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EULER OBSTRUCTION AND DEFECTS OF FUNCTIONS ON SINGULAR VARIETIES

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Abstract

Several authors have proved Lefschetz type formulas for the local Euler obstruction. In particular, a result of this type has been proved that turns out to be equivalent to saying that the local Euler obstruction, as a constructible function, satisfies the local Euler condition (in bivariant theory) with respect to general linear forms. The purpose of the paper is to determine what prevents the local Euler obstruction from satisfying the local Euler condition with respect to functions which are singular at the considered point. This is measured by an invariant (or 'defect') of such functions. An interpretation of this defect is given in terms of vanishing cycles, which allows it to be calculated algebraically. When the function has an isolated singularity, the invariant can be defined geometrically, via obstruction theory. This invariant unifies the usual concepts of the Milnor number of a function and the local Euler obstruction of an analytic set.

Introduction

This work is a natural continuation of [**3**], in which a Lefschetz-type formula for the local Euler obstruction is proved. More precisely, let $(X,0)$ be an equidimensional complex analytic singularity germ in an open set $U \subset \mathbb{C}^N$. We endow $(X,0)$ with a Whitney stratification $\{V_i\}$ and consider a general complex linear form $l : \mathbb{C}^N \longrightarrow \mathbb{C}$. The formula of [**3**] says that the local Euler obstruction of X at 0 satisfies.

$$\mathrm{Eu}_X(0) = \sum_i \chi(V_i \cap \mathbb{B}_\varepsilon \cap l^{-1}(t_0)) \cdot \mathrm{Eu}_X(V_i),$$

where $\mathbb{B}_\varepsilon$ is a small closed ball around 0 in $\mathbb{C}^N$ (we could also use an open ball, see Remark 1.4), $t_0 \in \mathbb{C} \setminus \{0\}$ is sufficiently near $\{0\}$, and $\mathrm{Eu}_X(V_i)$ is the Euler obstruction of X at any point of the stratum V_i.

As we explain below, this formula is actually equivalent to saying that the local Euler obstruction, as a constructible function on X, satisfies the local Euler condition with respect to general linear forms. The purpose of this work is to understand what prevents the local Euler obstruction from satisfying the local Euler condition with respect to functions with a singularity at $0 \in X$.

We consider first an analytic function $f : X \longrightarrow \mathbb{C}$ with an isolated singularity at 0. We introduce (in Section 2) an invariant $\mathrm{Eu}_{f,X}(0)$ called the *local Euler obstruction* of f at 0. To define this invariant, let us consider the Nash bundle $\widetilde{T}$ over the Nash transform $\widetilde{X}$ of X. Roughly speaking, $\mathrm{Eu}_{f,X}(0)$ is the obstruction for constructing a section of $\widetilde{T}$ that extends a lifting of $\overline{\nabla}_X f$, the complex conjugate of the gradient

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vector field of f on X (see Section 2 for a precise definition). In Section 3, we prove (Theorem 3.1) that

$$\mathrm{Eu}_X(0) = \left(\sum_i \chi(V_i \cap \mathbb{B}_\varepsilon \cap f^{-1}(t_0)) \cdot \mathrm{Eu}_X(V_i) \right) + \mathrm{Eu}_{f,X}(0).$$

In Sections 3 and 4, we give a proof of this formula, which is similar to the proof in [**3**]. The main difference is that the invariant $\mathrm{Eu}_{f,X}(0)$ does not vanish in general and its contribution to the local Euler obstruction $\mathrm{Eu}_X(0)$ has to be taken into account. The more natural way to prove our formula is to work directly on X, which is somehow simpler than working on the Nash transform $\widetilde{X}$, as in [**3**]. It seems to us that this approach gives interesting insights even for the case of general linear forms.

