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RESEARCH ARTICLE

# Shapiro's problem on polynomials with large partial sums of coefficients

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**Abstract**

Given a polynomial $\sum_\nu a_\nu X^\nu$ of degree $< d$, bounded by one on the unit disk, how large can $|a_0 + a_1 + \ldots + a_n|$ $(n < d)$ get? This question dates back at least to the 1952 thesis work of H. S. Shapiro. In 1978, D. J. Newman gave an *exact* answer for $d = 2(n+1)$, but there does not seem to have been further progress on the question since. We study variations on exact answers for some related coefficient sums, and answer the original question in an asymptotic sense, provided that $n$ is 'not too large' in terms of $d$. The latter is achieved via a 'quantitative' Eneström–Kakeya theorem, while the former is based on certain identities for carefully selected Lagrange interpolators. From the interpolation approach we also obtain a general inequality for coefficient sums $|t_0 a_0 + \ldots + t_{d-1} a_{d-1}|$ for *arbitrary* complex numbers $t_0, \ldots, t_{d-1}$. This inequality fails to be sharp in general, yet it is in some cases and also yields non-trivial bounds for Shapiro's problem for some choices of $n$ and $d$.

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## 1. Introduction

For any function $f$ on the complex unit disk $\mathbb{D}$, let $\|f\|_\infty = \sup\{\,|f(z)| : z \in \mathbb{D}\,\}$. In his 1952 PhD thesis, H. S. Shapiro [19, §2.8] studied the following problem (*cf.* also his Master's thesis [18] with the same title):

**Problem 1.1** (Shapiro, 1952). For integers $d > n \geq 0$, determine the maximum $\mathscr{M}_{n,d}$ of

$$|a_0 + a_1 + \ldots + a_n|, \tag{1.1}$$

over all polynomials $f = a_0 + a_1 X + \ldots + a_{d-1}X^{d-1} \in \mathbb{C}[X]$ of degree strictly less than $d$ with $\|f\|_\infty \leq 1$.

Clearly, Problem 1.1 may be viewed as determining the *operator norm* of a certain linear functional (here the underlying space being $\mathbb{C}[X]^{<d}$, the space of polynomials of degree $< d$). This relates the present problem to a vast family of problems in functional analysis and adjacent fields, the scope of which we cannot possibly frame here in a just manner.

More concretely, the motivation for Shapiro's question seems to have been (at least in part) a result of Landau, who had determined the maximum of (1.1) when instead $f$ ranges over *all 1-bounded holomorphic functions* on the unit disk (see [8] or [9, pp. 20–23]):

**Theorem 1.2** (Landau, 1913). *Let $n$ be a non-negative integer. Then, for any holomorphic function $f$ on $\mathbb{D}$ with $\|f\|_\infty \leq 1$ and power series expansion $f(z) = a_0 + a_1 z + a_2 z^2 + \ldots$, one has*

$$|a_0 + a_1 + \ldots + a_n| \leq \sum_{\nu=0}^{n} \binom{-1/2}{\nu}^2 =: \mathscr{L}_n \quad \left(= \frac{\log n}{\pi} + O(1)\right), \tag{1.2}$$

*where*

$$\binom{-1/2}{\nu} = \prod_{\kappa=0}^{\nu-1} \frac{-1/2-\kappa}{\nu-\kappa} = (-1)^\nu \frac{1\cdot 3\cdot \ldots \cdot (2\nu-1)}{2\cdot 4\cdot \ldots \cdot 2\nu}. \tag{1.3}$$

*Moreover, '$\leq$' in (1.2) is attained with equality precisely for the rational function $f_n$ given by*

$$f_n(z) = \frac{z^n P_n(1/z)}{P_n(z)}, \quad \text{where} \quad P_n(z) = \sum_{\nu=0}^{n} \binom{-1/2}{\nu}(-z)^\nu, \tag{1.4}$$

*and the functions $\eta f_n$, where $\eta$ is any unimodular constant.*

**Remark.** Landau [8, 9] proves the asymptotic in (1.2) with an error term $o(\log n)$ rather than $O(1)$. The proof is based on Stirling's formula. Newman [11] cites Landau's result with the stronger error term $O(1)$. Regardless, the latter version is immediate from Landau's proof upon using a stronger form of Stirling's formula.

Let us now restrict attention to those $f$ in Theorem 1.2 which are polynomials of degree $< d$. By compactness, the set of all possible numbers produced by the left hand side of (1.2) admits a maximum $\mathscr{M}_{n,d}$ (recall Problem 1.1). As the extremal function $f_n$ in Theorem 1.2 is only a polynomial for $n = 0$, we have the *strict* inequality

$$\mathscr{M}_{n,d} < \mathscr{L}_n \quad (\leq \mathscr{L}_d). \tag{1.5}$$

(For a reasonably self-contained justification of this inequality, see Proposition 4.1 below.) Despite this, in a stroke of genius, Newman [11] was able to compute the numbers $\mathscr{M}_{n-1,2n}$ *exactly*:

**Theorem 1.3** (Newman, 1978). *Let $n$ be a non-negative integer. Then, for every polynomial $f = \sum_\nu a_\nu X^\nu \in \mathbb{C}[X]^{<2n}$ with $\|f\|_\infty \leq 1$, one has*

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$$|a_0 + a_1 + \ldots + a_{n-1}| \le \frac{1}{2} + \frac{1}{n} \sum_{\omega^n = -1} \frac{1}{|\omega - 1|}. \tag{1.6}$$

*(Here the sum ranges over all complex numbers $\omega$ with $\omega^n = -1$.) Moreover, '$\le$' in (1.6) is attained with equality for some polynomial satisfying the above conditions; in other words,*

$$\mathscr{M}_{n-1,2n} = \frac{1}{2} + \frac{1}{n} \sum_{\omega^n = -1} \frac{1}{|\omega - 1|} \quad \left( = \frac{\log n}{\pi} + O(1) \right). \tag{1.7}$$

We conclude this section with a short remark on the real case, *i.e.,* inequalities for coefficient sums of real polynomials that are bounded by 1 on the interval $[-1, 1]$. In 1892, Markov established sharp bounds for the derivatives of such polynomials (see [10, p. 258]). The question analogous to (1.1) seems to have first been considered by Reimer and Zeller [14] who established sharp bounds for even/odd polynomials (see also [13] and the references therein). As in the above complex setting, the current state of knowledge concerning the real case seems to be similarly fragmented and leaves much to be desired. Regardless, the methods that have been used to study the real situation have a somewhat different flavour, and in the present investigation we shall restrict our attention to the complex case.

## 2. Results

### 2.1. Approximate formulæ

If $n$ is kept fixed, then it follows from a (more general) result of Shapiro [19, Theorem 13 in §2.7] that $\mathscr{M}_{n,d} = \mathscr{L}_d - O_n(\epsilon_n^d)$ as $d \to \infty$. (Here $\epsilon_n$ is some positive number $< 1$.) Our first result provides an approximate lower bound for $\mathscr{M}_{n,d}$ when *both* parameters $n < d$ grow:

**Theorem 2.1.** *Let $d > n$ be positive integers. Then there exists a polynomial $\tilde{f}_{n,d} = \sum_\nu a_\nu X^\nu \in \mathbb{C}[X]^{<d}$ with $\|\tilde{f}_{n,d}\|_\infty \le 1$ such that*

$$|a_0 + a_1 + \ldots + a_n| = \left(1 - O\left\{n^3 \exp\left(-\frac{d}{5n}\right)\right\}\right) \sum_{\nu=0}^{n} \binom{-1/2}{\nu}^2.$$

On combining this with Theorem 1.2 we immediately deduce the following result:

**Corollary 2.2.** $\mathscr{M}_{n,d} = \dfrac{\log n}{\pi} + O(1)$ *for $d > 16 n \log n$ and $n \to \infty$.*

Newman [11, p. 187] also remarks that by (1.5) and (1.2) one has

$$\mathscr{M}_{n,d} \le \frac{\log d}{\pi} + O(1),$$

and proceeds to ask whether *'[the left hand side] ever gets this large'* (as $n$ varies from 0 to $d - 1$). He points out that (1.7) *answers this in the affirmative*. We remark that our Corollary 2.2 furnishes the following estimate:

**Corollary 2.3.** $\mathscr{M}_{n,d} = \dfrac{\log d}{\pi} + O(\log\log d)$ *provided that* $n \asymp \dfrac{d}{\log d}$.

