $u$, defined in (2-20), is bounded, and has the same sign as $H_F$. If $|S_F(x^{\top})|_{\nu}$ is bounded, then $M^n$ is anisotropic minimal. If, in addition, $|u|_{\nu} \geq \varepsilon\delta/(\varepsilon - n(n-1)H_{2;F})$ for some constants $\varepsilon, \delta > 0$, where $\delta$ is such that $|u|_{\nu} \leq \delta$, then $x(M^n)$ is a hyperplane of $\mathbb{R}^{n+1}$.*

## 4. Extensions to higher-order anisotropic mean curvatures

In this subsection, we apply the maximum principles established in Lemmas 4 and 5 to obtain new rigidity results for complete noncompact hypersurfaces in $\mathbb{R}^{n+1}$. Our focus now shifts to the influence of higher-order anisotropic mean curvatures $H_{r+1;F}$ for $r \in \{1, \ldots, n-1\}$, in conjunction with the angle function $f_{\xi}$ defined in (2-19).

**4.1. Via convergence at infinity.** The next result establishes a rigidity theorem for hypersurfaces with vanishing higher-order anisotropic mean curvature, constituting a direct generalization of Theorem 1 to the higher-order setting. Recall that a hypersurface $x : M^n \looparrowright \mathbb{R}^{n+1}$ is called *anisotropic r-minimal* if its $(r+1)$ th anisotropic mean curvature vanishes identically.

THEOREM 13. *Let $F : \mathbb{S}^n \to (0, +\infty)$ be a smooth function satisfying (1-1) and let $x : M^n \looparrowright \mathbb{R}^{n+1}$ be a connected complete noncompact hypersurface. Suppose that there exists a vector $\xi \in \mathbb{R}^{n+1}$, with $|\xi|_{\nu} = 1$, such that the function $f_{\xi}$, defined in (2-19), is positive on $M^n$. If $S_F \circ P_{r;F}$ is positive semidefinite for some $r \in \{1, \ldots, n-1\}$, and $\nu$ converges to $\xi$ at infinity, then $x : M^n \looparrowright \mathbb{R}^{n+1}$ is anisotropic r-minimal.*

PROOF. Initially, we note that since $S_F \circ P_{r;F}$ is positive semidefinite, from (2-14),

$$\operatorname{tr}(S_F \circ P_{r;F}) = b_r H_{r+1;F} \geq 0, \tag{4-1}$$

where $b_r = (n-r)\binom{n}{r}$. Now, from item (c) of Lemma 2 and (4-1),

$$\operatorname{div}_{\nu}(P_{r;F}(\nabla^{\nu} \ell_{\xi})) = L_{r;F}(\ell_{\xi}) = b_r H_{r+1;F} f_{\xi} \geq 0, \tag{4-2}$$

where the functions $\ell_{\xi}$ and $f_{\xi}$ are defined in (2-18) and (2-19), respectively.

Furthermore, since $S_F \circ P_{r;F}$ is positive semidefinite, we can use items (a) and (b) of Lemma 2 and consider the function $f = 1 - f_{\xi}$; then

$$\begin{aligned} \langle \nabla^{\nu} f, P_{r;F}(\nabla^{\nu} \ell_{\xi}) \rangle_{\nu} &= -\langle \nabla^{\nu} f_{\xi}, P_{r;F}(\nabla^{\nu} \ell_{\xi}) \rangle_{\nu} \\ &= \langle S_F(\xi^{\top}), P_{r;F}(\xi^{\top}) \rangle_{\nu} \\ &= \langle (S_F \circ P_{r;F})(\xi^{\top}), \xi^{\top} \rangle_{\nu} \geq 0. \end{aligned} \tag{4-3}$$

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Since $\nu$ converges to $\xi$ at infinity, the function $f = 1 - f_\xi$ converges to zero at infinity and so from (4-2) and (4-3), applying Lemma 4 to the vector field $X = P_{r;F}(\nabla^\nu \ell_\xi)$ and the function $f$,

$$\operatorname{div}_\nu(P_{r;F}(\nabla^\nu \ell_\xi)) = 0 \quad \text{on } M^n \setminus f^{-1}(0).$$

From (4-2), this implies $b_r H_{r+1;F} f_\xi = 0$ on $M^n \setminus f^{-1}(0)$. Since $f_\xi > 0$ on $M^n$, it follows that $H_{r+1;F} = 0$ on $M^n \setminus f^{-1}(0)$.