In Section 5 we look at the situation where the function f may have a non-isolated singularity at 0. In this case our geometric definition of the invariant $\mathrm{Eu}_{f,X}(0)$ does not make sense. However, one can define the following *defect*.

$$D_{f,X}(0) := \mathrm{Eu}_X(0) - \left(\sum_i \chi(V_i \cap B_\varepsilon \cap f^{-1}(t_0)) \cdot \mathrm{Eu}_X(V_i) \right).$$

This is well defined even when the singularity is non-isolated, and we use a formal, derived category argument to prove that the defect $D_{f,X}(0)$ has a nice interpretation in terms of vanishing cycles (Theorem 5.3). The proof that we give is very short, but it uses a number of basic results, including the index theorem of Brylinski, Dubson and Kashiwara [**5**]. From the defect point of view, what Theorem 3.1 tells us is that, in the case of isolated singularities, $D_{f,X}(0) = \mathrm{Eu}_{f,X}(0)$.

The vanishing cycle interpretation that we give for $D_{f,X}(0)$ allows us to calculate it algebraically. In the case of isolated singularities, we may use another index theorem (proved independently by Ginsburg [**9**], Lê [**13**] and Sabbah [**26**]) to conclude that, up to sign, $D_{f,X}(0)$ is equal to the intersection number of the conormal variety of X with the image of the differential $d\widetilde{f}$ at the point $(0, d_0\widetilde{f})$ (Corollary 5.4).

Thus when X is $\mathbb{C}^d$ and f has an isolated singularity at 0, we obtain $D_{f,X}(0) = (-1)^d\mu$. That is, $D_{f,X}(0)$ coincides (up to sign) with the usual Milnor number of f (compare with Remark 3.4). We can also define the invariant $\mathrm{Eu}_{f,X}(0)$ when f is the function distance to 0 on X; in this case it coincides with the usual local Euler number $\mathrm{Eu}_X(0)$. Hence $\mathrm{Eu}_{f,X}(0)$ unifies these two important notions. The definition of the defect $D_{f,X}(0)$ also makes sense when f is a constant function, and in this case the algebraic interpretation says that $D_{f,X}(0) = \mathrm{Eu}_X(0)$.

Moreover, in Section 6, even when f has non-isolated singularities, we obtain algebraic formulas for $D_{f,X}(0)$ by using the work of Massey. When X is affine space, we use the results of [**19**] on Lê numbers. When X is arbitrary, we use the recent results of [**20**] on Lê–Vogel numbers.

It would be interesting to understand the relationship of this invariant with the *Milnor number* for functions on singular spaces, introduced by Goryunov in [**12**] for functions on curves (see also [**22**]). This latter invariant has an obvious generalization to functions on isolated complete intersection germs, either as the number of critical points (on a Milnor fibre) of a generic perturbation of the given function, or, equivalently, as the GSV-index of the gradient vector field of the given function [**10**].

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1. *Euler obstruction and the Euler condition*

Let us recall the result of [**3**]. Let $(X,0)$ be an equidimensional complex analytic singularity germ of dimension d in an open set $U\subset\mathbb{C}^N$. Let $G(d,N)$ denote the Grassmanian of complex d-planes in $\mathbb{C}^N$. On the regular part $X_{\rm reg}$ of X, there is a map $\sigma: X_{\rm reg}\longrightarrow U\times G(d,N)$, defined by $\sigma(x)=(x,T_x(X_{\rm reg}))$. The Nash transformation $\widetilde{X}$ of X is the closure of $\mathrm{Im}(\sigma)$ in $U\times G(d,N)$. It is a complex analytic space endowed with an analytic projection $\nu:\widetilde{X}\longrightarrow X$, which is a biholomorphism on $\nu^{-1}(X_{\rm reg})$. Let us denote by $U(d,N)$ the tautological bundle over $G(d,N)$, and by $\mathbb{U}$ the corresponding locally trivial extension bundle over $U\times G(d,N)$. The Nash bundle $\widetilde{T}$ on $\widetilde{X}$ is the restriction of $\mathbb{U}$ to $\widetilde{X}$.

