$$\geqslant \frac{3vr^3}{\prod_{i=1}^4 s_i}\left[2\times 4\sqrt[4]{\left(\prod_{i=1}^4 s_i\right)^3}+12\sqrt[12]{\left(\prod_{i=1}^4 s_i\right)^9}\right]+r^4$$

$$=\frac{r^4\sum_{i=1}^4 s_i}{\prod_{i=1}^4 s_i}\times 20\sqrt[4]{\left(\prod_{i=1}^4 s_i\right)^3}+r^4$$

$$\geqslant \frac{4r^4\sqrt[4]{\prod_{i=1}^4 s_i}}{\prod_{i=1}^4 s_i}\times 20\sqrt[4]{\left(\prod_{i=1}^4 s_i\right)^3}+r^4$$

$$=(3r)^4$$

so $IA_1 \times IA_2 \times IA_3 \times IA_4 \geqslant (3r)^4$ is proved.

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109.15 Sampling from the hypergeometric probability distribution

The goal of this paper is to provide an intuitive probabilistic argument for the following fact, taken from Chapter 3 of [1]:

Show that the hypergeometric probability function approaches the binomial in the limit as $\frac{r}{N} \to p$ as $N \to \infty$. That is, show that

$$\lim_{N\to\infty}\frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}}=\binom{n}{y}p^y(1-p)^{n-y}$$

for $\frac{r}{N} \to p$.

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The chief importance of this result arises in statistics, where the hypergeometric distribution appears in sampling from a population without replacement. Then the limit theorem stated above amounts to the assertion that for a sufficiently large popuation it makes no significant difference whether we sample with, or without, replacement. The binomial distribution arises from sampling with replacement, of course. This is convenient for applied statistics, where some experiments may not allow replacement; (i.e. when the act of sampling alters or destroys the object being sampled). And the binomial distribution is much more easily handled than the hypergeometric distribution.

The proof we provide essentially involves sampling from the hypergeometric probability distribution in two different ways. First we briefly review the properties of the hypergeometric random variable.

Suppose that an urn contains a total of N marbles, r of which are red and the remaining $N - r$ marbles are black. We select n marbles at random, and the random variable of interest is Y, the number of red marbles in the sample (of size n). This random variable has what is known as the *hypergeometric probability distribution*.

There are a total of $\binom{N}{n}$ subsets of marbles that we may select, and exactly $\binom{r}{y} \times \binom{N-r}{n-y}$ of these subsets will contain y red marbles, and therefore $n - y$ black marbles. This correspondence with equally likely subsets shows that the probability function $p(y) = P(Y = y)$ is given by the following formula:

$$p(y) = \frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}}, \tag{1}$$

where y is an integer 0, 1, 2, … , n, subject to the restrictions $y \leqslant r$ and $n - y \leqslant N - r$.

Now suppose that we select the marbles *one at a time* instead of altogether as a subset of size n. If we draw out the n marbles one at a time, what is the probability of selecting exactly y red and $n - y$ black marbles?

Solution: We first suppose that we draw out y red marbles consecutively, followed by $n - y$ drawings of only black marbles. The probability of selecting this one particular sequence of marbles is given by

$$\left\{\frac{r}{N}\cdot\frac{r-1}{N-1}\cdot\frac{r-2}{N-2}\cdots\frac{r-(y-1)}{N-(y-1)}\right\}\cdot\left\{\frac{N-r}{N-y}\cdot\frac{N-r-1}{N-y-1}\cdots\frac{N-r-(n-y-1)}{N-(n-1)}\right\}. \tag{2}$$

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Now there are exactly $\binom{n}{y}$ sequences of y red and $n - y$ black marbles, and the probability of choosing any such sequence is also given by (2). Selecting any other sequence of y red and $n - y$ black marbles just reorders the numerators of the factors that appear in (2), so each sequence is therefore equally likely. Thus, summing over all $\binom{n}{y}$ sequences, the total probability of selecting y red and $n - y$ black marbles is given by the following expression:

$$\binom{n}{y}\cdot\left\{\frac{r}{N}\cdot\frac{r-1}{N-1}\cdot\frac{r-2}{N-2}\cdot\cdots\frac{r-(y-1)}{N-(y-1)}\right\}\cdot\left\{\frac{N-r}{N-y}\cdot\frac{N-r-1}{N-y-1}\cdots\frac{N-r-(n-y-1)}{N-(n-1)}\right\}. \quad (3)$$

Now the fact that the expression in (1) equals that in (3) is immediate, because choosing a subset uniformly at random from all subsets of size n, is the same as choosing a subset of size n by a sequence of consecutive removals made uniformly at random. To see this, imagine the N balls placed randomly in a circle, and then select n of them as a sequence of neighbours starting from a random point on the circle. By uniformity, or by symmetry, that selects any one of the subsets of size n with equal probability. The two methods of selection are thus equivalent, and we conclude that both sampling procedures result in the same probability $p(y)$.

We now assume that $\frac{r}{N} \to p$ as $N \to \infty$. But it is now very simple to evaluate $N \to \infty$ in (3) where $\frac{r}{N} \to p$, and we obtain

$$\lim_{N\to\infty} p(y) = \lim_{N\to\infty}\binom{n}{y}\left(\frac{r}{N}\right)^{y}\left(1 - \frac{r}{N}\right)^{n-y} = \binom{n}{y}p^{y}(1-p)^{n-y}.$$

We therefore conclude from (1) that

$$\lim_{N\to\infty}\frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}} = \binom{n}{y}p^{y}(1-p)^{n-y},$$

where $\frac{n}{N} \to p$ as $N \to \infty$. This completes the proof.

We also mention that there are many other far more difficult combinatorial identitities that may be proved by doing the same probability experiment in several different ways; it is a standard method.

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