This may be regarded as *another positive answer to Newman's question* up to lower order terms.

**Remark 2.4** (Improvements to the error terms). We have refrained from attempting to get the sharpest error term in Theorem 2.1 that our method allows for. Also the number 16 ($> 3 \cdot 5$) in Corollary 2.2 is not optimal in this regard. It would be particularly pleasing if Corollary 2.2 could be established for $d \gg n$, but we do not know if this can be achieved. We refer to §5.4 below for further discussion pertaining to potential improvements.

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### 2.2. *Exact formulæ*

Before stating further results, let us provide more context. Shapiro [19] also considers extremal problems for more general sums than in (1.1), namely the question of maximising

$$|t_0 a_0 + \ldots + t_{d-1} a_{d-1}| \tag{2.1}$$

over all polynomials $f = \sum_\nu a_\nu X^\nu \in \mathbb{C}[X]^{<d}$ with $\|f\|_\infty \leq 1$, where $\boldsymbol{t} = (t_0, \ldots, t_{d-1}) \in \mathbb{C}^d$ is fixed. The outcome of his theory is a better theoretical understanding of the behaviour of the extremal polynomials (for the above maximisation problem) on the unit circle. In particular, for those $\boldsymbol{t}$ for which the extremal polynomials are not given by monomials, he finds that for some $r \leq d$ there are complex numbers $z_1, \ldots, z_r, u_1, \ldots, u_r$ with $|z_1| = \ldots = |z_r| = 1$ such that

$$t_0 a_0 + \ldots + t_{d-1} a_{d-1} = \sum_{\nu=1}^{r} u_\nu f(z_\nu) \tag{2.2}$$

for all $f$ as above, and the answer to the maximisation of (2.1) is given *exactly* by $|u_1| + \ldots + |u_r|$ (see [19, §§2.3–4]).

In Shapiro's terminology, the $z_\nu$ are called the *nodes* associated to the functional given by $\boldsymbol{t}$. Observe that (2.2) may be interpreted as decomposing said functional using point evaluations at the $z_\nu$ (see also (6.5) below), yet there may be many more such decompositions with point evaluations not at the nodes. Despite this promising insight, Shapiro notes that *'[he] has been unable to devise any method for finding the nodes'* ([19, §2.4 2]).[1]

Here we point out that Newman's method [11] furnishes a (thin) host of examples for which the nodes *can* be computed. The next theorem (proved in §6.3 below) follows rather easily from a careful reading of Newman's argument:[2]

**Theorem 2.5.** *Let $n$ and $d$ be positive integers such that $d \geq 2n - 1$. Moreover, let*

$$\boldsymbol{t}^n = (t_{0 \bmod 2n}, \ldots, t_{n-1 \bmod 2n}) \in \mathbb{C}^n$$

*be arbitrary. Let $t_{n+\nu \bmod 2n} = -t_{\nu \bmod 2n}$ for $\nu = 0, \ldots, n-1$. Then, for every polynomial $f = \sum_\nu a_\nu X^\nu \in \mathbb{C}[X]^{<d}$ with $\|f\|_\infty \leq 1$, one has*

$$\left| \sum_{\nu=0}^{d-1} t_{\nu \bmod 2n}\, a_\nu \right| \leq \frac{1}{n} \sum_{\omega^n = -1} \left| \sum_{\nu=0}^{n-1} \frac{t_{\nu \bmod 2n}}{\omega^\nu} \right|. \tag{2.3}$$

*Moreover, there is a choice of $f$ as above for which '$\leq$' in (2.3) is attained with equality. In fact, this $f$ may even be chosen to have degree $\leq 2n - 2$.*

**Remark.** Evidently, Theorem 2.5 falls short of implying Theorem 1.3. Here the latter theorem corresponds to the choice $\boldsymbol{t} = (1, \ldots, 1, 0, \ldots, 0) \in \mathbb{C}^d$ in (2.1), with equally many 1's and 0's. However, Theorem 2.5 does handle $\boldsymbol{t} = (1, \ldots, 1, -1, \ldots, -1)$ and Newman [11] provides a rather ingenious argument for bridging this gap.

We note that the vectors $\boldsymbol{t}$ in (2.1) to which Theorem 2.5 applies are of the form

$$\boldsymbol{t} = \Big( \boxed{\boldsymbol{t}^n}, \boxed{-\boldsymbol{t}^n}, \ldots, \boxed{\pm \boldsymbol{t}^n \ ///} \Big) \in \mathbb{C}^d,$$

[1] Note, however, the partial progress in [19, §2.8] for some particular examples.
[2] Newman does not state this as a result, yet it would be hard to conceive that he would not have been aware of this.

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where '///' indicates that suitably many last few entries in the final block are to be omitted as to have $\boldsymbol{t}$ belong to $\mathbb{C}^d$. We are also able to prove the following variant of Theorem 2.5 which handles the more symmetric case of vectors $\boldsymbol{t}$ of the form

$$\boldsymbol{t} = \Big( \boxed{\boldsymbol{t}^n}, \boxed{\boldsymbol{t}^n}, \ldots, \boxed{\boldsymbol{t}^n \ ///} \Big) \in \mathbb{C}^d :$$

**Theorem 2.6.** *Let $n$ and $d$ be positive integers such that $d \geq 2n-1$. Moreover, let*

$$\boldsymbol{t}^n = (t_{0 \bmod n}, \ldots, t_{n-1 \bmod n}) \in \mathbb{C}^n$$

*be arbitrary. Then, for every polynomial $f = \sum_\nu a_\nu X^\nu \in \mathbb{C}[X]^{<d}$ with $\|f\|_\infty \leq 1$, one has*

$$\left| \sum_{\nu=0}^{d-1} t_{\nu \bmod n} \, a_\nu \right| \leq \frac{1}{n} \sum_{\omega^n=1} \left| \sum_{\nu=0}^{n-1} \frac{t_{\nu \bmod n}}{\omega^\nu} \right|. \tag{2.4}$$

*Moreover, there is a choice of $f$ as above for which '$\leq$' in (2.4) is attained with equality. In fact, this $f$ may even be chosen to have degree $\leq 2n-2$.*

As an application of (either of) the two theorems we prove the following cute inequality:

**Corollary 2.7.** *Let $0 \leq k < n$ be integers. Then, for any polynomial $f \in \mathbb{C}[X]^{<2n+k}$,*

$$\left| \frac{f^{(k)}(0)}{k!} \right| + \left| \frac{f^{(n+k)}(0)}{(n+k)!} \right| \leq \|f\|_\infty. \tag{2.5}$$

*Moreover, equality occurs for some polynomial of degree $\leq 2n-2$.*

### *2.3. A general bound*

In both Theorem 2.5 and Theorem 2.6 we are able to handle an $n$-dimensional subspace of functionals on $\mathbb{C}[X]^{<d}$. Specifically for $d \in \{2n-1, 2n\}$ the sum of these subspaces turns out to be the *whole* dual space of $\mathbb{C}[X]^{<d}$. The underlying reason is that (in the notation from §2.2) every vector $\boldsymbol{t} \in \mathbb{C}^d$ admits a decomposition of the form

$$\boldsymbol{t} = \Big( \boxed{\boldsymbol{t}_+^n}, \boxed{\boldsymbol{t}_+^n \ ///} \Big) + \Big( \boxed{\boldsymbol{t}_-^n}, \boxed{-\boldsymbol{t}_-^n \ ///} \Big)$$

for suitable vectors $\boldsymbol{t}_+^n, \boldsymbol{t}_-^n \in \mathbb{C}^n$ (see (6.11) below).