Consider a complex analytic stratification $\{V_i\}$ of $U\subset\mathbb{C}^N$, adapted to X. We assume that $\{0\}$ is a stratum and the stratification satisfies the Whitney conditions [**31**]. We choose a representative of the germ of $(X,0)$ sufficiently small so that $\{0\}$ is in the closure of all the strata.

A *stratified vector field* v on X is a continuous section of the restriction $TU|_X$ of TU to X such that, if $x\in V_i\cap X$, then $v(x)\in T_x(V_i)$. Such a vector field is *radial* at $a\in X$ if, for each sufficiently small ball $\mathbb{B}_\varepsilon(a)\subset U$ around a with $\varepsilon>0$, and for each boundary point $x\in\mathbb{S}_\varepsilon(a)=\partial\mathbb{B}_\varepsilon(a)$, the vector $v(x)$ is pointing outwards from the ball $\mathbb{B}_\varepsilon(a)$. Such a vector field necessarily has an isolated zero at a.

One has the following lemma [**2**, Proposition 9.1].

LEMMA 1.1. *Let v be a stratified non-zero vector field on $A\subset X$. Then v can be lifted to a section $\widetilde{v}$ of $\widetilde{T}$ along $\nu^{-1}(A)$.*

PROPOSITION–DEFINITION 1.2 ([**2**, Proposition 10.1], see [**17**] for the original definition). *Let a be a point in X and let v be a radial vector field defined on $X\cap\mathbb{B}_\varepsilon(a)$. Let $\widetilde{v}$ be the lifting of v to $\nu^{-1}(X\cap\mathbb{S}_\varepsilon(a))$ as a section of $\widetilde{T}$. The local Euler obstruction* $\mathrm{Eu}_X(a)$ *is the obstruction to the extension of $\widetilde{v}$ as a nowhere-zero section of $\widetilde{T}$ inside $\nu^{-1}(X\cap\mathbb{B}_\varepsilon(a))$.*

More precisely, let $\mathcal{O}(\widetilde{v})\in H^{2d}(\nu^{-1}(X\cap\mathbb{B}_\varepsilon(a)),\nu^{-1}(X\cap\mathbb{S}_\varepsilon(a)))$ be the obstruction cocycle for extending $\widetilde{v}$ as a nowhere-zero section of $\widetilde{T}$ inside $\nu^{-1}(X\cap\mathbb{B}_\varepsilon(a))$. The local Euler obstruction $\mathrm{Eu}_X(a)$ is the evaluation of $\mathcal{O}(\widetilde{v})$ on the fundamental class of the pair $(\nu^{-1}(X\cap\mathbb{B}_\varepsilon(a)),\nu^{-1}(X\cap\mathbb{S}_\varepsilon(a)))$. The Euler obstruction is an integer. It is constant on each stratum of the Whitney stratification [**2**, **6**].

We consider a complex linear form $l:U\longrightarrow\mathbb{C}$, so that $0\in l^{-1}(0)$, and we assume that the kernel of l, $\ker(l)$, is transversal to every generalized tangent space T at 0, that is, to every limit of tangent spaces $T_{x_n}(V_i)$, for every V_i and every sequence $x_n\in V_i$ converging to 0. Such a linear form is said to be *general* (with respect to X). From now on, we will denote by $\mathbb{B}_\varepsilon$ the ball $\mathbb{B}_\varepsilon(0)$.

THEOREM 1.3 [**3**, Theorem 3.1]. *Let $(X,0)$ and the Whitney stratification $\{V_i\}$ be as above, and let $l:U\longrightarrow\mathbb{C}$ be a general linear form. Then*

$$\mathrm{Eu}_X(0)=\sum_i\chi(V_i\cap\mathbb{B}_\varepsilon\cap l^{-1}(t_0))\cdot\mathrm{Eu}_X(V_i),$$

where ε is sufficiently small, $t_0\in\mathbb{C}\setminus\{0\}$ is close to 0, and $\mathrm{Eu}_X(V_i)$ is the Euler obstruction of X at any point of the stratum V_i.

