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A deformation approach to the BFK formula

Romain Speciel

Abstract. Understanding how spectral quantities localize on manifolds is a central theme in geometric spectral theory and index theory. Within this framework, the BFK formula, obtained by Burghelea, Friedlander, and Kappeler in 1992, describes how the zeta-regularized determinant of an elliptic operator decomposes as the underlying manifold is cut into pieces. In this article, we present a novel proof of this result. Inspired by the work of Brüning and Lesch on the eta invariant of Dirac operators, we derive the BFK formula by interpolating continuously between boundary conditions and understanding the variation of the determinant along this deformation.

1 Introduction

The zeta-regularized determinant of the Laplacian, introduced in [14], is a global spectral invariant of smooth Riemannian manifolds with important interpretations and applications in geometry, topology, and mathematical physics (see Chapter 5 of [15] for an overview and [13] for an elegant application). Explicitly computing this invariant is typically impossible, but a useful tool to understand its nature is the Mayer–Vietoris type gluing formula proven by Burghelea, Friedlander, and Kappeler in [3], commonly called the *BFK formula*. The BFK formula, which applies to a broad class of elliptic boundary problems on bundles, relates the determinant of an elliptic differential operator across different complementary boundary conditions via the determinant of an intermediate lower-order pseudodifferential operator on the boundary.

This formula is part of a broader body of work which seeks to understand how various global spectral quantities localize on manifolds. Such results have been successfully proved through various approaches; we provide several examples chronologically for context. In [3], where the original proof of the BFK formula is presented, the authors perturb the interior operator by adding to it a multiple of the identity, then explicitly compute the corresponding variation of the determinant in terms of the perturbation parameter. In [12], Müller studies how the eta invariant, another natural spectral quantity which arises in index theory, varies on a manifold with cylindrical boundary as the boundary gets pushed to infinity. In [11], Mazzeo and Melrose compute how the eta invariant varies under analytic surgery, in which a submanifold is pushed to infinity by taking degenerating metrics which limit to a

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b-metric. These ideas are extended by Hassell, Mazzeo, and Melrose in [6], and applied to handle analytic torsion, a spectral invariant with connections to the combinatorial structure of the underlying manifold. In [2], Bunke studies the adiabatic limit of the eta invariant of the Dirac operator associated with a pair of APS-type boundary conditions. In [4], Cappell, Lee, and Miller develop a splicing technique which is applied to study localization of low eigensolutions of a self-adjoint elliptic operator. And in [1], Brüning and Lesch study the Dirac operator on a manifold with boundary, continuously deform the domain of the operator by varying the boundary conditions, and compute how the eta invariant of the Dirac operator changes through this deformation.

In this article, we present a new proof of the BFK formula, inspired mainly by the arguments in [1]. We develop this deformation approach with the aim of obtaining a proof method which will more readily generalize to settings where we cut along submanifolds with geometric singularities such as corners. For clarity, we state the main result here but delay precise definitions to the next section.

Let A be an admissible elliptic differential operator of positive order on a compact manifold M with nonempty boundary N. Fix B_0 and B_1 two complementary boundary conditions, and take Q to be the induced correspondence operator on the boundary. Then, with A_i denoting the operator A confined to the domain $\ker B_i$, we have the following theorem.

Theorem 1.1 (The BFK Formula [3]) *The zeta-regularized determinants of A_0 and A_1 are related by*

$$\frac{\det A_1}{\det A_0} = c \cdot \det Q, \tag{1.1}$$

where c is a local quantity which can be computed in terms of the symbols of A, B_0, and B_1 along N.

The proof of Theorem 1.1 presented here proceeds by explicitly considering how the determinant varies along a family of operators A_t interpolating between A_0 and A_1, rather than by perturbing the interior operator directly, as originally done in [3].

Remark 1.2 To provide context to this approach, we describe here in more detail the strategy employed by Brüning and Lesch in [1]. The authors obtain a gluing law for the eta invariant associated with a Dirac operator on a manifold with boundary by rotating through a family of boundary conditions. More specifically, they consider a Hermitian vector bundle E over a compact Riemannian manifold M, and a first-order symmetric elliptic differential operator D acting on smooth sections of the bundle (the main example of which are Dirac operators). With $N \subset M$ a compact hypersurface, they note that D restricted to $C^\infty(E|_{M\setminus N})$ is no longer essentially self-adjoint, and consider therefore two natural boundary conditions: the APS boundary condition and the continuous transmission boundary condition. The authors then explicitly parametrize a path between these boundary conditions and show that the variation of the eta invariant across this path can be computed by studying the operator D, conjugated by appropriate rotations which map between the domains corresponding to each of the boundary conditions. (Details are outlined in Section 3 of [1].)

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Our proof carries this idea over to the zeta-regularized determinant. The main difference is that there is no longer a natural set of rotations carrying the domain corresponding to one boundary condition to that corresponding to the other boundary condition. Instead, we work directly with the resolvent, and obtain variational formulas by studying how this operator changes as we interpolate between B_0 and B_1.