Now, note that $f^{-1}(0) = \{p \in M^n : f_\xi(p) = 1\}$. If $p$ is an interior point of $f^{-1}(0)$, then $\nu \equiv \xi$ in a neighborhood of $p$, which implies $S_F \equiv 0$ in that neighborhood (as in the proof of Theorem 1). Consequently, $H_{r+1;F} = 0$ in that neighborhood. Since $H_{r+1;F}$ vanishes on $M^n \setminus f^{-1}(0)$ and on the interior of $f^{-1}(0)$, we conclude that $H_{r+1;F} = 0$ on all of $M^n$. □

To establish our next result, we introduce for $r \in \{1, \dots, n-1\}$, a pair of auxiliary objects derived from the anisotropic curvature operators. Let us define the differential operator $\mathcal{P}_{r;F}$ and the function $\mathcal{H}_{r;F}$ by

$$\mathcal{P}_{r;F} = b_r H_{r;F} P_{r-1;F} - b_{r-1} H_{r-1;F} P_{r;F} \quad \text{and} \quad \mathcal{H}_{r;F} = H_{r;F}^2 - H_{r-1;F} H_{r+1;F}, \tag{4-4}$$

where $b_r = (r+1)\binom{n}{r+1} = (n-r)\binom{n}{r}$ and $P_{r;F}$ denotes the $r$th anisotropic curvature operator defined in (2-13).

We recall that for higher-order anisotropic mean curvatures, the following inequality holds (see [20, Lemma 16]):

$$H_{r;F}^2 \geq H_{r-1;F} H_{r+1;F}. \tag{4-5}$$

In particular, following the notation established in (4-4), we have that $\mathcal{H}_{r;F} \geq 0$ on $M^n$.

REMARK 3. If $\mathcal{H}_{r;F} = 0$ on a complete noncompact hypersurface, then it is totally anisotropic umbilical and consequently a hyperplane (see [5, Proposition 2.3(a)] and [11, Proposition 1.1]).

THEOREM 14. *Let $F : \mathbb{S}^n \to (0, +\infty)$ be a smooth function satisfying (1-1) and let $x : M^n \looparrowright \mathbb{R}^{n+1}$ be a connected complete noncompact hypersurface. Suppose that, for some $r \in \{1, \dots, n-1\}$, $H_{r-1;F}$ and $H_{r;F}$ are constant, and there exists a vector $\xi \in \mathbb{R}^{n+1}$, with $|\xi|_\nu = 1$, such that the angle function $f_\xi$, defined in (2-19), is positive on $M^n$. If the operator $S_F \circ \mathcal{P}_{r;F}$ is positive semidefinite in the $\xi^\top$ direction and $\nu$ converges to $\xi$ at infinity, then $x(M^n)$ is a hyperplane of $\mathbb{R}^{n+1}$.*

PROOF. Consider the smooth operator $\mathcal{P}_{r;F}$ defined in (4-4) and the tangent vector field $X = \mathcal{P}_{r;F}(\xi^\top)$ on $M^n$. From Inequality (4-5) and item (c) of Lemma 2,

$$\operatorname{div}_\nu(X) = b_r H_{r;F} L_{r-1;F}(\ell_\xi) - b_{r-1} H_{r-1;F} L_{r;F}(\ell_\xi) = b_r b_{r-1} \mathcal{H}_{r;F} f_\xi \geq 0. \tag{4-6}$$

However, by the Cauchy–Schwarz inequality, the function $f = 1 - f_\xi$ is nonnegative on $M^n$. Moreover, since $S_F \circ \mathcal{P}_{r;F}$ is positive semidefinite in the $\xi^\top$ direction, using item (a) of Lemma 2,

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$$\langle \nabla^{\nu} f, X\rangle_{\nu} = -\langle \nabla^{\nu} f_{\xi}, X\rangle_{\nu} = \langle (S_F \circ \mathcal{P}_{r;F})(\xi^{\top}), \xi^{\top}\rangle_{\nu} \geq 0. \tag{4-7}$$

Now, suppose that $f$ is not identically zero. Since $\nu$ converges to $\xi$ at infinity, the function $f = 1 - f_{\xi}$ converges to zero at infinity. Therefore, by (4-6), (4-7), and Lemma 4, we conclude that $\operatorname{div}_{\nu}(X) = 0$ on $M^n \setminus f^{-1}(0)$.

Note that $f^{-1}(0) = \{p \in M^n : f_{\xi}(p) = 1\}$. If $p$ is an interior point of $f^{-1}(0)$, then $\nu \equiv \xi$ in a neighborhood of $p$, which implies $\xi^{\top} = 0$ in that neighborhood. Consequently, $X = \mathcal{P}_{r;F}(\xi^{\top}) = 0$ and, thus, $\operatorname{div}_{\nu}(X) = 0$ in that neighborhood. Since $\operatorname{div}_{\nu}(X)$ vanishes on $M^n \setminus f^{-1}(0)$ and on the interior of $f^{-1}(0)$, we conclude that $\operatorname{div}_{\nu}(X) = 0$ on all of $M^n$.