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REMARK 1.4. In the statement of Theorem 1.3, the closed ball may be replaced by an open ball, and/or the Euler characteristic may be replaced by the compactly-supported Euler characteristic (as in, for instance, [**1**, **25**]); locally, these all agree.

By Poincaré duality, $\chi(V_i \cap \mathbb{B}_\varepsilon \cap l^{-1}(t_0)) = \chi_c(V_i \cap \mathbb{B}^\circ_\varepsilon \cap l^{-1}(t_0))$, and then one may use either formal derived category statements or a result of Sullivan [**30**] to conclude that $\chi_c(V_i \cap \mathbb{B}^\circ_\varepsilon \cap l^{-1}(t_0)) = \chi(V_i \cap \mathbb{B}^\circ_\varepsilon \cap l^{-1}(t_0))$. See the detailed discussion in [**28**].

Alternatively, one may show that $\chi(V_i \cap \mathbb{B}_\varepsilon \cap l^{-1}(t_0)) = \chi(V_i \cap \mathbb{B}^\circ_\varepsilon \cap l^{-1}(t_0))$ by applying Thom's first isotopy lemma to the function distance-squared from the origin. One uses here the a_f (or, here, a_l) argument of Lê from [**16**, Theorem 1.1] to show that $\partial\mathbb{B}_\varepsilon$ transversely intersects the Whitney strata of $X \cap l^{-1}(t_0)$ near the origin.

Theorem 1.3 can be stated through the framework of bivariant theory [**8**]. We follow the formulation given in [**1**]. For this we recall that a function $\alpha : X \longrightarrow \mathbb{Z}$ is *constructible* if, for each $n \in \mathbb{Z}$, the set $\alpha^{-1}(n)$ is constructible, that is, it is obtained as a finite number of unions, intersections and differences of analytic subsets of X. Given a constructible function $\alpha : X \longrightarrow \mathbb{Z}$ and $A \subset X$, the weighted Euler–Poincaré characteristic $\chi(A; \alpha)$ is defined by

$$\chi(A; \alpha) := \sum_{n \in \mathbb{Z}} n \cdot \chi(A \cap \alpha^{-1}(n)),$$

provided that, for all n, $\chi(A \cap \alpha^{-1}(n))$ is defined and is non-zero for a finite number of n.

If $\{V_i\}$ is a stratification of X such that the value of α is constant on each stratum, we denote by $\alpha(V_i)$ its value in any point of V_i. We have, equivalently,

$$\chi(A; \alpha) = \sum_i \chi(A \cap V_i) \cdot \alpha(V_i),$$

provided that A is such that all of the $\chi(A \cap V_i)$ exist and are non-zero for a finite number of i. Of course, locally, we may always assume that there are a finite number of V_i.

Given an analytic map $f : X \longrightarrow \mathbb{C}$ and a constructible function $\alpha : X \longrightarrow \mathbb{Z}$, we say that α *satisfies the local Euler condition with respect to* f if, for all $x \in X$,

$$\alpha(x) = \chi(\mathbb{B}_\varepsilon(x) \cap f^{-1}(t); \alpha),$$

where $\mathbb{B}_\varepsilon(x)$ is a small closed ball in $U \subset \mathbb{C}^N$, $t \in \mathbb{D}(f(x)) \setminus \{f(x)\}$ and $\mathbb{D}(f(x))$ is a small disc in $\mathbb{C}$ centered at $f(x)$ (see [**1**, **25**]). Note that, for α to satisfy the local Euler condition with respect to f, we require the right-hand side of the above equality to be well defined, that is, to be independent of the choices of small, positive ε and the value of t in a relatively small disk around $f(x)$.

From Theorem 1.3, we quickly obtain the following.