In general, this does not, unfortunately, enable us to obtain *sharp* bounds for (2.1). However, we still get upper bounds:

**Corollary 2.8.** *Let $d$ be a positive integer and write $D = d$ or $= d-1$ according to whether $d$ is even or odd. Let $\boldsymbol{t} = (t_0, \ldots, t_{d-1}) \in \mathbb{C}^d$ be arbitrary. Then, for every polynomial $f = \sum_\nu a_\nu X^\nu \in \mathbb{C}[X]^{<d}$ with $\|f\|_\infty \leq 1$, one has*

$$|t_0 a_0 + \ldots + t_{d-1} a_{d-1}| \leq \frac{1}{D} \sum_{\omega^D=1} \left| \sum_{\nu=0}^{d-1} \frac{t_\nu}{\omega^\nu} \right|. \tag{2.6}$$

Lastly, let us use Shapiro's example (1.1) as a benchmark for Corollary 2.8.

**Remark 2.9.** For $d = 2n$ and $\boldsymbol{t} = (1_{\times n}, 0_{\times n}) \in \mathbb{C}^d$ the right hand side (2.6) turns out to be precisely Newman's bound (1.6). However, we cannot quite deduce Newman's result (Theorem 1.3) from Corollary 2.8, because our bound (2.6) is not guaranteed to be sharp. The fact that the bound (2.6) may indeed *fail* to be sharp can be seen from the next example (namely the fact that there are white parts in Figure 1).

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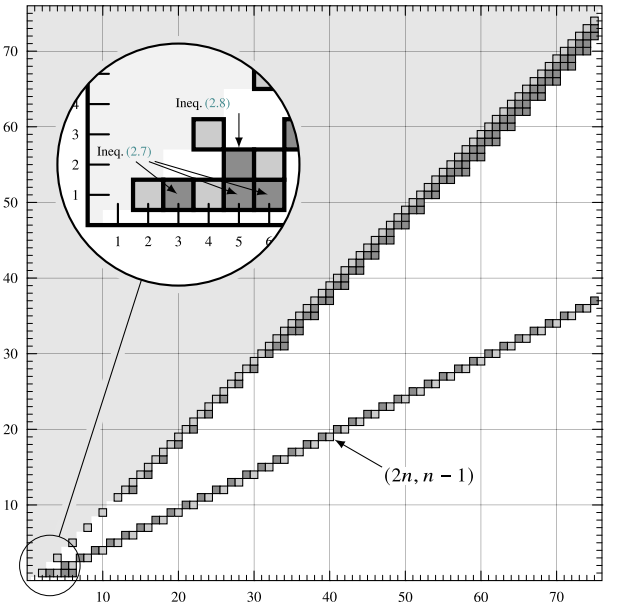



***Figure 1.*** *A numerical experiment showing when Corollary 2.8 produces stronger upper bounds on the maximum $\mathscr{M}_{n,d}$ from Problem 1.1 than previously known (cf. Example 2.10). The plot shows the points $(d,n)$, $0 \le n < d \le 75$, for which the upper bound (2.6) furnished by Corollary 2.8 for $\boldsymbol{t} = (1_{\times(n+1)}, 0_{\times(d-n-1)}) \in \mathbb{C}^d$ is strictly smaller than the bound $\mathscr{L}_n$ for $\mathscr{M}_{n,d}$ coming from Landau's theorem (see (1.5)).*

**Example 2.10.** Using a computer, we have investigated when Corollary 2.8 yields non-trivial information about (1.5): for $0 \le n < d \le 75$ we have computed the upper bound (2.6) ($\mathscr{C}_{n,d}$ say) furnished by Corollary 2.8 for $\boldsymbol{t} = (1_{\times(n+1)}, 0_{\times(d-n-1)}) \in \mathbb{C}^d$. For instance, we obtain the following (hitherto unknown) upper bounds for $\mathscr{M}_{1,d}$ ($d = 3, 5, 6$):

$$\left.\begin{aligned} \mathscr{M}_{1,3} \le \mathscr{C}_{1,3} &= \tfrac{1}{2} + \tfrac{1}{\sqrt{2}} = 1.2071\ldots \\ \mathscr{M}_{1,5} \le \mathscr{C}_{1,5} &= \tfrac{2}{3} + \tfrac{1}{\sqrt{3}} = 1.2440\ldots \\ \mathscr{M}_{1,6} \le \mathscr{C}_{1,6} &= \tfrac{2}{3} + \tfrac{1}{\sqrt{3}} \end{aligned}\right\} < 1.25 = \tfrac{5}{4} = \mathscr{L}_1. \tag{2.7}$$

Furthermore, we have

$$\mathscr{M}_{2,5} \le \mathscr{C}_{2,5} = \tfrac{4}{3} = 1.3\dot{3} < 1.390625 = \tfrac{89}{64} = \mathscr{L}_2. \tag{2.8}$$

Figutre 1 shows the tuples $(d,n)$ for which $\mathscr{C}_{n,d} < \mathscr{L}_n$. Note that we have the trivial cases $\mathscr{M}_{0,d} = \mathscr{M}_{d-1,d} = 1$, and $\mathscr{M}_{n-1,2n}$ is already covered by Newman's result (recall Remark 2.9, though). In the figure we have greyed these out.

## 3. Plan of the paper

In §4 we show that Landau's bound (1.2) is not sharp for polynomials. In §5 we prove Theorem 2.1. The key result in our approach turns out to be a 'quantitative Eneström–Kakeya theorem', stated as Theorem 5.4 below. In §6 we extend Newman's approach. The key outcome of this is given as Corollary 6.5 below. In §6.3, we use that result to prove Theorem 2.5 and Theorem 2.6. Finally, in §6.4 we prove the two corollaries enunciated in §2.2 and §2.3.

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Let us also mention a few points about *notation*: we use both the Landau symbol $O(\cdot)$ and Vinogradov's notation $\ll$ in the usual fashion. The implicit constants are always absolute, unless the contrary is explicitly indicated using subscripts (see, *e.g.*, (5.2)).

## 4. Polynomials and Landau's bound

### *4.1. Landau's argument*

In this section we give a self-contained proof that Landau's bound (1.2) is not sharp for polynomials. Of course, this result is not new. In fact, it follows directly from Landau's characterisation of the extremal functions in Theorem 1.2 (see [8], but not [9], where the required characterisation is omitted). Nevertheless, we believe that revisiting Landau's argument is instructive. We return to this point in §4.2 below.