Returning to (4-6),

$$b_r b_{r-1} \mathcal{H}_{r;F} f_{\xi} = 0 \quad \text{on } M^n.$$

Since $f_{\xi} > 0$ and $b_r, b_{r-1} > 0$, it follows that

$$\mathcal{H}_{r;F} = 0 \quad \text{on } M^n.$$

This implies that $M^n$ is totally anisotropic umbilical. Since $M^n$ is complete and noncompact, from Remark 3, it must be a hyperplane of $\mathbb{R}^{n+1}$.

If $f$ is identically zero, then $f_{\xi} \equiv 1$, which means $\nu \equiv \xi$ on $M^n$. This immediately implies that $S_F \equiv 0$ and, thus, $x(M^n)$ is a hyperplane of $\mathbb{R}^{n+1}$. □

The following result can be interpreted as a version of Theorem 3 for higher-order anisotropic mean curvatures. Here, we impose conditions on the sign of $f_{\xi}$ and on the quadratic form $\langle \mathcal{P}_{r;F}(\xi^{\top}), \xi^{\top}\rangle_{\nu}$, where the operator $\mathcal{P}_{r;F}$ is defined in (4-4).

THEOREM 15. *Let $F : \mathbb{S}^n \to (0, +\infty)$ be a smooth function satisfying (1-1) and let $x : M^n \hookrightarrow \mathbb{R}^{n+1}$ be a connected complete noncompact hypersurface. Suppose that, for some $r \in \{1, \ldots, n\}$, $H_{r-1;F}$ and $H_{r;F}$ are constant and there exists a vector $\xi \in \mathbb{R}^{n+1}$ such that the angle function $f_{\xi}$ is positive (respectively, negative) on $M^n$. If $\sup_M(\ell_{\xi}) \leq c$ for some positive constant $c$, the operator $\mathcal{P}_{r;F}$, given in (4-4), is negative (respectively, positive) semidefinite in the $\xi^{\top}$ direction, and $\ell_{\xi}$ converges to $c$ at infinity, then $x(M^n)$ is a hyperplane of $\mathbb{R}^{n+1}$.*

PROOF. Since $\sup_M(\ell_{\xi}) \leq c$ for some positive constant $c$, the function $f = c - \ell_{\xi}$ is nonnegative on $M^n$. Moreover, since $\mathcal{P}_{r;F}$ defined in (4-4) is negative semidefinite in the $\xi^{\top}$ direction, using item (a) of Lemma 2 and the tangent vector field $X = \mathcal{P}_{r;F}(\xi^{\top})$ on $M^n$,

$$\langle \nabla^{\nu} f, X\rangle_{\nu} = -\langle \mathcal{P}_{r;F}(\xi^{\top}), \xi^{\top}\rangle_{\nu} \geq 0. \tag{4-8}$$

Now, suppose that $f$ is not identically zero. Since $f$ converges to zero at infinity, by (4-6), (4-8), and Lemma 4, we conclude that $\operatorname{div}_{\nu}(X) = 0$ on $M^n \setminus f^{-1}(0)$.

Note that $f^{-1}(0) = \{p \in M^n : \ell_{\xi}(p) = c\}$. To conclude the proof in this case, we argue as in the final part of the proof of Theorem 14.

For the case where $f_{\xi}$ is negative and $\mathcal{P}_{r;F}$ is positive semidefinite in the $\xi^{\top}$ direction, we consider the vector field $X = -\mathcal{P}_{r;F}(\xi^{\top})$ and apply the same argument.

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Finally, if $f$ is identically zero, then $\ell_\xi \equiv c$, which means $x(M^n)$ is contained in a hyperplane orthogonal to $\xi$. Since $M^n$ is complete and connected, $x(M^n)$ must be a hyperplane of $\mathbb{R}^{n+1}$. □

**4.2. Via polynomial $\nu$-volume growth.** The following result is a version of Theorem 8 for higher-order anisotropic mean curvatures. Thus, we need to consider the hypothesis of polynomial $\nu$-volume growth.