THEOREM 1.5. *In an open neighborhood of the origin, the local Euler obstruction, as a constructible function, satisfies the local Euler condition with respect to general linear forms.*

Proof. A generic linear form will have (at worst) an isolated stratified critical point at the origin. Thus there is an open neighborhood W of the origin such

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that, for all $x \in W - \{0\}$, l is a stratified submersion near x. By Thom's first isotopy lemma, the local Euler obstruction satisfies the local Euler condition with respect to a stratified submersion. However, the condition also holds at the origin by Theorem 1.3. □

REMARK 1.6. Recently, Schürmann [**28**] proved that, with the same hypotheses as in Theorem 1.3, if a constructible function α on X is a linear combination of the functions $\mathrm{Eu}_{\overline{V}_j}(*)$, then one has

$$\alpha(0) = \sum_i \chi(V_i \cap \mathbb{B}_\varepsilon \cap l^{-1}(t_0)) \cdot \alpha(V_i).$$

In other words, α satisfies the local Euler condition with respect to general linear forms. The proof in [**28**] uses the index theorem for the vanishing cycle functor. We notice that this result can also be proved using Theorem 1.3, by applying it to the closure of each stratum $\overline{V}_j$.

2. *Local Euler obstruction of a function*

Let us recall some well known concepts about singularity theory that originate in the work of Thom.

Let $(X, 0)$ be a complex analytic germ as above, contained in an open subset U of $\mathbb{C}^N$ and endowed with a complex analytic Whitney stratification $\{V_i\}$. We assume further that $\{0\}$ is a stratum and that every stratum contains 0 in its closure. For every point $x \in X$, we will denote by $V_i(x)$ the stratum containing x. Let $f : X \longrightarrow \mathbb{C}$ be a holomorphic function which is the restriction of a holomorphic function $\widetilde{f} : U \longrightarrow \mathbb{C}$. We recall [**11**] that a *critical point* of f is a point $x \in X$ such that $d\widetilde{f}(x)(T_x(V_i(x))) = 0$. We say, following [**11**, **13**], that f has an isolated singularity at $0 \in X$ relative to the given Whitney stratification if f has no critical points in a punctured neighborhood of 0 in X.

Let us denote by $\overline{\nabla}\widetilde{f}(x)$ the gradient vector field of $\widetilde{f}$ at a point $x \in U$, defined by

$$\overline{\nabla}\widetilde{f}(x) := \left(\overline{\frac{\partial \widetilde{f}}{\partial x_1}}, \ldots, \overline{\frac{\partial \widetilde{f}}{\partial x_N}} \right),$$

where the bar denotes complex conjugation. From now on we assume that f has an isolated singularity at $0 \in X$. This implies that the kernel $\ker(d\widetilde{f})$ is transverse to $T_x(V_i(x))$ in any point $x \in X \setminus \{0\}$. Therefore, at each point $x \in X \setminus \{0\}$, we have

$$\mathrm{Angle}\langle \overline{\nabla}\widetilde{f}(x), T_x(V_i(x)) \rangle < \pi/2,$$

so the projection of $\overline{\nabla}\widetilde{f}(x)$ on $T_x(V_i(x))$, denoted by $\widehat{\zeta}_i(x)$, is not zero.

Let V_j be a stratum such that $V_i \subset \overline{V}_j$, and let $\pi : \mathcal{U}_i \longrightarrow V_i$ be a tubular neighborhood of V_i in U. Following the construction of Schwartz in [**27**, § 2], we see that the Whitney condition (a) implies that at each point $y \in V_j \cap \mathcal{U}_i$, the angle of $\widehat{\zeta}_j(y)$ and the angle of the parallel extension of $\widehat{\zeta}_i(\pi(y))$ are small. This property implies that these two vector fields are homotopic on the boundary of $\mathcal{U}_i$. Therefore, we can glue together the vector fields $\widehat{\zeta}_i$ to obtain a stratified vector field on X, denoted by $\overline{\nabla}_X f$. This vector field is homotopic to $\overline{\nabla}\,\widetilde{f}|_X$ and one has $\overline{\nabla}_X f \neq 0$ unless $x = 0$.