In Section 2, we provide rigorous definitions and assumptions for our theorem, and place the BFK formula in context by describing as an example the case of the Laplacian on a compact Riemannian manifold. In Section 3, we prove several relationships linking the Poisson extension operator P, the operator Q which serves as an analog of the Dirichlet-to-Neumann map, and the resolvent R. In particular, we establish in Proposition 3.3 a surprising integral identity which later serves as the key for our proof. Finally, in Section 4, we use properties of the trace functional and identities from the meromorphic functional calculus to deduce a link between the zeta functions of A_0, A_1 and the trace of the operator Q. The main technical challenges arise here in the subtlety of trace class operators, and this is handled by introducing high powers of regularizing operators and then applying unique continuation to deduce results globally. Finally, we use these identities to deduce the BFK formula.

2 Setup

We now describe in detail our setting, which is analogous to that of [3]. Let M be a compact manifold of dimension $d \geq 2$ with nonempty boundary $N = \partial M$, and take $E \to M$ to be a smooth vector bundle over M. Suppose $A\colon C^\infty(E) \to C^\infty(E)$ is an elliptic differential operator of order $\omega > 0$. Let $F_j \to N$, $1 \leq j \leq k$, be smooth vector bundles over the boundary, and consider for each j a pair of boundary differential operators $B_i^j\colon C^\infty(E) \to C^\infty(F_j)$, for $i = 0, 1$, such that $\operatorname{ord} B_1^j - \operatorname{ord} B_0^j$ is positive and constant in j, and $\operatorname{ord} B_i^j < \omega$. Writing

$$B_t^j = (1-t)B_0^j + tB_1^j \quad \text{and} \quad B_t = (B_t^1, \dots, B_t^k), \tag{2.1}$$

we require that B_0 and B_1 be *complementary*, meaning that the intersection of their null spaces consists of sections of $C^\infty(E)$ which vanish up to order $\omega - 1$ on N.

Next, we assume that (A, B_t) is an invertible elliptic boundary problem in the sense of [7] for $0 \leq t \leq 1$ and hence extends to a bijective continuous operator

$$(A, B_t)\colon H^\omega(E) \to L^2(E) \oplus H^{\omega-\omega_1(t)-1/2}(F_1) \oplus \cdots \oplus H^{\omega-\omega_k(t)-1/2}(F_k), \tag{2.2}$$

where $\omega_j(t) = \operatorname{ord} B_t^j$. Importantly, note that $\omega_j(t)$ is discontinuous at $t = 0$.

Example 2.1 (The Laplacian on a manifold with boundary) The following setup is our main motivating example. Suppose M is equipped with a Riemannian metric g and denote by Δ_g the corresponding (positive) Laplacian acting on $C^\infty(M)$. Define the Dirichlet and Neumann boundary operators, denoted $\mathcal{D}$ and $\mathcal{N}$, respectively, by

$$\mathcal{D}u = u|_{\partial M} \quad \text{and} \quad \mathcal{N}u = \left.\frac{\partial u}{\partial n}\right|_{\partial M}, \tag{2.3}$$

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where n is the outward-pointing unit normal at the boundary. With E and F_1 the trivial line bundles over their respective spaces (i.e., considering the Laplacian acting simply on real-valued functions), set $B_0 = \mathcal{D}$ and $B_1 = \mathcal{N}$. Finally, to ensure invertibility, set $A = \Delta_g + \varepsilon$ for some positive ε. The discussed assumptions are then clearly satisfied.

Let A_t denote the operator which acts as A on its domain $\mathrm{dom}(A_t) = \ker(B_t) \subset H^\omega(E)$, and define the corresponding *resolvent operator* by

$$\begin{aligned} R_t(z)\colon L^2(E) &\to H^\omega(E) \\ u &\mapsto (A - z, B_t)^{-1}(u, 0) \end{aligned} \tag{2.4}$$

and *Poisson operator* by

$$\begin{aligned} P_t(z)\colon H^{\omega-\omega_1(t)-1/2}(F_1) \oplus \cdots \oplus H^{\omega-\omega_k(t)-1/2}(F_k) &\to H^\omega(E) \\ f = (f_1, \ldots, f_k) &\mapsto (A - z, B_t)^{-1}(0, f). \end{aligned} \tag{2.5}$$

Both of these families of operators depend meromorphically on z as a consequence of the analytic Fredholm theorem.

Away from the spectrum of A_0, the triple $(A - z, B_0, B_1)$ gives rise to a natural map between sections of $\oplus_j F_j$ which correspond to each other as the image under either B_0 or B_1 of a solution of $(A - z)u = 0$. This map is given by the operator $Q(z) = B_1 P_0(z)$, which we assume to be positive and self-adjoint when z is in the ray $\mathbb{R}_- = \{re^{i\pi} r \geq 0\}$. We shall write $Q = Q(0)$.