**Proposition 4.1.** *Let $d > n \geq 1$ be integers. Then there is some $\delta(n,d) > 0$ such that for every polynomial $f = \sum_\nu a_\nu X^\nu \in \mathbb{C}[X]^{<d}$ with $\|f\|_\infty \leq 1$, one has*

$$|a_0 + a_1 + \ldots + a_n| \leq \sum_{\nu=0}^{n} \binom{-1/2}{\nu}^2 - \delta(n,d). \tag{4.1}$$

*Proof.* By compactness of the unit ball in $(\mathbb{C}[X]^{<d}, \|\cdot\|_\infty)$, there exists a polynomial $f = \sum_\nu a_\nu X^\nu \in \mathbb{C}[X]^{<d}$ with $\|f\|_\infty \leq 1$ for which (1.2) is maximal. Let $f$ be such a polynomial. Furthermore, let $P_n$ be given as in (1.4). Then, by means of comparison with the binomial series expansion

$$\left(\sum_{\nu=0}^{\infty} \binom{-1/2}{\nu} (-z)^\nu\right)^2 = \left(\frac{1}{\sqrt{1-z}}\right)^2 = \frac{1}{1-z} = \sum_{\nu=0}^{\infty} z^\nu \quad (\text{for } z \in \mathbb{C},\ |z| < 1),$$

it follows that

$$P_n^2(z) = 1 + z + z^2 + \ldots + z^n + \text{terms of higher order} \quad (\text{for } z \in \mathbb{C}),$$

so that Cauchy's formula, trivial estimation and Parseval's identity yield

$$|a_0 + a_1 + \ldots + a_n| = \left|\frac{1}{2\pi \mathrm{i}} \oint_{|z|=1} \frac{f(z)}{z^{n+1}} P_n^2(z) \mathrm{d}z\right| \leq \int_0^1 |P_n(\exp(2\pi \mathrm{i} t))|^2 \mathrm{d}t \tag{4.2}$$
$$= \text{right hand side of (1.2)}.$$

Therefore, in order to establish the existence of $\delta(n,d) > 0$ as in (4.1), it suffices to observe that '$\leq$' in (4.2) actually holds with '$<$'. This is simple, for if we had equality in (4.2) then $|f(z)| = 1$ for all $|z| = 1$ (in the first place, at least at the points $z$, $|z| = 1$, where $P_n$ does not vanish,[3] but then for all $|z| = 1$ by reason of continuity). However, this can only happen if $f$ is a monomal (see Lemma 4.3 below); yet in this case $|a_0 + a_1 + \ldots + a_n| = \|f\|_\infty = 1$, but the right hand side of (1.2) is $> 1$. □

**Remark 4.2.** It follows from a general result of Shapiro, [19, Theorem 13 in §2.7], that $\delta(n,d) \to 0$ for fixed $n$ and $d \to \infty$. Our Theorem 2.1 relaxes the condition on $n$.

The following result (used in the proof of Proposition 4.1) is a special case of the well-known classification of rational functions of modulus one on the unit circle (see, *e.g.*, [17, Ch. 14, Ex. 3 on p. 293]). We include a proof (without any claim of novelty) both for completeness' sake and for lack of a suitable reference.

[3] In fact, $P_n$ is non-vanishing throughout the closed unit disk; see Corollary 5.2 below.

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**Lemma 4.3.** *Let $f \in \mathbb{C}[X]$ be a polynomial such that $|f(z)| = 1$ for all $z \in \mathbb{C}$ with $|z| = 1$. Then $f$ is a monomial.*

*Proof.* Let $\bar{f}$ be the polynomial obtained from $f$ by replacing each coefficient by its complex conjugate. For $|z| = 1$ we have $\overline{z} = 1/z$. Consequently, the rational function $\bar{f}(1/X)f(X)$ equals one: (a) on the unit circle and, *a fortiori*, (b) everywhere (by the identity principle, for instance). Now if $f$ were to have a non-zero root $z \in \mathbb{C}$, then we reap the contradiction $1 = \bar{f}(1/z)f(z) = 0$. Hence, the zero locus of $f$ is either $\emptyset$ or $\{0\}$. In both cases, it follows that $f$ is a monomial. □

### *4.2. Reflections on Landau's argument*

The key to Proposition 4.1 may be described as resting on two observations:

1. The expression in question, $a_0 + a_1 + \ldots + a_n$, admits a nice integral representation, namely
$$a_0 + a_1 + \ldots + a_n = \frac{1}{2\pi \mathrm{i}} \oint_{|z|=1} \frac{f(z)}{z^{n+1}} \big(1 + z + z^2 + \ldots + z^n\big) \mathrm{d}z. \tag{4.3}$$
2. The 'integration kernel' $1 + z + z^2 + \ldots + z^n$ in (4.3) can be replaced by $P_n^2(z)$ (recall (4.2)). In the holomorphic case (Theorem 1.2) one can, in fact, use this to get *sharp* bounds via the functions $f_n$ given in (1.4).

It is clear that this approach does not readily generalise to determine the maximum $\mathscr{M}_{n,d}$ of the left hand side of (1.2) as $f$ ranges over all 1-bounded *polynomials* of degree $< d$: what we lack is sufficient understanding of the precise boundary behaviour of the corresponding extremal polynomials $f$.

Despite not having the required understanding of $f(z)$ for *all* $|z| = 1$, it is possible to get control on a finite number of such $z$ using an interpolation-based approach. In this way one can get results when the integral formula (4.3) can be replaced by a finite sum (involving point evaluations of $f$) with 'especially few' terms. Such phenomena of replacing integration of polynomials by finite sums with few terms is a well-known theme in numerical analysis, namely *Gauss quadrature*. In our case, this discrete analogue of (4.3) is given in the form of (6.5) in Corollary 6.5 below. We return to this in §6, when we discuss the exact formulæ presented in §2.2.

## 5. Asymptotic estimates

### *5.1. Proof of Theorem 2.1*

The strategy is to approximate Landau's extremal function $f_n$ by polynomials; this natural idea is already found in [19, Theorem 13 in §2.7], where it is applied to more general extremal problems. In the generality of the cited result, one cannot hope to achieve error bounds as explicit as the ones we obtain here. Our approach hinges upon bounding away the poles of $f_n$ from the unit disk (*cf.* Figure 2). To achieve this, we shall employ the well-known *Eneström–Kakeya theorem* [4, 7] (see also [2, p. 13]):

**Lemma 5.1** (Eneström, 1893; Kakeya, 1912). *Let $a_n X^n + \ldots + a_1 X + a_0$ be a polynomial with positive coefficients $a_0 \geq a_1 \geq \ldots \geq a_n > 0$. Then all its complex zeros are contained in the (closed) annulus with radii* $\min Q$ *and* $\max Q$*; here $Q = \{ a_\nu / a_{\nu+1} : 0 \leq \nu < n \}$.*

**Corollary 5.2.** *The poles $\zeta \in \mathbb{C}$ of $f_n$ given in (1.4) satisfy $|\zeta| \geq 1 + \dfrac{1}{2n+1}$.*

*Proof.* This follows from the definition (1.4) of $f_n$, (1.3), Lemma 5.1, and
$$-\binom{-1/2}{\nu} \Big/ \binom{-1/2}{\nu+1} = \frac{2\nu+2}{2\nu+1}.$$
□

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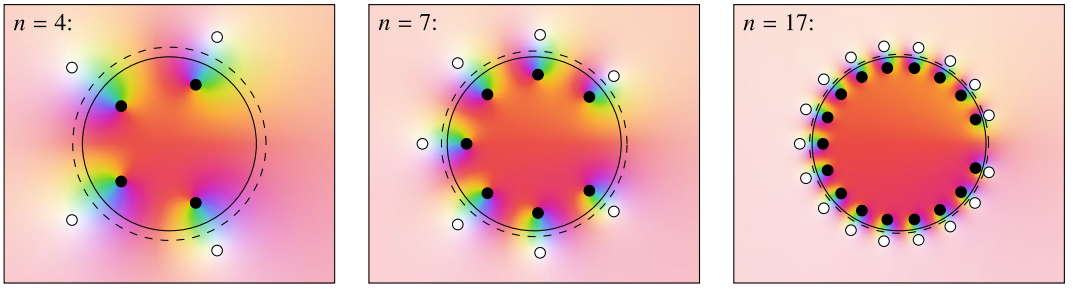



***Figure 2.*** *Zeros (black dots) and poles (white dots) of $f_n$ from (1.4) for $n \in \{4, 7, 17\}$. In each picture the two circles are centred about 0 and have radii 1 and $1 + 1/(2n+1)$, respectively.*

From the above one easily obtains decay estimates for the Taylor coefficients of the function $f_n$. An unpleasant feature of this rather naïve approach is that it does not yield an explicit dependence on $n$. This is a nuisance for our purpose, which we mitigate in Proposition 5.5.

