Mathematical Proceedings of the Cambridge Philosophical Society

VOL. 106 SEPTEMBER 1989 PART 2

Math. Proc. Camb. Phil. Soc. (1989), **106**, 193
Printed in Great Britain

Groups whose projective characters are Galois conjugate

BY R. J. HIGGS
Department of Mathematics, University College, Belfield, Dublin, Ireland

(*Received* 3 *October* 1988)

All groups G considered in this paper are finite and all representations of G are defined over the complex numbers. The reader unfamiliar with projective representations is referred to [6] for basic definitions and elementary results.

Introduction

Let α be a cocycle of G such that $\alpha(x,y)$ is a root of unity for all $x,y \in G$, let $\mathrm{Proj}(G,\alpha)$ denote the set of irreducible projective characters of G with cocycle α, and let $N \trianglelefteq G$. Also let $\mathscr{G}$ denote the Galois group of $\mathbb{C}$ over the field F, where $\mathbb{Q}(\alpha) \subseteq F$. Then $G \times \mathscr{G}$ acts on $\mathrm{Proj}(N, \alpha_N)$ via the action

$$\zeta^{(g,\sigma)}(x) = f_\alpha(x, g^{-1})(\zeta(gxg^{-1}))^\sigma$$

for $(g,\sigma) \in G \times \mathscr{G}$, and all $x \in N$, where $f_\alpha(x,g^{-1}) = \alpha(g, xg^{-1})\alpha(x,g^{-1})(\alpha(g,g^{-1}))^{-1}$. The object of this paper is to study the group G when the above action is transitive with respect to a normal subgroup N of G; if this is the case we shall call G a $(\mathscr{G},N,\alpha)$-group.

In a previous paper [2], the author showed that if G acts transitively on $\mathrm{Proj}(N,\alpha_N)$ by conjugation, and G/N is solvable, then G is also solvable. Our aim here is to partially generalize this result to obtain the following

THEOREM. *Suppose that G is a $(\mathscr{G},N,\alpha)$-group such that G/N is solvable and G does not involve $GL(3,2)$. Then G is solvable.*

The proof of this theorem demonstrates above all the fragility of the methods used to investigate groups of cental type by Howlett and Isaacs [3] compared to those developed by Liebler and Yellen [7]. Indeed it is exactly the failure of the aforementioned techniques that forces the imposition that G should not involve $GL(3,2)$ in the statement of the theorem. We conjecture that this condition can be removed. We commence by investigating some basic properties of $(\mathscr{G},N,\alpha)$-groups; our work here concludes with a simple result which removes some of the ambiguities in dealing with Galois conjugates of projective characters. After this a very brief

https://doi.org/10.1017/S0305004100078002 Published online by Cambridge University Press

account of the proof of the theorem is given, which is followed by an explanation of the apparent necessity for G to not involve $GL(3,2)$. Finally we note that a special case of the theorem occurs if $N = G$, which provides a result unrelated to that proved in [2].

Behaviour of $(\mathscr{G}, N, \alpha)$-groups

Let G be a group and let α be a cocycle of G of finite order. Then we can construct an α-covering group of G as described in (3·3·8) of [6]. It follows from this that there exists a coboundary $\delta: G \to \mathbb{Q}(\alpha)$ such that if $\beta(x,y) = \delta(x)\,\delta(y)\,(\delta(xy))^{-1}\alpha(x,y)$, then β is a class-function cocycle. Thus since G is a $(\mathscr{G}, N, \alpha)$-group if and only if it is a $(\mathscr{G}, N, \beta)$-group, we may assume for the duration of this paper that α is a class-function cocycle. Now let $N \trianglelefteq G$. We may define an equivalence relation $\sim_N$ on $\mathrm{Proj}\,(G, \alpha)$ by $\xi \sim_N \xi'$ if and only if $\xi'(1)\,\xi_N = \xi(1)\,\xi'_N$. The equivalence class $[\xi]_N$ of ξ consists of the elements of $\mathrm{Proj}\,(G, \alpha)$ which restrict to N as some multiple of the G-orbit on $\mathrm{Proj}\,(N, \alpha_N)$ whose elements are the irreducible constituents of ξ_N. We now define an action of $\mathscr{G}$ on the set of equivalence classes of $\mathrm{Proj}\,(G, \alpha)$ under $\sim_N$ by $([\xi]_N)^\sigma = [\xi^\sigma]_N$. Using these definitions we now make the following obvious observations.