Example 2.2 (The Dirichlet-to-Neumann map) We continue with Example 2.1. In that setting, the problem $(A = \Delta_g + \varepsilon, B_t = (1-t)\mathcal{D} + t\mathcal{N})$ is well known to be an elliptic boundary problem (see [7] for an extensive discussion, for example). The operator $Q(z)$ described above corresponds to the standard Dirichlet-to-Neumann map on N with respect to interior extension operator $\Delta_g + \varepsilon - z$. In this setting, it is clear to see why the restriction $z \notin \mathrm{spec}\, A_0$ is necessary: $P_0(z)$ fails to be well defined precisely when the problem $(\Delta_g + \varepsilon - z)u = 0$ on M, $u = f$ on N does not have a unique solution. Indeed, if $u, \tilde{u}$ are two distinct solutions to the problem, observe $u - \tilde{u}$ is a z-eigenfunction of A_0 (and, in the other direction, the existence of such an eigenfunction obstructs uniqueness of extensions). Next, we confirm that $Q(z)$ is positive and self-adjoint on $\mathbb{R}_-$: by Green's identity,

$$\left\langle Q(z)f, f \right\rangle_{L^2(N)} = \left\| \nabla P_0(z) f \right\|^2_{L^2(M)} + (\varepsilon - z) \left\| P_0(z) f \right\|^2_{L^2(M)} > 0 \tag{2.6}$$

since $z < 0$, and similarly

$$\begin{aligned} \left\langle f, Q(z)g \right\rangle_{L^2(N)} - \left\langle Q(z)f, g \right\rangle_{L^2(N)} &= \left\langle B_0 P_0(z) f, B_1 P_0(z) g \right\rangle_{L^2(N)} \\ &\quad - \left\langle B_1 P_0(z) f, B_0 P_0(z) g \right\rangle_{L^2(N)} \\ &= (z - \bar{z}) \left\langle P_0(z) f, P_0(z) g \right\rangle_{L^2(M)}, \end{aligned} \tag{2.7}$$

which vanishes when $z \in \mathbb{R}$.

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Remark 2.3 Note that the computations presented above hold more generally when A is a symmetric divergence-form elliptic operator, thus the assumption that $Q(z)$ is self-adjoint on $\mathbb{R}_-$ is automatically satisfied in those cases. In general, however, this assumption will be necessary for Proposition 3.3 to hold.

In order to ensure that our operators have well-defined zeta-regularized determinants, we shall require that they satisfy additional spectral properties which guarantee certain path integrals in $\mathbb{C}$ to be well defined and convergent. This is captured by the following definition.

Definition 2.1 For B any boundary differential operator such that (A, B) is an elliptic boundary problem, the angle θ is a *principal angle* for (A, B) if the following conditions are satisfied:

(1) *(Agmon angle condition)* For every $p \in M$ and $\xi \in T_p^*(M)\backslash 0$, we have

$$\operatorname{spec} \sigma_\omega(A)(p, \xi) \cap \{re^{i\theta} r \geq 0\} = \varnothing, \tag{2.8}$$

where $\sigma_\omega(A)$ denotes the principal symbol of A.

(2) *(Shapiro–Lopatinski condition)* In a collar neighborhood $N \times [0, \infty)$ of the boundary, let (x, y) denote the induced coordinates and (η, τ) the corresponding coordinates on the cotangent bundle. Write $\alpha = (\alpha', \alpha_d)$, a multi-index of length d, and say $A = \sum_\alpha a_\alpha(p) D^\alpha$ in coordinates. Then, given $r \geq 0$ and $f_j \in F_j$ for $1 \leq j \leq k$, there exists a unique solution $u(x, y)$ to

$$\sum_{|\alpha|=\omega} a_\alpha(x, 0)(\eta^{\alpha'}) D_y^{\alpha_d} u(x, y) = re^{i\theta} u(x, y) \tag{2.9}$$

satisfying both $\lim_{y\to\infty} u(x, y) = 0$ and

$$\sum_{l=\omega-\omega_j}^{\omega} \sigma_{\omega_j}(B_j)(x, \eta) D_y^{\omega-l} u(x, 0) = f_j \tag{2.10}$$

for every j.

We assume that π is a principal angle for (A, B_t) for every $0 \leq t \leq 1$, which allows us to define complex powers of A_t as in the Seeley calculus. To this end, let γ be a contour in the complex plane around the ray $\mathbb{R}_-$ traveling clockwise around the origin, so that it positively winds around every point of the spectrum (see Figure 1). Taking the branch cut of z^{-s} along $\mathbb{R}_-$, define

$$A_t^{-s} = \frac{i}{2\pi} \int_\gamma z^{-s} R_t(z)\, dz. \tag{2.11}$$

When $\operatorname{Re}(s)$ is sufficiently large, this operator is well defined and trace class. Its trace, denoted by $\zeta_t(s)$, extends meromorphically to the whole complex plane and is regular at $s = 0$ (we refer the reader to [18, 19] for details). Finally, we may now define

$$\det A_t = e^{-\zeta_t'(0)}, \tag{2.12}$$

the *zeta-regularized determinant* of A_t.