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Intuitively, what we are doing in the construction of $\overline{\nabla}_X f$ is taking, for each stratum V_i of X, the gradient vector field of the restriction of f to V_i, and then gluing all these vector fields together.

DEFINITION 2.1. Let $\nu : \widetilde{X} \longrightarrow X$ be the Nash transform of X. Let $\widetilde{\zeta}$ be the lifting of $\overline{\nabla}_X f$ as a section of the Nash bundle $\widetilde{T}$ over $\widetilde{X}$ without singularity over $\nu^{-1}(X \cap \mathbb{S}_\varepsilon)$. Let $\mathcal{O}(\widetilde{\zeta}) \in H^{2d}(\nu^{-1}(X \cap \mathbb{B}_\varepsilon), \nu^{-1}(X \cap \mathbb{S}_\varepsilon))$ be the obstruction cocycle for the extension of $\widetilde{\zeta}$ as a nowhere-zero section of $\widetilde{T}$ inside $\nu^{-1}(X \cap \mathbb{B}_\varepsilon)$. We define the *local Euler obstruction of f on X* at 0, denoted by $\mathrm{Eu}_{f,X}(0)$, to be the evaluation of $\mathcal{O}(\widetilde{\zeta})$ on the fundamental class of the pair $(\nu^{-1}(X \cap \mathbb{B}_\varepsilon), \nu^{-1}(X \cap \mathbb{S}_\varepsilon))$.

The invariant $\mathrm{Eu}_{f,X}(0)$ is an integer. We notice that all these definitions and constructions also work when f is the restriction to X of a real analytic function on the ambient space. For instance, we can take f to be the function distance to 0 on X; then $\overline{\nabla}_X f$ is a radial vector field as in Proposition–Definition 1.2 and the invariant $\mathrm{Eu}_{f,X}(0)$ is the usual local Euler obstruction of X at 0.

We remark that the usual Hermitian metric on $\mathbb{C}^N$ defines a Riemannian metric which allows us to identify the real vector bundles $T\mathbb{C}^N$ and $T^*\mathbb{C}^N$. The latter is the holomorphic cotangent bundle and under the above identification $d\widetilde{f}$ corresponds to the conjugate gradient vector field.

A reason for considering the conjugate gradient vector field $\overline{\nabla}_X f$, and not the usual gradient vector field $\nabla_X f$, is given in the following lemma, where f can be taken to be the restriction to X of either a real or a complex analytic function on the ambient space.

LEMMA 2.2. *The vector field $\overline{\nabla}_X f$ is the lifting, up to homotopy, of a vector field on $\mathbb{C}$, via $d\widetilde{f}$.*

Proof. The gradient vector field satisfies

$$d\widetilde{f}(\overline{\nabla}\widetilde{f}(x)) = \|\overline{\nabla}\widetilde{f}(x)\|^2 \in \mathbb{R} \setminus \{0\} \qquad \text{for } x \in X \setminus \{0\}.$$

This means that it is the lifting, up to scaling, of a constant vector field on a small disk $\mathbb{D}_\eta \subset \mathbb{C}$. □

From now on we consider only complex analytic functions. If 0 is a smooth point of X and also a regular point of f, then $\mathrm{Eu}_{f,X}(0) = 0$. In Proposition 2.4, we prove that this is the case in a more general situation.

DEFINITION 2.3. Let $(X, 0) \subset (U, 0)$ be a germ of analytic set in $\mathbb{C}^N$ equipped with a Whitney stratification, and let $f : (X, 0) \longrightarrow (\mathbb{C}, 0)$ be an analytic map that is a restriction of a regular holomorphic function

$$\widetilde{f} : (U, 0) \longrightarrow (\mathbb{C}, 0).$$

We say that 0 *is a general point of* f if the hyperplane $\ker d\widetilde{f}(0)$ is transversal in $\mathbb{C}^N$ to every generalized tangent space at 0, that is, to every limit of tangent spaces $T_{x_n}(V_i)$, for every V_i and every sequence $x_n \in V_i$ converging to 0.