PROPOSITION 1. *The following statements are equivalent*:

(i) *G is a $(\mathscr{G}, N, \alpha)$-group*;

(ii) *$\mathscr{G}$ acts transitively on the G-orbits of* $\mathrm{Proj}\,(N, \alpha_N)$;

(iii) *$\mathscr{G}$ acts transitively on the set of equivalence classes of* $\mathrm{Proj}\,(G, \alpha)$ *under* $\sim_N$.

From this we deduce

COROLLARY 1. *Let G be a $(\mathscr{G}, N, \alpha)$-group, and let $K \trianglelefteq G$ with $K \subseteq N$. Then G is a $(\mathscr{G}, K, \alpha)$-group.*

Proof. Let $\xi \in \mathrm{Proj}\,(G, \alpha)$. Then $[\xi]_K$ is a union of equivalence classes of $\mathrm{Proj}\,(G, \alpha)$ under $\sim_N$, but by Proposition 1 $\mathscr{G}$ acts transitively on $\{[\xi]_N : \xi \in \mathrm{Proj}\,(G, \alpha)\}$. Thus $\mathscr{G}$ acts transitively on $\{[\xi]_K : \xi \in \mathrm{Proj}\,(G, \alpha)\}$, and Proposition 1 yields that G is a $(\mathscr{G}, K, \alpha)$-group.

We note in passing that if G is a $(\mathscr{G}, N, \alpha)$-group and α_N is cohomologous to the trivial cocycle, then N must be abelian, and so our theorem will automatically be true in this case. However we now turn our attention to the behaviour of the Sylow subgroups of a $(\mathscr{G}, N, \alpha)$-group.

PROPOSITION 2. *Let p be a prime number, let G be a $(\mathscr{G}, N, \alpha)$-group, and let S be a Sylow p-subgroup of G. Then S is a $(\mathscr{G}, N \cap S, \alpha_S)$-group.*

Proof. The proof of this result is essentially the same as that of (1·3) of [2].

We shall now show that certain quotients of inertia subgroups also inherit the property of being a '$(\mathscr{G}, N, \alpha)$-group' from G. However in order to prove this we need a preliminary result which is set up in a far more general context.

LEMMA 1. *Let α be a cocycle of G, and let $K, N \trianglelefteq G$ with $K \subseteq N$. Now let $\mu \in \mathrm{Proj}\,(K, \alpha_K)$, let $V = I_G(\mu)$, and let $U = I_N(\mu)$. Finally let*

$$\mathscr{A} = \{\xi \in \mathrm{Proj}\,(G, \alpha) : \langle \xi_K, \mu \rangle \neq 0\},$$

and

$$\mathscr{B} = \{\gamma \in \mathrm{Proj}\,(V, \alpha_V) : \langle \gamma_K, \mu \rangle \neq 0\},$$

https://doi.org/10.1017/S0305004100078002 Published online by Cambridge University Press

where $\langle , \rangle$ *denotes the inner product of projective characters. Then*

(i) *induction defines a bijection from* $\mathscr{B}$ *onto* $\mathscr{A}$,

(ii) *the image of* $[\gamma]_U$ *under induction is* $[\gamma^G]_N$ *for* $\gamma \in \mathscr{B}$.