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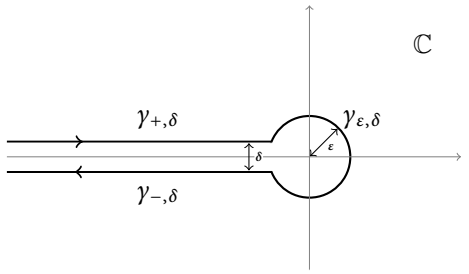


Figure 1: The contour γ used in the proof of Theorem 1.1.

Example 2.4 (The determinant of the Laplacian) We continue with Examples 2.1 and 2.2. Since A_t, which is simply $\Delta_g + \varepsilon$ restricted to the Robin-boundary condition domain $\ker((1-t)\mathcal{D} + t\mathcal{N})$, is a positive self-adjoint operator and its principal symbol is $|\xi|^2$, conditions 1 and 2 in the definition of principal angle are satisfied with $\theta = \pi$. We may then define A_t^{-s}, and the Weyl law tells us that this will be a bounded (and in fact trace class) operator when $\mathrm{Re}(s) > d/2$. In fact, denoting the eigenvalues of A_t by $\lambda_1, \lambda_2, \ldots$, we may rewrite

$$\mathrm{Tr}(A_t^{-s}) = \sum_j \lambda_j^{-s} \tag{2.13}$$

to then note the connection to the standard zeta function. We may then seek to take $\varepsilon \to 0$ and regularize the limit in order to get a well-defined determinant of the genuine Laplacian, rather than its shift A. This works, and is described in the proof of Theorem B$*$ in [3]. See also [9] for further specialization to Laplace-type operators.

Next, we seek to extend our definition of zeta-regularized determinant to matrices of operators since $Q(z)$, which appears in the right-hand side of Equation (1.1), is a $k \times k$ matrix of ΨDOs. The necessary properties are described in the following definition.

Definition 2.2 We say the operator $Q = (Q_{ij}) : \oplus_{j=1}^k C^\infty(F_j) \to \oplus_{j=1}^k C^\infty(F_j)$ is *of order* $(\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_k)$ if $\mathrm{ord}\, Q_{ij} = \alpha_i + \beta_j$. Define then the *principal symbol* of Q to be the matrix $\sigma(Q) = \left(\sigma_{\alpha_i+\beta_j}(Q_{ij})\right)$, and say Q is *elliptic in the sense of Agmon, Douglis, and Nirenberg* if for any $p \in M$ and $\xi \in T_p^*(M)\backslash 0$, the map $\sigma(Q)(p,\xi)$ is invertible.

We shall show in Proposition 3.2 that $Q(z)$ is indeed elliptic in the sense of Agmon, Douglis, and Nirenberg when $z \notin \mathrm{spec}\, A_0$. With minor modification, the notion of a principal angle carries over to such operators, and we assume that π is also a principal angle for $Q(z)$, hence, the determinant of $Q(z)$ is well defined (again only when $z \notin \mathrm{spec}\, A_0$). The details of these definitions can be found in Section 2 of [3].

Remark 2.5 For simplicity, we have taken π to be the principal angle for all involved operators. In particular, this is the case for the discussed example of the Laplacian on

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a manifold, as the Dirichlet-to-Neumann operator is positive. However, all proofs presented below can be modified to the general case without too much complication: one must ensure that the path γ used in the Seeley calculus is adapted to the principal angle, and that the dependence of the determinants on the principal angle be made explicit. The proof in [3], for example, accounts for these details. Our proof, however, has an additional assumption: we make use of the fact that $Q(z)$ is self-adjoint when $z \in \mathbb{R}_-$ in Proposition 3.3. To adapt this condition to a general principal angle, one may ask that $Q(z)$ be self-adjoint when z lies on the ray corresponding to the angle.

Remark 2.6 (Decomposing the determinant along a cut) The original statement of the BFK formula in [3] differs slightly from our formulation in Theorem 1.1; we outline in this remark how the traditional perspective may be recovered from our approach in the case of the Laplacian.

Let M be a smooth Riemannian manifold *without* boundary and take N an oriented submanifold of codimension 1, which we assume for simplicity to have trivial normal bundle. Denote by M_{cut} the compact manifold obtained by cutting M along N, with boundary $N^- \sqcup N^+$. Define the sum and difference operators by, respectively,

$$\begin{aligned} &\sigma, \delta \colon C^\infty(N^-) \oplus C^\infty(N^+) \to C^\infty(N), \\ &\sigma(f,g) = f+g, \quad \delta(f,g) = f-g. \end{aligned} \tag{2.14}$$

Now, consider the two complementary boundary conditions $B_0 = \mathcal{D}$ and $B_1 = \mathcal{N}$ on M_{cut} as before, and denote by $B_i^{\pm}$, $i = 0, 1$, their composition with the restriction map to $N^{\pm}$. We introduce for $\tau \in \mathbb{R}$ the twisted boundary operator

$$\begin{aligned} T_\tau \colon C^\infty(M_{\mathrm{cut}}) &\to C^\infty(N^-) \oplus C^\infty(N^+) \cong C^\infty(N) \oplus C^\infty(N) \\ u &\mapsto \left(\delta(B_0^- u, B_0^+ u) + \tau \cdot \delta(B_1^- u, B_1^+ u), \sigma(B_1^- u, B_1^+ u)\right). \end{aligned} \tag{2.15}$$

The two boundary operators B_0 and T_τ form a complementary pair of boundary conditions when $\tau \neq 0$.