THEOREM 16. *Let $F : \mathbb{S}^n \to (0, +\infty)$ be a smooth function satisfying (1-1) and let $x : M^n \looparrowright \mathbb{R}^{n+1}$ be a complete, connected hypersurface with polynomial $\nu$-volume growth. Suppose that, for some $r \in \{1, \dots, n-1\}$, $S_F \circ P_{r;F}$ is negative semidefinite. If $|P_{r;F}|_\nu$ is bounded and the angle function $f_\xi$, defined in (2-19), satisfies*

$$f_\xi \geq \frac{|\xi|_\nu}{1 - H_{r+1;F}} \tag{4-9}$$

*for some vector $\xi \in \mathbb{R}^{n+1}$, then $x(M^n)$ is a hyperplane of $\mathbb{R}^{n+1}$.*

PROOF. Consider the vector field $X = -P_{r;F}(\xi^\top)$, which is bounded on $M^n$ since $|P_{r;F}|_\nu$ is bounded and $|\xi^\top|_\nu \leq |\xi|_\nu < +\infty$, and since $S_F \circ P_{r;F}$ is negative semidefinite from (2-14),

$$\operatorname{tr}(S_F \circ P_{r;F}) = (r+1)\binom{n}{r+1} H_{r+1;F} \leq 0,$$

which implies $H_{r+1;F} \leq 0$. Therefore, $1 - H_{r+1;F} \geq 1 > 0$.

From (4-9), item (c) of Lemma 2, and considering $f = |\xi|_\nu - f_\xi$, we have

$$\operatorname{div}_\nu(X) = \operatorname{div}_\nu(-P_{r;F}(\nabla^\nu \ell_\xi)) \geq (r+1)\binom{n}{r+1}(|\xi|_\nu - f_\xi) \geq (r+1)\binom{n}{r+1} f \geq 0. \tag{4-10}$$

However, since $S_F \circ P_{r;F}$ is negative semidefinite, using item (b) of Lemma 2,

$$\langle \nabla^\nu f, X\rangle_\nu = \langle \nabla^\nu f_\xi, P_{r;F}(\xi^\top)\rangle_\nu = -\langle S_F(\xi^\top), P_{r;F}(\xi^\top)\rangle_\nu = -\langle (S_F \circ P_{r;F})(\xi^\top), \xi^\top\rangle_\nu \geq 0. \tag{4-11}$$

Suppose that $M^n$ is a noncompact hypersurface; since $M^n$ has polynomial $\nu$-volume growth, by (4-10), (4-11), and using Lemma 5, we conclude that $f \leq 0$ on $M^n$. Since $f \geq 0$ by the Cauchy–Schwarz inequality, it follows that $f_\xi = |\xi|_\nu$.

By the equality case of the Cauchy–Schwarz inequality, this implies that $\nu$ and $\xi$ are collinear. Since $|\nu|_\nu = 1$, we have $\xi = |\xi|_\nu \nu$. Therefore, $\nu$ is constant and, consequently, $S_F \equiv 0$. Thus, $M^n$ is totally geodesic. Since $M^n$ is complete and connected, $x(M^n)$ must be a hyperplane of $\mathbb{R}^{n+1}$. □

From some modifications in the hypotheses of the previous theorem, we obtain the following version of Theorem 10 for higher-order anisotropic mean curvatures.

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THEOREM 17. *Let $F : \mathbb{S}^n \to (0, +\infty)$ be a smooth function satisfying (1-1) and let $x : M^n \hookrightarrow \mathbb{R}^{n+1}$ be a complete, connected hypersurface with polynomial $\nu$-volume growth. Suppose that $|S_F|_\nu$ is bounded and the $(r+2)$ th anisotropic mean curvature satisfies $H_{r+2;F} < 1$ for some $r \in \{1, \dots, n-2\}$. If $P_{r;F}$ is positive semidefinite and there is a vector $\xi \in \mathbb{R}^{n+1}$ such that the operator $H_{r+1;F}S_F$ is negative semidefinite in the direction $\xi^\top$ and the angle function $f_\xi$, defined in (2-19), satisfies*

$$f_\xi \geq \frac{|\xi|_\nu}{1 - H_{r+2;F}},$$

*then $x(M^n)$ is a hyperplane of $\mathbb{R}^{n+1}$.*

PROOF. Consider the vector field

$$X = -P_{r;F}(\nabla^\nu f_\xi) - \binom{n}{r+1} H_{r+1;F}\xi^\top.$$

Using Lemma 2,

$$\begin{aligned}
\operatorname{div}_\nu(X) &= \operatorname{div}_\nu(-P_{r;F}(\nabla^\nu f_\xi)) - \binom{n}{r+1}\langle \nabla^\nu H_{r+1;F}, \xi^\top\rangle_\nu - \binom{n}{r+1} H_{r+1;F}\operatorname{div}_\nu(\xi^\top) \\
&= \left(n\binom{n}{r+1} H_F H_{r+1;F} f_\xi - b_{r+1}H_{r+2;F}f_\xi + \binom{n}{r+1}\langle \nabla^\nu H_{r+1;F}, \xi\rangle_\nu\right) \\
&\quad - \binom{n}{r+1}\langle \nabla^\nu H_{r+1;F}, \xi^\top\rangle_\nu - n\binom{n}{r+1} H_{r+1;F}H_F f_\xi \\
&= -b_{r+1}H_{r+2;F}f_\xi.
\end{aligned}$$