**Corollary 5.3.** *Fix an integer $n \geq 1$ and let*

$$f_n(z) = \sum_{\nu=0}^{\infty} b_{n,\nu} z^{\nu} \tag{5.1}$$

*be the power series expansion (about the origin) of the rational function $f_n$ from (1.4). Then, for every $\epsilon \in (0, 1)$,*

$$b_{n,\nu} \ll_{n,\epsilon} \left(1 + \frac{1}{2n+1} - \epsilon\right)^{-\nu}. \tag{5.2}$$

*Proof.* The radius $R$ of convergence of the power series expansion (5.1) is precisely $\min_{\zeta} |\zeta|$, where $\zeta$ ranges over the poles of $f_n$. The lemma now follows upon combining Corollary 5.2 and the well-known Cauchy–Hadamard formula for $R$. □

*Proof of Theorem 2.1.* Suppose for the moment that we had a version of (5.2) at our disposal in which the dependence of the implied constant was spelled out in terms of $n$. Precisely, we shall work with the inequality

$$\left|b_{n,\nu}\right| \ll n^2 \left(1 + \frac{1}{4n}\right)^{-\nu}. \tag{5.3}$$

which we establish in Proposition 5.5 below.

We shall employ (5.3) to see how well the polynomial

$$f_{n,d} = \sum_{\nu=0}^{d-1} b_{n,\nu} X^{\nu}.$$

approximates $f_n$ (see (5.1)). Indeed, for all $|z| \leq 1$,

$$\left|f_n(z) - f_{n,d}(z)\right| = \left|\sum_{\nu=d}^{\infty} b_{n,\nu} z^{\nu}\right| \ll n^3 \left(1 + \frac{1}{4n}\right)^{-d} \ll n^3 \exp\left(-\frac{d}{5n}\right) =: E(n,d).$$

In particular,

$$\left\|f_{n,d}\right\|_{\infty} \leq \|f_n\|_{\infty} + O(E(n,d)) = 1 + O(E(n,d)).$$

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Therefore, $\tilde{f}_{n,d} = f_{n,d}/\|f_{n,d}\|_\infty$ is a polynomial with $\|\tilde{f}_{n,d}\|_\infty = 1$. If $d > n$, then the first $n+1$ coefficients of the polynomial $f_{n,d}$ sum to the right hand side of (1.2). In particular, we have constructed a polynomial of degree $< d$, bounded by 1 in absolute value on the unit disk, whose first $n+1$ coefficients sum to

$$\frac{1}{\|f_{n,d}\|_\infty}\mathscr{L}_n \geq \frac{1}{1+O(E(n,d))}\mathscr{L}_n \geq \big(1-O(E(n,d))\big)\mathscr{L}_n$$

in absolute value. □

### *5.2. A quantitative Eneström–Kakeya theorem*

For our desired explicit version of Corollary 5.3, the original Eneström–Kakeya theorem is insufficient. Fortunately, it is an easy matter to extract a suitable quantitative statement of the standard proof of said theorem. Despite its easy proof, we believe that the result obtained may be of independent interest:

**Theorem 5.4** (Quantitative Eneström–Kakeya). *Let $p = a_nX^n + \ldots + a_1X + a_0$ be a polynomial with positive coefficients $a_0 \geq a_1 \geq \ldots \geq a_n > 0$. Let $Q = \{a_\nu/a_{\nu+1} : 0 \leq \nu < n\}$ and put $r = \min Q \leq \max Q = R$. Then, for $z \in \mathbb{C}\setminus\{r,R\}$, we have the following inequalities:*

(1) $\dfrac{r-|z|}{|r-z|} \leq \dfrac{|p(z)|}{a_0} \leq \dfrac{r+|z|}{|r-z|}$ *for* $|z| \leq r$,

(2) $\dfrac{|z|-R}{|z-R|} \leq \dfrac{|p(z)|}{a_n|z|^n} \leq \dfrac{R+|z|}{|R-z|}$ *for* $|z| \geq R$.

*Proof.* (We follow [2, pp. 12–13] with minimal adjustments in order to extract the desired inequalities.) For $\zeta \in \mathbb{C}$ with $|\zeta| \leq 1$ we have

$$\begin{aligned}|(1-\zeta)p(r\zeta)-a_0| &= \big|(ra_1-a_0)\zeta + \ldots + (ra_n-a_{n-1})r^{n-1}\zeta^n - a_nr^n\zeta^{n+1}\big| \\ &\leq (a_0-ra_1)|\zeta| + \ldots + (a_{n-1}-ra_n)r^{n-1}|\zeta|^n + a_nr^n|\zeta|^{n+1} \\ &\leq \big\{(a_0-ra_1)+\ldots+(a_{n-1}-ra_n)r^{n-1}+a_nr^n\big\}|\zeta| \\ &= a_0|\zeta|.\end{aligned} \tag{5.4}$$

Then $\zeta = z/r$ $(\neq 1)$ satisfies $|\zeta| \leq 1$ and we find that

$$|(1-\zeta)p(r\zeta)| \geq a_0(1-|\zeta|) \quad \text{and} \quad |(1-\zeta)p(r\zeta)| \leq a_0(1+|\zeta|).$$

This shows (1).

For (2) one considers $\zeta = z/R$ for $|z| \geq R$ and estimates

$$|(1-\zeta)p(R\zeta) + a_n(R\zeta)^n\zeta|$$

similarly as in (5.4). □

### *5.3. Growth of the Taylor coefficients of Landau's extremal functions $f_{n,d}$*

Recall that our proof of Theorem 2.1, given in §5.1, crucially hinges on a certain 'upgrade' of Corollary 5.3 in which the dependence of the bound (5.2) on $n$ is made explicit (see (5.2)). We are now ready to prove this explicit version:

**Proposition 5.5.** *Assume the hypotheses of Corollary 5.3. Then*

$$|b_{n,\nu}| \ll n^2\left(1+\frac{1}{4n}\right)^{-\nu}. \tag{5.5}$$

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*Proof.* Recall that, by Corollary 5.2, $f_n$ is holomorphic in the open disk with radius $r = 1 + 1/(2n+1)$ centred about the origin. Consequently, for any $0 < \rho < r$, Cauchy's estimate for Taylor coefficients yields

$$\left|b_{n,\nu}\right| \le \rho^{-\nu} \max_{|z|=\rho} |f_n(z)| = \rho^{n-\nu} \max_{|z|=\rho} \left|\frac{P_n(1/z)}{P_n(z)}\right|.$$

We take $\rho = 1 + 1/4n$ and estimate this further using Theorem 5.4. Then (5.5) follows easily. □