Now, following the previous examples, take $A = \Delta_g + \varepsilon$ for some fixed $\varepsilon > 0$. Applying Theorem 1.1, we obtain

$$\frac{\det(A, T_\tau)}{\det(A, B_0)} = c(\tau) \det Q_\tau, \tag{2.16}$$

where $Q_\tau = T_\tau P_0(0)$. However, all terms in which τ appears are continuous in τ. Indeed, the constant $c(\tau)$ depends on the symbol of Q_τ, which varies continuously, and similarly Q_τ tends continuously to Q_0, the twisted Dirichlet-to-Neumann map corresponding to M_{cut}. (The continuous dependence of the determinant on a parameter τ is discussed in particular in Section 4 of [9].) Furthermore, T_0 corresponds precisely to the transmission condition across N, and by elliptic regularity, we thus have

$$\det(A, T_0) = \det(A) \tag{2.17}$$

(where the first determinant is understood on M_{cut}, and the second on M). We therefore recover from Equation (2.16) the main result of [3].

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Remark 2.7 Theorem 1.1 is a generalization of the main result in [10], which considers explicitly the setting described in our series of examples. Furthermore, the proofs in both [3, 10] proceed by obtaining explicit expressions of the variation $\frac{\partial}{\partial t}\left(\log\det(A+t,B_1)-\log\det(A+t,B_0)\right)$, using the fact that for specific families of operators $X(t)$, such as $(A+t,B_0)$, $(A+t,B_1)$, and $Q(t)$, the property

$$\frac{\partial}{\partial t}\log\det(X(t)) = \operatorname{Tr}(X'(t)X^{-1}(t)) \tag{2.18}$$

can be regularized to avoid problems with the trace class, and can be combined with properties of the trace to obtain Equation (1.1). The details in the case of the Laplacian are outlined in an approachable fashion in [9]. We will instead keep A unchanged on the interior and vary the boundary condition continuously from B_0 to B_1.

3 Properties of the operators *P*, *Q*, and *R*

The Poisson extension operator P, the boundary correspondence operator Q, and the resolvent R are closely tied to each other. In this section, we establish several fundamental properties which will later be needed to complete the proof of Theorem 1.1. To begin, we study the variations of our operators.

Proposition 3.1 *With $B' = B_1 - B_0$, we have*

$$\frac{\partial}{\partial t}R_t(z) = -P_t(z)B'R_t(z) \quad \text{and} \quad \frac{\partial}{\partial z}P_t(z) = R_t(z)P_t(z). \tag{3.1}$$

Proof These are straightforward computations. For the first statement, fix $u \in C^\infty(E)$, $0 < t < 1$, and $z \notin \operatorname{spec}(A_t)$, and define

$$R_{t+\varepsilon}(z)u =: v(\varepsilon) = v_0 + \varepsilon\cdot v_1 + O(\varepsilon^2). \tag{3.2}$$

By definition, $B_{t+\varepsilon}v(\varepsilon) = 0$ and $(A-z)v(\varepsilon) = u$. Differentiating both of these conditions at $\varepsilon = 0$ yields $B_t v_1 = -B'v_0$ and $(A-z)v_1 = 0$, from which we conclude

$$v_1 = -P_t(z)B'v_0, \tag{3.3}$$

and the result follows since $v_0 = R_t(z)u$.

The second part of the claim follows by a similar argument. This time, fix $f \in C^\infty(F_1)\oplus\cdots\oplus C^\infty(F_k)$, $0 \le t \le 1$, and $z \notin \operatorname{spec}(A_t)$, and define

$$P_t(z+h)f =: v(h) = v_0 + h\cdot v_1 + O(|h|^2). \tag{3.4}$$

By definition, $B_t v(h) = f$ and $(A-z-h)v(h) = 0$. Differentiating both of these conditions at $h = 0$ yields $B_t v_1 = 0$ and $(A-z)v_1 = v_0$, from which we conclude

$$v_1 = R_t(z)v_0, \tag{3.5}$$

and the result follows since $v_0 = P_t(z)f$. ∎

Next, recall that in order to define the determinant of our generalized Dirichlet-to-Neumann operator

$$Q(z)\colon C^\infty(F_1)\oplus\cdots\oplus C^\infty(F_k) \to C^\infty(F_1)\oplus\cdots\oplus C^\infty(F_k), \tag{3.6}$$

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we must first check that this system of ΨDOs satisfies an adapted ellipticity condition, as explained in the introduction. The following proposition confirms this.