Since $H_{r+2;F} < 1$ by hypothesis, we have $1 - H_{r+2;F} > 0$ and

$$\operatorname{div}_\nu(X) \geq b_{r+1}(|\xi|_\nu - f_\xi).$$

Thus,

$$\operatorname{div}_\nu(X) \geq b_{r+1} f. \tag{4-12}$$

However, using item (b) of Lemma 2,

$$\begin{aligned}
\langle \nabla^\nu f, X\rangle_\nu &= \langle -\nabla^\nu f_\xi, -P_{r;F}(\nabla^\nu f_\xi) - \binom{n}{r+1} H_{r+1;F}\xi^\top\rangle_\nu \\
&= \langle \nabla^\nu f_\xi, P_{r;F}(\nabla^\nu f_\xi)\rangle_\nu + \binom{n}{r+1} H_{r+1;F}\langle \nabla^\nu f_\xi, \xi^\top\rangle_\nu \\
&= \langle S_F(\xi^\top), (P_{r;F} \circ S_F)(\xi^\top)\rangle_\nu - \binom{n}{r+1} H_{r+1;F}\langle S_F(\xi^\top), \xi^\top\rangle_\nu \\
&= \langle \xi^\top, (S_F^2 \circ P_{r;F})(\xi^\top)\rangle_\nu - \binom{n}{r+1} H_{r+1;F}\langle S_F(\xi^\top), \xi^\top\rangle_\nu.
\end{aligned} \tag{4-13}$$

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Since $P_{r;F}$ is positive semidefinite and $H_{r+1;F}S_F$ is negative semidefinite in the direction $\xi^\top$, both terms in (4-13) are nonnegative. Therefore,

$$\langle \nabla^\nu f, X\rangle_\nu \geq 0.$$

Moreover, $|X|_\nu$ is bounded because $|\xi^\top|_\nu \leq |\xi|_\nu < +\infty$, $|S_F|_\nu$ is bounded by hypothesis (which implies that $|P_{r;F}|_\nu$ is bounded), and $H_{r+1;F}$ is bounded since $|S_F|_\nu$ is bounded.

Suppose that $M^n$ is a noncompact hypersurface; since $M^n$ has polynomial $\nu$-volume growth, by (4-12), (4-13), and Lemma 5, we conclude that $f \leq 0$ on $M^n$. Since $f \geq 0$ by the Cauchy–Schwarz inequality, it follows that $f \equiv 0$ on $M^n$.

By the equality case of the Cauchy–Schwarz inequality, this implies that $\nu$ and $\xi$ are collinear. Since $|\nu|_\nu = 1$, we have $\xi = |\xi|_\nu \nu$. Therefore, $\nu$ is constant and, consequently, $S_F \equiv 0$. Thus, $M^n$ is totally geodesic. Since $M^n$ is complete and connected, $x(M^n)$ must be a hyperplane of $\mathbb{R}^{n+1}$. □

The following result establishes a version of Theorem 14 in the context of polynomial volume growth. We use the operator $\mathcal{P}_{r;F}$ and the function $\mathcal{H}_{r;F}$ defined in (4-4).

THEOREM 18. *Let $F : \mathbb{S}^n \to (0, +\infty)$ be a smooth function satisfying (1-1) and let $x : M^n \looparrowright \mathbb{R}^{n+1}$ be a complete, connected hypersurface with polynomial $\nu$-volume growth. Suppose that $|S_F|_\nu$ is bounded and that there exists a vector $\xi \in \mathbb{R}^{n+1}$ such that the angle function $f_\xi$, defined in (2-19), is positive on $M^n$. If, for some $r \in \{1, \dots, n-1\}$, $H_{r-1;F}$ and $H_{r;F}$ are constant and the operator $S_F \circ \mathcal{P}_{r;F}$ is negative semidefinite in the $\xi^\top$ direction, then $\mathcal{H}_{r;F}$ is not globally bounded away from zero. In particular, if $\mathcal{H}_{r;F}$ is constant, then $x(M^n)$ is a hyperplane of $\mathbb{R}^{n+1}$.*

PROOF. Consider the operator $\mathcal{P}_{r;F}$, the function $\mathcal{H}_{r;F}$ defined in (4-4), and the tangent vector field $X = \mathcal{P}_{r;F}(\xi^\top)$ on $M^n$. Note that $|X|_\nu$ is bounded on $M^n$ because $|\xi^\top|_\nu \leq |\xi|_\nu < +\infty$, and the boundedness of $|S_F|_\nu$ implies that $|P_{j;F}|_\nu$ and $|H_{j;F}|$ are bounded for all $j \in \{1, \dots, n\}$.