We notice that for every f as above, the general points of f form a non-empty open set on each (open) stratum of X, essentially by Sard's theorem. We also remark

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that Definition 2.3 provides a coordinate-free way of looking at general linear forms. In fact the previous definition is equivalent to saying that, with an appropriate local change of coordinates, $\widetilde{f}$ is a linear form in U, and it is general with respect to X (compare with [**3**, Lemma 1.3]).

PROPOSITION 2.4. *Let 0 be a general point of $f:(X,0)\longrightarrow(\mathbb{C},0)$. Then*

$$\mathrm{Eu}_{f,X}(0)=0.$$

Proof. The proof of this proposition is implicit within [**3**, proof of 2.3] and it can also be obtained as a corollary of Theorem 3.1 and [**3**, Theorem 3.1]. However, for completeness, we outline the proof here. As a first step, define the map

$$\widetilde{T}\subset(U\times G(d,N))\times\mathbb{C}^N\xrightarrow{\widetilde{F}}\mathbb{D}_\eta\subset\mathbb{C}$$

by $\widetilde{F}(x,P,y)=d\widetilde{f}_x(y)$. As 0 is a general point of f, then $\widetilde{K}=\widetilde{T}\cap\widetilde{F}^{-1}(0)$ is a sub-bundle of $\widetilde{T}$ of (complex) codimension 1 and $d\widetilde{F}$ maps the orthogonal complement of $\widetilde{K}$ isomorphically over $T(\mathbb{D}_\eta)$ (see [**3**] for details).

Now, as we show in the proof of Lemma 4.2, every $\widetilde{f}$ defining an isolated singularity at 0 in X determines a sub-bundle Q of $T\mathbb{C}^N|_{X-\{0\}}$ everywhere transversal to $\ker(d\widetilde{f})$, and the restriction of $d\widetilde{f}$ to Q is an isomorphism between Q and $T(\mathbb{D}_\eta)$.

This implies that each nowhere-zero vector field on $\mathbb{D}_\eta$ lifts compatibly to a vector field on $X-\{0\}$ and also as a section of $\widetilde{T}|_X$. The final step is to notice, as we did in Lemma 2.2, that the gradient vector field $\overline{\nabla}_X f$ can be obtained by lifting such a vector field. □

PROPOSITION 2.5. *Let $f:(X,0)\longrightarrow(\mathbb{C},0)$ be holomorphic at 0. Then there exists a Zariski open subset Ω_f of the space of complex linear forms on $\mathbb{C}^N$ such that for all $l\in\Omega_f$, the point 0 is general for the map $f+\lambda l:X\longrightarrow\mathbb{C}$, for all sufficiently small $\lambda\in\mathbb{C}^*$.*

Proof. This is [**21**, Lemma 5.1]. Notice that in the particular case where $\{0\}$ is an isolated stratified critical point of f, one can use the proof of [**14**, Morsification Theorem 2.2]. □

3. *The theorem*

THEOREM 3.1. *Let $f:(X,0)\longrightarrow(\mathbb{C},0)$ have an isolated singularity at $0\in X$. One has*

$$\mathrm{Eu}_X(0)=\left(\sum_i\chi(V_i\cap\mathbb{B}_\varepsilon\cap f^{-1}(t_0))\cdot\mathrm{Eu}_X(V_i)\right)+\mathrm{Eu}_{f,X}(0).$$

Theorem 3.1 is proved using Lemma 3.2. To state this lemma, let us fix some notation. We choose $\varepsilon>0$ sufficiently small so that every sphere $\mathbb{S}_\gamma$ in U centered at 0 and with radius $\gamma\leqslant\varepsilon$ intersects transversally every stratum in $X\setminus\{0\}$. For each $t\in\mathbb{C}$, set $Y_t:=f^{-1}(t)$. Choose $\eta>0$ small enough so that for each t in the disk $\mathbb{D}_\eta$ of radius η around $0\in\mathbb{C}$, the hypersurface Y_t intersects transversally the sphere $\mathbb{S}_\varepsilon$. Now choose ε' with $0<\varepsilon'<\varepsilon$, and a point $t_0\in\mathbb{D}_\eta$ such that Y_{t_0} does not meet the sphere $\mathbb{S}_{\varepsilon'}$. We notice that the strata V_i intersect $Y_{t_0}:=f^{-1}(t_0)$ transversally and provide a Whitney stratification of this space.