Proposition 3.2 *The operator $Q(z)$ is of order*

$$(-\operatorname{ord} B_0^1, \dots, -\operatorname{ord} B_0^k; \operatorname{ord} B_1^1, \dots, \operatorname{ord} B_1^k) \tag{3.7}$$

and is elliptic in the sense of Agmon, Douglis, and Nirenberg.

Proof We show the result for $Q = Q(0)$, since A may be replaced by $A - z$ away from the spectrum. Identifying $N \times [0,1)$ with a neighborhood U of $N \subset M$, we induce coordinates (x, y), with $x \in N$ and $y \in [0,1)$, on U. With $D_y = -i\frac{\partial}{\partial y}$, we may write A uniquely as

$$A = \sum_0^\omega A_j D_y^j, \tag{3.8}$$

where each A_j is a differential operator of order $\omega - j$ in x, which may be viewed as an operator on $C^\infty(E|_N)$. Let $S: \oplus_\omega C^\infty(E|_N) \to C^\infty(E)$ be given by

$$S(u_1, \dots, u_\omega) = A^{-1}(u_1 \otimes \delta, \dots, u_\omega \otimes \delta^{(\omega)}), \tag{3.9}$$

where $\delta^{(j)} = D_y^j \delta$ with δ the Dirac distribution in y, and $\cdot \otimes \delta^{(j)}: C^\infty(E|_N) \to \mathscr{D}'(E)$ maps smooth sections u of E over N to $(\cdot \otimes \delta^{(j)})(u)(x,y) = u(x)\delta^{(j)}(y)$. (Note that A^{-1} here may be defined by embedding M into a closed manifold and extending A, for example.) From [5, Theorem 2.4.i], S is well defined and its range consists of sections which are smooth up to the boundary. Define then the operator $T: \oplus_\omega C^\infty(E|_N) \to \oplus_{j=1}^k C^\infty(F_j)$ by

$$T = B_0 S \begin{pmatrix} A_1 & & \cdots & & A_\omega \\ A_2 & & \cdots & A_\omega & 0 \\ \vdots & & & & \vdots \\ A_\omega & 0 & \cdots & & 0 \end{pmatrix}. \tag{3.10}$$

Invertibility of the principal symbol of T is in fact equivalent to the ellipticity of the boundary value problem (A, B_0), following the Boutet de Monvel calculus.

Now, let P be a parametrix for T and define $Q' = B_1 S P$, which is elliptic in the sense of Agmon, Douglis, and Nirenberg. Observe $Q - Q'$ is smoothing, hence, Q is as claimed in the statement of the proposition, and its order is as claimed from the above construction. ∎

Crucial to Section 4 is the following proposition, which defines and then relates $\log Q(z)$ to the integral of the Poisson operators.

Proposition 3.3 *There exists a neighborhood U of the ray $\mathbb{R}_-$ on which $\log Q(z)$ is well defined and holomorphic. Furthermore, when $z \in U$,*

$$\int_0^1 B' P_t(z)\, dt = \log Q(z). \tag{3.11}$$

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Proof As a consequence of Proposition 3.2 and the assumption that $Q(z)$ is positive and self-adjoint when $z \in \mathbb{R}_-$, it follows from the Spectral Theorem that the operator $Q(z)$ has an orthonormal basis of eigenvectors with positive eigenvalues when $z \in \mathbb{R}_-$. We may therefore define $\log Q(z)$ by the functional calculus, with the standard branch of the logarithm. Perturbation theory of holomorphic families of operators then ensures that such a basis exists for all $z \in U$, with U some neighborhood of $\mathbb{R}_-$ (see page 368 of [8]).

Now, both sides of (3.11) are holomorphic families of operators on U, hence, it suffices to show equality on $\mathbb{R}_-$ and then apply the identity theorem to deduce the result. Fix therefore some $z \in \mathbb{R}_-$, and suppose f is a λ-eigenfunction of $Q(z)$ with $\lambda > 0$. Set $u = P_0(z)f$. Since

$$B_t u = (1-t)B_0 u + tB_1 u = (1 + t(\lambda - 1))f, \tag{3.12}$$

it follows that

$$P_t(z)f = \frac{1}{(1 + t(\lambda - 1))} \cdot u, \tag{3.13}$$

and so

$$B' P_t(z) f = \frac{\lambda - 1}{(1 + t(\lambda - 1))} f. \tag{3.14}$$

Finally, integrating in t yields

$$\left(\int_0^1 B' P_t(z)\, dt \right) f = \left(\int_0^1 \frac{\lambda - 1}{(1 + t(\lambda - 1))}\, dt \right) f = \log(\lambda) f. \tag{3.15}$$

Conclude that Equation (3.11) holds for every eigenfunction, and thus holds in general, concluding the proof. ∎

4 The BFK formula

We may now turn our attention to the proof of our main theorem. We begin by computing the difference of the zeta functions corresponding to A_0 and A_1 by interpolating between our boundary conditions.