Then, from (4-5) and item (c) of Lemma 2,

$$\begin{aligned} \operatorname{div}_\nu(X) &= b_r H_{r;F} L_{r-1;F}(\ell_\xi) - b_{r-1} H_{r-1;F} L_{r;F}(\ell_\xi) \\ &= b_r b_{r-1}(H_{r;F}^2 - H_{r-1;F} H_{r+1;F}) f_\xi \\ &= b_r b_{r-1}\, \mathcal{H}_{r;F}\, f_\xi \geq 0. \end{aligned} \tag{4-14}$$

Now, suppose by contradiction that $\mathcal{H}_{r;F}$ is globally bounded away from zero. Then, there exists a constant $\delta > 0$ such that $\mathcal{H}_{r;F} \geq \delta$ on $M^n$. From (4-14),

$$\operatorname{div}_\nu(X) = b_r b_{r-1}\, \mathcal{H}_{r;F}\, f_\xi \geq b_r b_{r-1}\, \delta\, f_\xi. \tag{4-15}$$

Furthermore, since $S_F \circ \mathcal{P}_{r;F}$ is negative semidefinite in the $\xi^\top$ direction, using item (b) of Lemma 2,

$$\langle \nabla^\nu f_\xi, X\rangle_\nu = -\langle (S_F \circ \mathcal{P}_{r;F})(\xi^\top), \xi^\top\rangle_\nu \geq 0. \tag{4-16}$$

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Suppose that $M^n$ is a noncompact hypersurface; since $M^n$ has polynomial $\nu$-volume growth, by (4-15), (4-16), and Lemma 5, it follows that $f_\xi \leq 0$ on $M^n$. This contradicts the hypothesis that $f_\xi > 0$. Therefore, $\mathcal{H}_{r;F}$ cannot be globally bounded away from zero.

In particular, if $H_{r+1;F}$ is constant, then $\mathcal{H}_{r;F}$ is constant. Since it cannot be bounded away from zero, we must have $\mathcal{H}_{r;F} = 0$. From (4-5), this implies that $M^n$ is totally anisotropic umbilical. Therefore, since $M^n$ is complete and noncompact, $x(M^n)$ must be a hyperplane of $\mathbb{R}^{n+1}$. □

REMARK 4. In Theorem 18, we assume that $S_F \circ \mathcal{P}_{r;F}$ is negative semidefinite in the $\xi^\top$ direction and that the angle function $f_\xi$ is strictly positive. It is immediate from the proof that the result also holds if $S_F \circ \mathcal{P}_{r;F}$ is positive semidefinite in the $\xi^\top$ direction and $f_\xi$ is strictly negative.

## 5. Estimating the index of anisotropic relative nullity

According to [9], for $p \in M^n$, the *anisotropic relative nullity space* $\mathcal{N}(p)$ of $M^n$ at $p$ is defined by

$$\mathcal{N}(p) = \{w \in T_pM : w \in \ker(S_F(p))\},$$

where $\ker(S_F(p))$ denotes the kernel of the $F$-Weingarten operator $S_F(p)$. The *index of anisotropic relative nullity* $\omega(p)$ of $M^n$ at $p$ is the dimension of $\mathcal{N}(p)$, that is,

$$\omega(p) = \dim(\mathcal{N}(p)),$$

and the *index of minimum anisotropic relative nullity* $\omega_0$ of $M^n$ is defined by

$$\omega_0 = \min_{p \in M} \omega(p).$$

For anisotropic $r$-minimal hypersurfaces, the anisotropic relative nullity provides important information about the local product structure and rigidity properties. Previous work by Wang and He [20] showed that if an anisotropic minimal hypersurface $M^n$ has an open nonempty set $W$ of points omitted by the tangent hyperplanes, then $M^n$ must be part of a hyperplane in $\mathbb{R}^{n+1}$. Furthermore, for anisotropic $r$-minimal hypersurfaces with $H_{r;F}$ nowhere vanishing, they proved that $M^n$ has anisotropic relative nullity $n - r$. This theme was also addressed in [8, Theorem 4].