### 5.4. *Remark on the zeros of* $P_n$

As we have seen, information on the location of the roots of $P_n$ is vital to our proof of Theorem 2.1. Precisely, we should like to have good lower bounds on the absolute value of the roots of $P_n$. Let

$$r(n) = \min\{\, |z| : \text{roots } z \text{ of } P_n \,\}.$$

Then Corollary 5.2 tells us that $r(n) - 1 \gg 1/n$. Towards the other extreme, we can give the following upper bound:

**Proposition 5.6.** *We have* $r(n) \le 1 + O\left(\dfrac{\log n}{n}\right)$ *for* $n \ge 2$.

*Proof.* (The proof is based on an observation we have learned from work by Erdős and Turán [5, §6].) Observe that

$$1 = |P_n(0)| = \left|\binom{-1/2}{n}\right| \prod_z |0 - z| \ge \left|\binom{-1/2}{n}\right| r(n)^n,$$

where $z$ ranges over all $n$ roots of $P_n$ (taking multiplicity into account[4]). The absolute value of the binomial coefficient above is asymptotic to $1/\sqrt{\pi n}$ (by Stirling's formula; *cf.* [9, pp. 22–23]). In particular, we have the inequality $r(n)^n \le 2\sqrt{\pi n} < n$ for all sufficiently large $n > 4\pi$. Hence, for those $n$,

$$r(n) < \exp\left(\frac{\log n}{n}\right) < 1 + \frac{\log n}{n}.$$

In particular, we have the bound claimed in the proposition for *all* $n \ge 2$. □

Consequently, for $n \ge 2$, we have

$$\frac{1}{n} \ll r(n) - 1 \ll \frac{\log n}{n}. \tag{5.6}$$

Based on some limited numerical evidence, one may be tempted to conjecture that the lower bound is closer to the truth; we have been unable to resolve this.

**Open problem 5.7.** Improve on either of the two bounds in (5.6) to the extent that one of the following two statements follows:

$$\limsup_{n\to\infty} (r(n) - 1)n = \infty, \quad \liminf_{n\to\infty} \frac{(r(n) - 1)n}{\log n} = 0.$$

Ideally, establish an asymptotic formula for $(r(n) - 1)n$ as $n \to \infty$.

[4] As observed subsequently in Remark 5.8, the statement about multiplicities is unnecessary.

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**Remark 5.8.** We mention two other interesting facts about the roots of $P_n$:

1. they are all simple, as can be seen by applying the Eneström–Kakeya theorem to the derivative of $P_n$;
2. every point on the unit circle is an accumulation point of zeros of $P_n$ for $n$ restricted to a suitable subsequence of the positive integers; this follows from a classical theorem of Jentzsch [6] on sections of power series with finite radius of convergence (see also the subsequent influential refinements by Szegö [20] and Erdős–Turán [5], as well as the monography [1]).

## 6. Interpolation with roots of unity

This section is devoted to the proof of both Theorem 2.5 and Theorem 2.6. We handle both simultaneously. Throughout this section fix a positive integer $n$ and let

$$\Omega = \{\, \omega \in \mathbb{C} : \omega^n \pm 1 = 0 \,\}. \tag{6.1}$$

In the sequel, $\omega$, $\varpi$ and $\xi$ will always denote elements of $\Omega$. Occurrences of '$\pm$' or '$\mp$' always refer to the sign chosen in (6.1).

### *6.1. Lagrange interpolators for roots of unity*

Write

$$L_{\Omega,\omega} = \prod_{\substack{\varpi\in\Omega\\ \varpi\neq\omega}} \frac{X-\varpi}{\omega-\varpi} \tag{6.2}$$

for the *Lagrange interpolating polynomial* with respect to $\Omega$ and $\omega$. Clearly, for $\varpi \in \Omega$,

$$L_{\Omega,\omega}(\varpi) = \begin{cases} 1 & \text{if } \varpi = \omega, \\ 0 & \text{if } \varpi \neq \omega. \end{cases} \tag{6.3}$$

The key observation is the following neat property of the Lagrange interpolators with respect to our special choice for $\Omega$:

**Proposition 6.1.** $\sum_{\omega\in\Omega} \left|L_{\Omega,\omega}(z)\right|^2 = 1$ *for all* $z \in \mathbb{C}$ *with* $|z| = 1$.

**Remarks.**

1. Proposition 6.1 for '+' in the definition of $\Omega$ is precisely Lemma 2 of [11].
2. Proposition 6.1 is the crucial ingredient for obtaining Corollary 6.5 below. The latter result is, as we shall see, a common generalization of Theorem 2.5 and Theorem 2.6.
3. The statement of Proposition 6.1 may become invalid if $\Omega$ from (6.1) is replaced by some other set, even if that set consists only of roots of unity. For instance, for $\Omega = \{1, \mathrm{i}\}$, one easily checks that $2\sum_{\omega\in\Omega}\left|L_{\Omega,\omega}(z)\right|^2 = |z-1|^2 + |z-\mathrm{i}|^2$. It would seem desirable to have a characterisation of the sets $\Omega$ for which Proposition 6.1 holds.

We base the proof of Proposition 6.1 on three lemmas.

**Lemma 6.2.**

(1) $\mp\dfrac{n}{\omega} = \displaystyle\prod_{\substack{\varpi\in\Omega\\ \varpi\neq\omega}} (\omega-\varpi)$,

(2) $\mp\dfrac{(n-1)n}{\omega^2} = 2\displaystyle\sum_{\substack{\xi\in\Omega\\ \xi\neq\omega}} \prod_{\substack{\varpi\in\Omega\\ \varpi\neq\omega,\xi}} (\omega-\varpi)$.

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*Proof.* The first statement is certainly classical (see, *e.g.*, [12, VI, §9, Problem 74]); it follows from

$$\mp\frac{n}{\omega} = n\omega^{n-1} = \frac{\mathrm{d}}{\mathrm{d}X}(X^n \pm 1)\Big|_{X=\omega} = \frac{\mathrm{d}}{\mathrm{d}X}\prod_{\varpi\in\Omega}(X-\varpi)\Big|_{X=\omega} = \prod_{\substack{\varpi\in\Omega\\ \varpi\neq\omega}}(X-\varpi)\Big|_{X=\omega}.$$

Likewise, the second statement follows from a similar evaluation of the second derivative of $X^n \pm 1$ at $\omega \in \Omega$. □

**Lemma 6.3.** $L_{\Omega,\omega} = \pm\dfrac{X^n \pm 1}{\omega - X}\dfrac{\omega}{n}$ *in* $\mathbb{C}(X)$.

*Proof.* The product over the denominators in (6.2) is $\mp\omega/n$ by Lemma 6.2 (1). On the other hand, clearly, the product over the numerators equals $(X^n - 1)/(X - \omega)$. This establishes the lemma. □

**Lemma 6.4.** $\displaystyle\sum_{\omega\in\Omega}\frac{\omega}{(X-\omega)^2} = \mp\frac{n^2X^{n-1}}{(X^n\pm 1)^2}$ *in* $\mathbb{C}(X)$.

*Proof.* The claimed identity clearly follows from the polynomial identity

$$\sum_{\omega\in\Omega}\omega\prod_{\substack{\varpi\in\Omega\\ \varpi\neq\omega}}(X-\varpi)^2 = \mp n^2X^{n-1}. \tag{6.4}$$

upon dividing both sides by $(X^n \pm 1)^2 = \prod_{\omega\in\Omega}(X-\omega)^2$.