Proof. It is convenient to let

$$\mathscr{C} = \{\zeta \in \mathrm{Proj}\,(N, \alpha_N) : \langle \zeta_K, \mu \rangle \neq 0\},$$

and

$$\mathscr{D} = \{\phi \in \mathrm{Proj}\,(U, \alpha_U) : \langle \phi_K, \mu \rangle \neq 0\}.$$

We now begin by noting that $U = V \cap N \trianglelefteq V$. By (6·11) of [4], $\gamma \mapsto \gamma^G$ defines a bijection from $\mathscr{B}$ onto $\mathscr{A}$. Moreover if $\gamma^G = \xi$ then γ is the unique constituent of ξ_V in $\mathscr{B}$. It follows that $\phi \in \mathscr{D}$ is a constituent of $\xi_U = (\xi_V)_U$ if and only if ϕ is a constituent of γ_U. Let $\gamma_U = f(\phi_1 + \ldots + \phi_m)$, where $\phi = \phi_1, \ldots, \phi_m$ are the distinct V-conjugates of ϕ, and f is some natural number. Then since $\xi_U = (\xi_N)_U$ we have from the bijection described above that $\zeta_1 = \phi_1^N, \ldots, \zeta_m = \phi_m^N$ are exactly the distinct elements of $\mathscr{C}$ which are constituents of ξ_N.

Now let $(\gamma')^G = \xi'$ for $\gamma' \in \mathscr{B}$. Then $\xi' \in [\xi]_N$ if and only if $\zeta_1, \ldots, \zeta_m$ are the constituents of $(\xi')_N$ in $\mathscr{C}$, and this is true from above if and only if $\phi_1, \ldots, \phi_m$ are the constituents of γ'_U in $\mathscr{D}$, that is, if and only if $\gamma' \in [\gamma]_U$.

The application of this result to our situation is immediate.

PROPOSITION 3. *Suppose that* G *is a* $(\mathscr{G}, N, \alpha)$*-group and let* $K \trianglelefteq G$ *with* $K \subseteq N$. *Let* $\mu \in \mathrm{Proj}\,(K, \alpha_K)$, *let* $V = I_G(\mu)$, *and let* $U = I_N(\mu)$. *Then for some special cocycle* β *of* V/K *and for each pair* τ, τ' *of* $\mathrm{Proj}\,(V/K, \beta)$, *there exists* $\sigma \in \mathscr{G}$ *and a linear character* λ_σ *of* V/K *such that*

$$\tau'(1)(\lambda_\sigma \tau^\sigma)_{U/K} = \tau(1)\tau'_{U/K}.$$

In particular V/K *is a* $(\mathscr{H}, (U/K)', \beta)$*-group, where* $\mathscr{H}$ *is the Galois group of* $\mathbb{C}$ *over* $\mathbb{Q}(\beta)$.

Proof. We shall use the notation introduced in Lemma 1. By Clifford's theorem there exists a unique cohomology class $[\beta']$ in the Schur multiplier of V/K, and a bijection from $\mathrm{Proj}\,(V/K, \beta) \to \mathscr{B}$ for any $\beta \in [\beta']$ defined by $\tau \mapsto \tau\pi$, where $\pi \in \mathrm{Proj}\,(V, \alpha_V \beta^{-1})$ such that $\pi_K = \mu$. Now $\tau_{U/K} = e(\psi_1 + \ldots + \psi_t)$, where e is a natural number and $\psi_1, \ldots, \psi_t$ are the distinct V/K conjugates of $\psi_1 \in \mathrm{Proj}\,(U/K, \beta_{U/K})$. It follows from this and the bijection described above that $\psi_i \pi_U$ for $i = 1, \ldots, t$ are the distinct irreducible constituents of $(\tau\pi)_U$. Thus $\tau' \in [\tau]_{U/K}$ if and only if $\tau'\pi \in [\tau\pi]_U$. So $[\tau]_{U/K} \mapsto [\tau\pi]_U$ defines a bijection from the equivalence classes of $\mathrm{Proj}\,(V/K, \beta)$ under $\sim_{U/K}$ onto the equivalence classes of $\mathscr{B}$ under $\sim_U$.