In this branch, we establish the following lower estimate for the index of minimum anisotropic relative nullity.

THEOREM 19. *Let $F : \mathbb{S}^n \to (0, +\infty)$ be a smooth function satisfying (1-1) and let $x : M^n \hookrightarrow \mathbb{R}^{n+1}$ be an anisotropic $r$-minimal, complete, connected hypersurface for some $r \in \{1, \ldots, n-2\}$. Suppose that there exists a vector $\xi \in \mathbb{R}^{n+1}$ with $|\xi|_\nu = 1$ such that the angle function $f_\xi$, defined in (2-19), is positive on $M^n$. If $H_{r;F} \neq 0$ and $H_{r+2;F}$ is nonnegative, $P_{r;F}$ is negative semidefinite in the $\xi^\top$ direction, and $\nu$ converges to $\xi$ at infinity, then $\omega_0 \geq n - r$.*

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PROOF. From the definition of the operator $L_{r;F}$ in (2-17) and item (d) of Lemma 2,

$$\operatorname{div}_\nu(P_{r;F}(\nabla^\nu f_\xi)) = L_{r;F}(f_\xi) = b_{r+1}H_{r+2;F}f_\xi \geq 0, \tag{5-1}$$

since $H_{r+2;F} \geq 0$ and $f_\xi > 0$ by hypothesis.

By the Cauchy–Schwarz inequality, the function $f = 1 - f_\xi$ is nonnegative on $M^n$. Suppose $f$ is not identically zero. Since $P_{r;F}$ is negative semidefinite in the $\xi^\top$ direction, using item (b) of Lemma 2,

$$\begin{aligned}\langle \nabla^\nu f, P_{r;F}(\nabla^\nu f_\xi)\rangle_\nu &= -\langle \nabla^\nu f_\xi, P_{r;F}(\nabla^\nu f_\xi)\rangle_\nu \\ &= -\langle (S_F^2 \circ P_{r;F})(\xi^\top), \xi^\top\rangle_\nu \geq 0.\end{aligned} \tag{5-2}$$

Since $\nu$ converges to $\xi$ at infinity, the function $f = 1 - f_\xi$ converges to zero at infinity. Therefore, by (5-1), (5-2), and Lemma 4, we conclude that $H_{r+2;F} = 0$ on $M^n$ (arguing as in the proof of Theorem 14).

Now, suppose that $H_r$ does not vanish on $M^n$. By [9, Theorem 5.3] (see also [10]), the distribution $p \mapsto \mathcal{N}(p)$ of minimal relative nullity of $M^n$ is smooth and integrable with complete leaves, which are totally geodesic in $M^n$ and in $\mathbb{R}^{n+1}$.

Therefore, the result follows from the characterization of complete totally geodesic submanifolds of $\mathbb{R}^{n+1}$ as hyperplanes of suitable dimension. Consequently, $\omega_0 \geq n - r$. □

Using similar ideas, we now obtain an estimate for the minimum anisotropic relative nullity index under polynomial volume growth conditions.

THEOREM 20. *Let $F : \mathbb{S}^n \to (0, +\infty)$ be a smooth function satisfying (1-1) and let $x : M^n \looparrowright \mathbb{R}^{n+1}$ be an anisotropic $r$-minimal, complete, connected hypersurface with polynomial $\nu$-volume growth for some $r \in \{1, \ldots, n-2\}$. Suppose that there exists a vector $\xi \in \mathbb{R}^{n+1}$ such that the angle function $f_\xi$, defined in (2-19), is positive on $M^n$. If $H_{r+2;F}$ is nonnegative, $P_{r;F}$ is positive semidefinite in the $\xi^\top$ direction, and $|S_F|_\nu$ is bounded on $M^n$, then $H_{r+2;F}$ is not globally bounded away from zero. In particular, if $H_{r;F} \neq 0$ and $H_{r+2;F}$ is constant, then $\omega_0 \geq n - r$.*

PROOF. Suppose, for contradiction, that $H_{r+2;F}$ is globally bounded away from zero. Then, there exists $\delta > 0$ such that $H_{r+2;F} \geq \delta$ on $M^n$. From (2-17) and item (d) of Lemma 2,

$$\operatorname{div}_\nu(P_{r;F}(\nabla^\nu f_\xi)) = L_{r;F}(f_\xi) = b_{r+1}H_{r+2;F}f_\xi \geq b_{r+1}\delta f_\xi > 0. \tag{5-3}$$