In order to establish (6.4), observe that both sides equal one another for all $\omega \in \Omega$, by virtue of Lemma 6.2 (1). Furthermore, also the derivatives of both sides of (6.4) are equal for all $\omega \in \Omega$: indeed, we have

$$\frac{\mathrm{d}}{\mathrm{d}X}\sum_{w\in\Omega} w\prod_{\substack{\varpi\in\Omega\\ \varpi\neq w}}(X-\varpi)^2\Big|_{X=\omega} = \omega\sum_{\substack{\xi\in\Omega\\ \xi\neq\omega}} 2(\omega-\xi)\prod_{\substack{\varpi\in\Omega\\ \varpi\neq\omega,\xi}}(\omega-\varpi)^2 = \frac{(n-1)n^2}{\omega^2},$$

where the last equality follows from (both parts of) Lemma 6.2. Hence, the difference of both sides of (6.4) either vanishes or is a polynomial of degree $\geq 2n-1$. Evidently, the latter is not the case, so we have (6.4). □

*Proof of Proposition 6.1.* Clearly, we may suppose that $z \notin \Omega$ (by (6.3) or by continuity, for instance). Since $|z| = 1$, it holds that $\overline{z} = 1/z$. Thus, by Lemma 6.3,

$$\sum_{\omega\in\Omega}\left|L_{\Omega,\omega}(z)\right|^2 = \sum_{\omega\in\Omega}\frac{z^n\pm 1}{\omega - z}\frac{\omega}{n}\frac{z^{-n}\pm 1}{\omega^{-1}-z^{-1}}\frac{\omega^{-1}}{n} = \mp\frac{(z^n\pm 1)^2}{n^2z^{n-1}}\sum_{\omega\in\Omega}\frac{\omega}{(z-\omega)^2}.$$

The result is now immediate from Lemma 6.4. □

### *6.2. Application to functionals: the key result*

In the sequel, we shall write $_(\omega)$ for the *evaluation-at-$\omega$ functional* $\mathbb{C}[X] \to \mathbb{C}$, mapping every polynomial $f$ to $f(\omega)$. Proposition 6.1 now affords a proof of the following result which turns out to be nothing but a unified version of Theorem 2.5 and Theorem 2.6:

**Corollary 6.5.** *Let $n$ and $\Omega$ be as above and let $d \geq 2n-1$ be an arbitrary integer. Let $\ell\colon \mathbb{C}[X]^{<d} \to \mathbb{C}$ be a functional of the form*

$$\ell = \sum_{\omega\in\Omega} u_\omega\cdot_(\omega)|_{\mathbb{C}[X]^{<d}} \tag{6.5}$$

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*with scalars $u_\omega \in \mathbb{C}$. Then the norm*

$$\|\ell\| = \sup\{\, |\ell(f)| : f \in \mathbb{C}[X]^{<d},\ \|f\|_\infty = 1 \,\}$$

*of $\ell$ is given by*

$$\|\ell\| = \sum_{\omega\in\Omega} |u_\omega|.$$

*Proof.* Let $f \in \mathbb{C}[X]^{<d}$ satisfy $\|f\|_\infty = 1$. Certainly,

$$|\ell(f)| \le \sum_{\omega\in\Omega} |u_\omega||f(\omega)| \le \sum_{\omega\in\Omega} |u_\omega|,$$

so it suffices to produce an $f$ with $\|f\|_\infty = 1$ for which the above inequalities hold as equalities. Such an $f$ is furnished by

$$f = \sum_{\omega\in\Omega}^{\star} \frac{\overline{u_\omega}}{|u_\omega|} L^2_{\Omega,\omega},$$

where $\sum^\star$ indicates that terms with $u_\omega = 0$ ought to be omitted. Indeed, since each $L^2_{\Omega,\omega}$ has degree $2n-2 < d$, the degree of $f$ is no higher than that. Moreover, $\|f\|_\infty = 1$ by Proposition 6.1, and (6.3) shows that

$$\ell(f) = \sum_{\omega\in\Omega}^{\star} \frac{\overline{u_\omega}}{|u_\omega|} u_\omega = \sum_{\omega\in\Omega} |u_\omega|. \qquad \square$$

**Remark 6.6.** In §2.2 we have already noted that the idea of maximising certain functionals by relating them to point evaluation functionals is crucial in [19, 11]. The interested reader is also referred to [16] for general reflections on such an approach or, *e.g.*, [15, Ch. 2.B] for further examples.

### *6.3. Application to functionals: cosmetic surgery*

The functionals $\ell$ to which Corollary 6.5 applies are given in (6.5) in terms of point evaluation functionals. For applications to (2.1) we should rather like to describe these functionals in terms of their action on monomials. This is a staight-forward exercise in linear algebra and is the object of the present subsection.

**Lemma 6.7.** *A linear functional $\ell\colon \mathbb{C}[X]^{<d} \to \mathbb{C}$ is of the form (6.5) if and only if $(u_\omega)_{\omega\in\Omega}$ satisfies the following system of linear equations:*

$$\sum_{\omega\in\Omega} u_\omega \omega^\kappa = \ell(X^\kappa), \quad 0 \le \kappa < d. \tag{6.6}$$

*Proof.* Consider the spaces $V = \mathbb{C}[X]^{<n}$ and $W = \mathbb{C}[X]^{<d}$. On $V$ pick the basis $\{X^0, X^1, \dots, X^{n-1}\}$ and consider its dual basis $\{x_0^*, x_1^*, \dots, x_{n-1}^*\} \subset V^*$ (*i.e.*, $x_\nu^*(X^\eta) = 1$ if $\nu = \eta$ and $= 0$ otherwise). Construct a basis $\{\tilde{x}_0^*, x_1^*, \dots, \tilde{x}_{d-1}^*\} \subset W^*$ in the same fashion. Now put the elements of $\Omega$ in any order, $\Omega = \{\omega_0, \dots, \omega_{n-1}\}$, say, and consider the linear map $T\colon V^* \to W^*$ sending each $x_\nu^*$ to $_(\omega_\nu)|_W$. Observe that any $\ell \in W^*$ admits the decomposition

$$\ell = \sum_{\kappa=0}^{d-1} \ell(X^\kappa)\, \tilde{x}_\kappa^* \tag{6.7}$$

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with respect to our chosen dual basis on $W^*$. From this we easily obtain a matrix representation of $T$; this is summed up in the following commutative diagram:

$$\begin{array}{ccccc} \mathbb{C}^n & \xrightarrow[(0,\dots,0,\underset{\nu}{1},0,\dots,0)\mapsto X^\nu]{\sim} & \mathbb{C}[X]^{<n} & \xrightarrow[X^\nu\mapsto x_\nu^*]{\sim} & V^* \\ \downarrow{\scriptstyle \left(\omega_\nu^\kappa\right)_{\substack{0\le\kappa<d\\0\le\nu<n}}} & & \downarrow{\scriptstyle X^\nu\mapsto\sum_{\kappa=0}^{d-1}\omega_\nu^\kappa X^\kappa} & & \downarrow{\scriptstyle T\quad x_\nu^*\mapsto_(\omega_\nu)|_W} \\ \mathbb{C}^d & \xrightarrow[(0,\dots,0,\underset{\kappa}{1},0,\dots,0)\mapsto X^\kappa]{\sim} & \mathbb{C}[X]^{<d} & \xrightarrow[X^\kappa\mapsto\tilde{x}_\kappa^*]{\sim} & W^*. \end{array} \tag{6.8}$$

Now $\ell \in W^*$ belongs to the image of $T$ if and only if there is a decomposition of the shape (6.5). Upon recalling (6.7) and (6.8), we deduce the asserted equivalency of (6.5) with (6.6). □

Observe that the left hand side of (6.6) is periodic:

1. with period $n$ if '$-$' is chosen in the definition (6.1) of $\Omega$,
2. with period $2n$ if '$+$' is chosen, and 'almost' $n$-periodic in the sense that a negative sign is picked up after $n$ steps (due to $\omega^n = -1$ in that case).