We shall assume henceforward that β is a special cocycle of V/K. We note that for the purposes of this paper a special cocycle β' may be interpreted as one whose multiplicative order in $\mathbb{C}$ is the same as the order of its cohomology class, for this then allows the construction of a β'-representation group of G as detailed on p. 99 of [6]. Now since G is a $(\mathscr{G}, N, \alpha)$-group, we may let $\xi = \xi^{\sigma_1}, \ldots, \xi^{\sigma_m}$ for $\sigma_i \in \mathscr{G}$ be representatives for the equivalence classes of $\mathscr{A}$ under $\sim_N$, so that, by Lemma 1, $\gamma = \gamma^{\sigma_1}, \ldots, \gamma^{\sigma_m}$ are representatives for the equivalence classes of $\mathscr{B}$ under $\sim_U$, where $\gamma^G = \xi$. Hence if we write $\gamma^{\sigma_i} = \tau_i \pi$ for $\tau_i \in \mathrm{Proj}\,(V/K, \beta)$, we have by the first paragraph that $\tau_1, \ldots, \tau_m$ are representatives for the equivalence classes of $\mathrm{Proj}\,(V/K, \beta)$ under $\sim_{U/K}$.

Let $\sigma \in \langle \sigma_1, \ldots, \sigma_m \rangle$. Then $\tau^\sigma \mapsto \tau^\sigma \pi^\sigma$ for $\tau \in \mathrm{Proj}\,(V/K, \beta)$ defines a bijection from $\mathrm{Proj}\,(V/K, \beta^\sigma)$ onto $\mathscr{B}$ with $\pi_K^\sigma = \mu$. Thus by the uniqueness of $[\beta]$ we must have $[\beta^\sigma] = [\beta]$, and since we are assuming that β is special we actually obtain that $\beta^\sigma = \beta$.

https://doi.org/10.1017/S0305004100078002 Published online by Cambridge University Press

We have shown that $\langle \sigma_1, \ldots, \sigma_m \rangle$ is a subgroup of $\mathscr{H}$, the Galois group of $\mathbb{C}$ over $\mathbb{Q}(\beta)$. Now we have $\pi^{\sigma_i} = \lambda_{\sigma_i}\pi$ for some linear character λ_{σ_i} of V/K, and hence $\tau_i = \lambda_{\sigma_i}\tau_1^{\sigma_i}$ for $1 \leqslant i \leqslant m$.

Finally we note that the equivalence classes of Proj $(V/K, \beta)$ under $\sim_{(U/K)'}$ are just unions of those under $\sim_{U/K}$, and so from above V/K is a $(\mathscr{H}, (U/K)', \beta)$-group since $(U/K)' \leqslant (V/K)'$.

We remark that it is the linear characters λ_σ which arise in Proposition 3 which cause the problems in the proof of the theorem. Indeed in general linear characters create a number of ambiguities concerning Galois conjugates of projective characters and the choice of cocycle within a cohomology class. This ambiguity may be demonstrated by considering S_4. As is well-known S_4 has two non-isomorphic representation groups of order 48. Constructing the projective characters of S_4 from these groups in the normal manner yields in one case that the two irreducible projective characters of degree 2 are complex conjugates of each other, whilst using the other group the corresponding characters are real-valued. This anomaly may be resolved as follows.

LEMMA 2. *Let E and F be any fields with $F \subseteq E$. Let α and β be E-valued cocycles of G with $F(\beta) \subseteq F(\alpha)$ and $[\alpha] = [\beta]$ in $H^2(G, E^*)$. Then if δ is the coboundary taking α to β and $\sigma \in \mathrm{Gal}\,(E/F(\alpha))$*

$$\lambda_\sigma \delta^\sigma = \delta,$$

where λ_σ is the linear character of G defined by $\lambda_\sigma = (\delta^\sigma)^{-1}\delta$.

We note that the conditions of Lemma 2 are satisfied in particular when α is a cocycle of G of finite order in $\mathbb{C}$, and β is a special cocycle of G (as defined in the proof of Proposition 3) which is cohomologous to α.

We are now in position to prove the theorem. The main difference between this proof and that in the author's previous paper [2] is that here all the G-orbits on Proj (K, α_K), for $K \trianglelefteq G$ with $K \subseteq N$, have the same length, whereas in the other proof we assumed that there was only one such G-orbit.