Consider the vector field $X = P_{r;F}(\nabla^\nu f_\xi)$. Since $|S_F|_\nu$ is bounded and $|\xi^\top|_\nu \leq |\xi|_\nu < +\infty$, it follows that $|X|_\nu$ is bounded on $M^n$. Furthermore, since $P_{r;F}$ is positive semidefinite in the $\xi^\top$ direction, using item (b) of Lemma 2,

$$\langle \nabla^\nu f_\xi, X\rangle_\nu = \langle (S_F^2 \circ P_{r;F})(\xi^\top), \xi^\top\rangle_\nu \geq 0. \tag{5-4}$$

Suppose that $M^n$ is a noncompact hypersurface; since $M^n$ has polynomial $\nu$-volume growth, by (5-3), (5-4), and Lemma 5, we conclude that $f_\xi \leq 0$ on $M^n$, which

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contradicts the hypothesis that $f_\xi > 0$. Therefore, $H_{r+2;F}$ cannot be globally bounded away from zero.

In particular, if $H_{r+2;F}$ is constant, then $H_{r+2;F} = 0$. To conclude the proof in this case, we argue as in the final part of the proof of Theorem 19.

For the case where $M^n$ is compact, the proof follows similarly to the end of the proof of Theorem 8. □

We conclude this section with an estimate for the relative nullity index using the support function $u = \langle x, \nu\rangle_\nu$ (see Lemma 3).

THEOREM 21. *Let $F : \mathbb{S}^n \to (0, +\infty)$ be a smooth function satisfying (1-1) and let $x : M^n \looparrowright \mathbb{R}^{n+1}$ be an anisotropic $r$-minimal, complete, connected hypersurface with polynomial $\nu$-volume growth for some $r \in \{1, \dots, n-2\}$. Suppose that the support function $u$, defined in (2-20), is positive on $M^n$. If $H_{r+2;F}$ is nonnegative, $P_{r;F}$ is positive semidefinite in the $x^\top$ direction, and $|S_F|_\nu$ is bounded on $M^n$, then $H_{r+2;F}$ is not globally bounded away from zero. In particular, if $H_{r;F} \neq 0$ and $H_{r+2;F}$ is constant, then $\omega_0 \geq n - r$.*

PROOF. Suppose, for contradiction, that $H_{r+2;F}$ is globally bounded away from zero. Then, there exists $\delta > 0$ such that $H_{r+2;F} \geq \delta$ on $M^n$. Since $H_{r+1;F} = 0$ ($r$-minimality), from item (b) of Lemma 3,

$$\operatorname{div}_\nu(P_{r;F}(\nabla^\nu u)) = L_{r;F}(u) = b_{r+1} H_{r+2;F} u \geq b_{r+1}\delta u > 0. \tag{5-5}$$

Consider the vector field $X = P_{r;F}(\nabla^\nu u)$. Since $|S_F|_\nu$ is bounded, $|X|_\nu$ is bounded on $M^n$. Moreover, since $P_{r;F}$ is positive semidefinite in the $x^\top$ direction, using item (a) of Lemma 3,

$$\begin{aligned}\langle \nabla^\nu u, X\rangle_\nu &= \langle \nabla^\nu u, P_{r;F}(\nabla^\nu u)\rangle_\nu \\ &= \langle S_F(x^\top), (P_{r;F} \circ S_F)(x^\top)\rangle_\nu \\ &= \langle (S_F^2 \circ P_{r;F})(x^\top), x^\top\rangle_\nu \geq 0.\end{aligned} \tag{5-6}$$

Suppose that $M^n$ is noncompact; since $M^n$ has polynomial $\nu$-volume growth, by (5-5), (5-6), and Lemma 5, we conclude that $u \leq 0$ on $M^n$, which contradicts the hypothesis that $u > 0$. Therefore, $H_{r+2;F}$ is not globally bounded away from zero. In particular, if $H_{r+2;F}$ is constant, then $H_{r+2;F} = 0$. To conclude the proof in this case, we argue as in the last part of the proof of Theorem 19. Finally, in the case where $M^n$ is compact, the proof follows similarly to the last part of the proof of Theorem 8. □

REMARK 5. It is worth pointing out that the results presented in this manuscript work, in particular, in the classical Euclidean setting, which corresponds to the case $F \equiv 1$. Furthermore, they can be regarded as extensions of those obtained in [4, Sections 4 and 5] for the context of the classical Euclidean mean curvatures and involving Euclidean geometric quantities. For other similar results, we also recommend [1, Section 3] and [2, Section 3].

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## Data availability statement

This manuscript has no associated data.

## Author contributions

All authors developed the theoretical formalism, performed the analytic calculations, discussed the results, and contributed to writing the manuscript.

## Competing interests

The authors have no competing interests to declare that are relevant to the content of this article.

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