This explains the particular choices for $\boldsymbol{t}$ in Theorem 2.5 and Theorem 2.6. In order to solve (6.6) explicitly, we shall make use of the following lemma:

**Lemma 6.8** (Orthogonality relations). *Let $\nu$ be an integer. Then*

$$\sum_{\omega\in\Omega}\omega^\nu = \begin{cases}\epsilon_{\Omega,\nu}n & \text{if } \nu \equiv 0 \bmod n,\\ 0 & \text{if } \nu \not\equiv 0 \bmod n,\end{cases}$$

*where $\epsilon_{\Omega,\nu} = 1$ if the sign '$-$' is chosen in (6.1) and $\epsilon_{\Omega,\nu} = (-1)^{\nu/n}$ otherwise.*

*Proof.* When the sign '$-$' is chosen in (6.1), then the lemma asserts nothing but the well-known *orthogonality relations* for roots of unity (see, *e.g.*, [3, p. 3]). On the other hand, if the sign '$+$' is chosen in (6.1), we may conclude as follows: let $\delta_{\nu\equiv 0 \bmod n} = 1$ or $= 0$ according to whether $\nu \equiv 0 \bmod n$ or not. Then, on appealing to the already established '$-$' case,

$$\sum_{\omega\in\Omega}\omega^\nu + n\,\delta_{\nu\equiv 0 \bmod n} = \sum_{\omega^n=-1}\omega^\nu + \sum_{\zeta^n=1}\zeta^\nu = \sum_{\zeta^{2n}=1}\zeta^\nu = 2n\,\delta_{\nu\equiv 0 \bmod 2n}.$$

This establishes the lemma. □

*Proof of Theorem 2.5 and Theorem 2.6.* Depending on the sign chosen in (6.1), define the linear functional $\ell\colon \mathbb{C}[X]^{<d} \to \mathbb{C}$ either via the sum on the left hand sides of (2.4) (*case* '$+$') or of (2.3) (*case* '$-$') respectively. Then, for $0 \le \nu < d$,

$$\ell(X^\nu) = \begin{cases} t_{\nu \bmod 2n} & \text{in case '+',}\\ t_{\nu \bmod n} & \text{in case '-'.}\end{cases} \tag{6.9}$$

Let $m$ be defined to be either $2n$ or $n$ depending on the sign chosen in (6.1). We aim to apply Corollary 6.5 and, to this end, we contend that $\ell$ admits a decomposition of the form (6.5). By Lemma 6.7 this means that we ought to solve the linear equations given in (6.6). In fact, we claim that the (unique) solution is given by

$$u_\omega = \frac{1}{n}\sum_{\nu=0}^{n-1}\frac{t_{\nu \bmod m}}{\omega^\nu} \quad (\omega \in \Omega).$$

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Indeed, for $0 \le \kappa < d$, we have

$$\sum_{\omega\in\Omega} u_\omega \omega^\kappa = \sum_{\omega\in\Omega} \frac{1}{n} \sum_{\nu=0}^{n-1} t_{\nu \bmod m} \omega^{\kappa-\nu} = \sum_{\nu=0}^{n-1} t_{\nu \bmod m} \frac{1}{n} \sum_{\omega\in\Omega} \omega^{\kappa-\nu} = \ell(X^\kappa),$$

where the last equality follows from Lemma 6.8 and (6.9). This shows that we may apply Corollary 6.5, and this completes the proof of the two theorems. □

### *6.4. Proof of the corollaries*

*Proof of Corollary 2.7.* We choose $\boldsymbol{t}^n \in \mathbb{C}^n$ in Theorem 2.6 as an appropriate standard unit vector. Then (2.4) yields[5]

$$\left| \frac{f^{(k)}(0)}{k!} + \frac{f^{(n+k)}(0)}{(n+k)!} \right| \le \|f\|_\infty. \tag{6.10}$$

This is *not quite* (2.5). However, on replacing $f$ by $f(\eta X)$, where $\eta$ is some suitably chosen unimodular constant, we may arrange for both numbers $f^{(k)}(0)$ and $f^{(n+k)}(0)$ to have equal argument in the complex plane. Hence, (6.10) implies (2.5). The sharpness of (2.5) follows, because (6.10) is guaranteed to be sharp by Theorem 2.6. □

*Proof of Corollary 2.8.* Note that $D$ is always even. We let $n = D/2$ so that either $d = 2n$ or $d = 2n-1$ depending on the parity of $d$. In the latter case, define $t_d = t_{2n-1} = 0$. Let $\langle \boldsymbol{t}, _ \rangle \colon \mathbb{C}[X]^{<d} \to \mathbb{C}$ denote the linear functional sending each $f = \sum_\nu a_\nu X^\nu \in \mathbb{C}[X]^{<d}$ to $\langle \boldsymbol{t}, f \rangle = t_0 a_0 + \ldots + t_{d-1} a_{d-1}$. We decompose $\boldsymbol{t}$ as follows:

$$\boldsymbol{t} = \begin{pmatrix} t_0 \\ \vdots \\ t_{n-1} \\ t_n \\ \vdots \\ t_{d-1} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} t_0 + t_n \\ \vdots \\ t_{n-1} + t_{2n-1} \\ t_0 + t_n \\ \vdots \\ t_{d-n-1} + t_{d-1} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} t_0 - t_n \\ \vdots \\ t_{n-1} - t_{2n-1} \\ -(t_0 - t_n) \\ \vdots \\ -(t_{d-n-1} - t_{d-1}) \end{pmatrix} =: \boldsymbol{t}_+ + \boldsymbol{t}_-. \tag{6.11}$$

Hence,

$$|t_0 a_0 + \ldots + t_{d-1} a_{d-1}| = |\langle (\boldsymbol{t}_+ + \boldsymbol{t}_-), f \rangle| \le |\langle \boldsymbol{t}_+, f \rangle| + |\langle \boldsymbol{t}_-, f \rangle|. \tag{6.12}$$

The first term on the right hand side of (6.12) can be estimated using Theorem 2.6; likewise, the second term can be estimated using Theorem 2.5. Specifically, we find that

$$|\langle \boldsymbol{t}_\pm, f \rangle| \le \frac{1}{n} \sum_{\omega^n=\pm 1} \left| \sum_{\nu=0}^{n-1} \frac{1}{2} \frac{t_\nu \pm t_{\nu+n}}{\omega^\nu} \right| = \frac{1}{D} \sum_{\omega^n=\pm 1} \left| \sum_{\nu=0}^{D-1} \frac{t_\nu}{\omega^\nu} \right| = \frac{1}{D} \sum_{\omega^n=\pm 1} \left| \sum_{\nu=0}^{d-1} \frac{t_\nu}{\omega^\nu} \right|,$$

where the penultimate equation is clear since either $d = D$ (if $d$ is even) or $d = D-1$ and $t_d = 0$ (by definition). Upon recalling (6.12) and noting that

$$\sum_{\omega^n=1} + \sum_{\omega^n=-1} = \sum_{\omega^D=1},$$

we finally infer the estimate (2.6). □

[5] If we had applied (2.3) instead, we would have obtained (6.10) with a '−' between the two fractions. This would equally well yield to the desired conclusion (2.5).

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