Proof of the theorem

Let G be a non-solvable group of minimal order such that G does not involve $GL(3,2)$ but is a $(\mathscr{G}, N, \alpha)$-group with G/N solvable. Let K be a fixed but arbitrary minimal normal subgroup of G with $K \subseteq N$. Then as in [2]

(*a*) K is an elementary abelian p-group for some prime p, and $[\alpha_K] = [1]$;

(*b*) if $\delta \in \mathrm{Proj}\,(K, \alpha_K)$, then $I_G(\delta)$ is solvable from Proposition 3;

(*c*) $C_G(K) \leqslant I_G(\delta)$ for all $\delta \in \mathrm{Proj}\,(K, \alpha_K)$, since either K contains no non-trivial α-regular elements or is generated by α-regular elements of G;

(*d*) $C_G(K)$ is the maximal solvable normal subgroup of G;

(*e*) if R is a non-solvable subnormal subgroup of G with $C = C_N(K) < R \leqslant N$, then R/C has a proper solvable subgroup of p-power index;

(*f*) $F = F(N) = O_p(N)$.

Now we are essentially at the same point that Liebler and Yellen reached in (2·6) of [7], and their proof of (2·7) of [7] finishes the proof of our theorem.

To explain why $GL(3,2)$ is excluded in the statement of the theorem we first note

https://doi.org/10.1017/S0305004100078002 Published online by Cambridge University Press

that there is a gap in the proof of (2·7) of [7], as detailed on p. 555 of [3]. However we cannot proceed with the technique used by the author in [1] and [2] as shown below. Let F be as above. Then we can obtain a subgroup H of G with $G = FH$, and characters χ of Proj (F, α_F), and ϕ of Proj (L, α_L), where $L = H \cap F$, which correspond under the bijection of (3·7) of [3]. Let $V = I_G(\chi)$; then $U = I_H(\phi) = V \cap H$. Now χ extends to $\pi \in \text{Proj}\,(V, \alpha_V \beta^{-1})$ as described in Proposition 3 and similarly ϕ extends to $\kappa \in \text{Proj}\,(U, \alpha_U \beta^{-1})$. Given $\sigma \in \mathscr{G}$ with $\chi^\sigma = \chi$ (and consequently $\phi^\sigma = \phi$), we have $\pi^\sigma = \lambda_\sigma \pi$ and $\kappa^\sigma = \mu_\sigma \kappa$, for linear characters $\lambda_\sigma, \mu_\sigma$ of $V/F \cong U/L$. The problem basically is that there is no way to guarantee that $\mu_\sigma = \lambda_\sigma$, and this is the only point at which the argument to show that H is solvable breaks down. It is worth recording that the proof of (3·8)(*c*) of [3] given in [5] relies on the use of character triple isomorphisms; the characters of such triples do not behave similarly under Galois conjugacy with respect to a fixed Galois group.

We conclude with perhaps the most important special case of the results in this paper.

COROLLARY 2. *Suppose G is a $(\mathscr{G}, G, \alpha)$-group and G does not involve $GL(3, 2)$. Then G is solvable.*

REFERENCES

[1] R. J. HIGGS. Groups with two projective characters. *Math. Proc. Cambridge Philos. Soc.* **103** (1987), 5–14.

[2] R. J. HIGGS. Groups which act transitively on the projective characters of a normal subgroup. *Math. Proc. Cambridge Philos. Soc.* **104** (1988), 429–434.

[3] R. B. HOWLETT and I. M. ISAACS. On groups of central type. *Math. Z.* **179** (1982), 555–569.

[4] I. M. ISAACS. *Character theory of finite groups*. Monographs and Textbooks in Pure and Applied Math. no. 69 (Academic Press, 1976).

[5] I. M. ISAACS. Character correspondences in solvable groups. *Advances in Math.* **43** (1982), 284–306.

[6] G. KARPILOVSKY. *Projective representations of finite groups*. Monographs and Textbooks in Pure and Applied Math. no. 94 (Marcel Dekker, 1985).

[7] R. A. LIEBLER and J. E. YELLEN. In search of nonsolvable groups of central type. *Pacific J. Math.* **82** (1979), 485–492.

https://doi.org/10.1017/S0305004100078002 Published online by Cambridge University Press