Proposition 4.1 *For* $\mathrm{Re}(s)$ *sufficiently large,*

$$\zeta_1(s) - \zeta_0(s) = \frac{s}{2\pi i} \int_\gamma z^{-s-1} \log\det Q(z)\, dz. \tag{4.1}$$

Remark 4.2 The proposition above resembles the characterization of *ζ-comparable operators* introduced in Section 3.3.4 of [16]. However, our setting is different since $R_1(z) - R_0(z)$ need not be trace class, hence, these operators are in fact not ζ-comparable. To see this, let us return to Example 2.1 and study the model case of the upper half-space $\mathbb{R}^n_+$ for $n \geq 3$. With

$$G(x, y, x', y') = \left(|x - x'|^2 + |y - y'|^2 \right)^{2-n} \tag{4.2}$$

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and

$$H(x,y,x',y') = \left(\left|x+x'\right|^2 + \left|y-y'\right|^2\right)^{2-n}, \tag{4.3}$$

the Green's function for the Dirichlet problem is $G - H$, and for the Neumann problem is $G + H$. The difference of these is therefore $2H$, which, when written in projective coordinates with $p = x/x'$ and $q = (y - y')/x$ then restricted to the $q = 0$ diagonal, equals $(x')^{2-n}(1+p)^{2-n}$. The corresponding operator is therefore not trace class, meaning that our setting does not fall into the framework of ζ-comparable operators.

To address the difficulty underscored by the above remark, we achieve regularization by taking high powers of $Q(z)^{-1}$. Since $Q(z)$ is assumed to be of positive order, these powers are eventually trace class.

Proof of Proposition 4.1 It follows from [17] that, since $Q(z)$ is a positive order ΨDO on N, a boundaryless closed manifold, the function $F(z,w) = \operatorname{Tr}\left(Q(z)^{-w}\right)$ is well defined and holomorphic in w when $\operatorname{Re}(w) > \dim N/\operatorname{ord} Q(z)$. Furthermore, since $Q(z)$ depends holomorphically on z, we may extend F meromorphically to $\mathbb{C}^2$. Note in particular that, by definition, $\frac{\partial F}{\partial w}(z,0) = -\log\det Q(z)$ since the determinant is defined precisely in terms of the derivative of this meromorphic extension.

Now, take U as in the proof of Proposition 3.3. Both $\det Q(z)$ and $\log Q(z)$ are well defined and holomorphic on U, and we may take the path γ to lie entirely in U. Since the trace class is an ideal, we have that for $\operatorname{Re}(w)$ sufficiently large

$$\frac{\partial F}{\partial w}(z,w) = -\operatorname{Tr}\left(\log Q(z) Q(z)^{-w}\right) = -\int_0^1 \operatorname{Tr}\left(B' P_t(z) Q(z)^{-w}\right) dt, \tag{4.4}$$

where the second equality follows from Proposition 3.3. Here, we may commute the trace and the integral as $B'P_t(z)Q(z)^{-w}$, $0 \le t \le 1$, is a compact path in the space of trace class operators (again because $Q(z)^{-w}$ is in the trace class).

We then integrate by parts in z and apply Proposition 3.1 to obtain

(4.5)

$$\begin{aligned}
\frac{s}{2\pi i}\int_\gamma z^{-s-1}\frac{\partial F}{\partial w}(z,w)\,dz &= \frac{1}{2\pi i}\int_\gamma \frac{\partial}{\partial z}\left(z^{-s}\right)\cdot\int_0^1 \operatorname{Tr}\left(B' P_t(z) Q(z)^{-w}\right) dt\,dz \\
&= \frac{-1}{2\pi i}\int_\gamma z^{-s}\int_0^1 \operatorname{Tr}\Big(B' R_t(z) P_t(z) Q(z)^{-w} \\
&\qquad + B' P_t(z)\frac{\partial}{\partial z}\left(Q(z)^{-w}\right)\Big)\,dt\,dz \\
&= T_1(s,w) + T_2(s,w)
\end{aligned}$$

when s is sufficiently large so that the terms at infinity tend to zero. Here, $T_1(s,w)$ and $T_2(s,w)$ represent the two terms of this expression. By commutativity of the trace (and once again thanks to the $Q(z)^{-w}$ term ensuring our integrand is trace class),

$$T_1(s,w) = \frac{-1}{2\pi i}\operatorname{Tr}\left(\int_\gamma z^{-s}\int_0^1 P_t(z) Q(z)^{-w} B' R_t(z)\,dt\,dz\right). \tag{4.6}$$

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We note, however, that this expression is regular at $w = 0$, and deduce that for s with sufficiently large real part (and hence for all s by analytic continuation), Proposition 3.1 gives

$$\begin{aligned} T_1(s,0) &= \operatorname{Tr}\left(\frac{1}{2\pi i}\int_\gamma z^{-s}\int_0^1 \frac{\partial}{\partial t}R_t(z)\,dt\,dz\right) \\ &= \operatorname{Tr}\left(\frac{i}{2\pi}\int_\gamma z^{-s}\left(R_0(z)-R_1(z)\right)\,dz\right) \\ &= \operatorname{Tr}\left(A_0^{-s}-A_1^{-s}\right) = \zeta_0(s)-\zeta_1(s). \end{aligned} \tag{4.7}$$

On the other hand, T_2 vanishes at $w = 0$ since $\frac{\partial}{\partial z}\left(Q(z)^0\right) = 0$. We conclude

$$\zeta_1(s)-\zeta_0(s) = \frac{-s}{2\pi i}\int_\gamma z^{-s-1}\frac{\partial F}{\partial w}(z,0)\,dz = \frac{s}{2\pi i}\int_\gamma z^{-s-1}\log\det Q(z)\,dz, \tag{4.8}$$

as desired. ∎

The final step toward our main theorem is to apply the well-known asymptotic expansion of determinants along a ray in the complex plane, proven in the appendix of [3] and stated for $\log\det Q(x)$ below.

Proposition 4.3 *The function* $\log\det Q(x)$ *admits an asymptotic expansion of the form*

$$\log\det Q(x) \sim \sum_{j=-(d-1)}^{\infty}\pi_j|x|^{-j/2} + \sum_{j=0}^{d-1} q_j|x|^{j/2}\log|x| \quad \text{as } x\to-\infty. \tag{4.9}$$

The coefficients π_j *and* q_j *can be evaluated in terms of the symbol of* Q*, and thus depend only on the germs of the symbols of* A, B_0, *and* B_1 *at* N.

We may now prove our theorem.

Proof of Theorem 1.1 We split the integral on the right-hand side of Equation (4.1) into three pieces by splitting the contour $\gamma \subset U$ into the ray $\gamma_{+,\delta}$ a distance $\delta/2$ above the negative real axis, a small loop $\gamma_{\varepsilon,\delta}$ of radius ε around the origin, and the ray $\gamma_{-,\delta}$ a distance $\delta/2$ below the negative real axis, as in Figure 1.

Then, take $\delta \to 0$ to obtain

$$\begin{aligned} \frac{s}{2\pi i}\int_\gamma z^{-s-1}\log\det Q(z)\,dz = \frac{s}{\pi}\sin(\pi s)\int_{-\infty}^{-\varepsilon}(-x)^{-s-1}\log\det Q(x)\,dx \\ + \frac{s}{2\pi i}\int_{\gamma_{\varepsilon,0}} z^{-s-1}\log\det Q(z)\,dz. \end{aligned} \tag{4.10}$$

By Proposition 4.3, we compute, initially for $\operatorname{Re}(s) > (d-1)/2$ but then by analytic continuation to the right half plane $\operatorname{Re}(s) > -(N+1)/2$ for any $N \geq 0$, that

$$\int_{-\infty}^{-\varepsilon}(-x)^{-s-1}\log\det Q(x)\,dx = \sum_{j=-(d-1)}^{N}\frac{\pi_j}{s+j/2} + \sum_{j=0}^{d-1}\frac{q_j}{(s-j/2)^2} + h(s), \tag{4.11}$$

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where $h(s)$ is holomorphic when $\mathrm{Re}(s) > -(N+1)/2$. Therefore,

$$\frac{\partial}{\partial s}\bigg|_{s=0}\left(\frac{s}{\pi}\sin(\pi s)\int_{-\infty}^{-\varepsilon}(-x)^{-s-1}\log\det Q(x)\,dx\right)=\pi_0. \tag{4.12}$$

To address the second term of Equation (4.10), recall that $\det Q(z)$ is holomorphic on U and so, by the residue theorem,

$$\begin{aligned}\frac{\partial}{\partial s}\bigg|_{s=0}\left(\frac{s}{2\pi i}\int_{\gamma_{\varepsilon,0}} z^{-s-1}\log\det Q(z)\,dz\right) &= \frac{1}{2\pi i}\int_{\gamma_{\varepsilon,0}}\frac{1}{z}\log\det Q(z)\,dz\\ &= -\log\det Q.\end{aligned} \tag{4.13}$$

(Note the negative sign above comes from the fact that the loop $\gamma_{\varepsilon,0}$ is traveled clockwise.) Combine these observations to conclude

$$\begin{aligned}\log\det A_1 - \log\det A_0 &= -\frac{\partial}{\partial s}\bigg|_{s=0}\big(\zeta_1(s)-\zeta_0(s)\big)\\ &= -\frac{\partial}{\partial s}\bigg|_{s=0}\left(\frac{s}{2\pi i}\int_{\gamma} z^{-s-1}\log\det Q(z)\,dz\right)\\ &= \log\det Q - \pi_0.\end{aligned} \tag{4.14}$$

The result follows after exponentiating. ∎

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