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LEMMA 3.2. *There is a stratified vector field w on $X_{\varepsilon',\varepsilon} = X \cap (\mathbb{B}_\varepsilon \setminus \mathrm{Int}(\mathbb{B}_{\varepsilon'}))$ such that the following hold.*

(i) *It coincides with $\overline{\nabla}_X f$ on $X \cap \mathbb{S}_{\varepsilon'}$ and its restriction to $X \cap \mathbb{S}_\varepsilon$ is radial.*

(ii) *It is tangent to Y_{t_0}.*

(iii) *w has only a finite number of zeros, and they are all contained in Y_{t_0}.*

(iv) *At each zero a, w is transversally radial to the stratum containing a (that is, it is transversal to the boundary of a tubular neighborhood of the stratum).*

Proof of Theorem 3.1 (assuming Lemma 3.2). We first notice that if ξ is a stratified vector field on a neighborhood of $\{0\}$ in X which is everywhere transversal to a small sphere $\mathbb{S}_\varepsilon$, then ξ is homotopic to a radial vector field, by elementary obstruction theory. Hence, to compute the Euler obstruction using Proposition–Definition 1.2, it is enough to consider vector fields transversal to $\mathbb{S}_\varepsilon$.

The restriction of the vector field w of Lemma 3.2 to $\partial(X_{\varepsilon',\varepsilon})$ is a stratified vector field, so it can be lifted as a section $\widetilde{w}$ of the Nash bundle $\widetilde{T}$ on $\nu^{-1}(\partial(X_{\varepsilon',\varepsilon}))$ by Lemma 1.1. Let us denote by $\mathrm{Obs}(\widetilde{w}, \nu^{-1}(X_\varepsilon))$ the obstruction for extending $\widetilde{w}$ to $\nu^{-1}(X_\varepsilon)$. One has

$$\mathrm{Obs}(\widetilde{w}, \nu^{-1}(X_\varepsilon)) = \mathrm{Obs}(\widetilde{w}, \nu^{-1}(X_{\varepsilon'})) + \mathrm{Obs}(\widetilde{w}, \nu^{-1}(X_{\varepsilon',\varepsilon})).$$

By statement Lemma 3.2(i), this formula becomes

$$\mathrm{Eu}_X(0) = \mathrm{Eu}_{f,X}(0) + \mathrm{Obs}(\widetilde{w}, \nu^{-1}(X_{\varepsilon',\varepsilon})).$$

By statement Lemma 3.2(iii), the contribution of $\mathrm{Obs}(\widetilde{w}, \nu^{-1}(X_{\varepsilon',\varepsilon}))$ is concentrated on $\nu^{-1}(Y_{t_0} \cap \mathbb{B}_\varepsilon)$. Statements (iii) and (iv), together with the theorem of proportionality [**2**, Théorème 11.1], imply that the contribution of each singularity x of w to $\mathrm{Obs}(\widetilde{w}, \nu^{-1}(X_{\varepsilon',\varepsilon}))$ is $\mathrm{Eu}_X(x)$-times the local Poincaré–Hopf index of w at x, regarded as a vector field on the stratum $V_i(x)$. Furthermore (by (ii) and (iv)), the sum of the Poincaré–Hopf indices of the restriction of w to $V_i \cap Y_{t_0}$ is $\chi(V_i \cap Y_{t_0} \cap \mathbb{B}_\varepsilon)$. Theorem 3.1 then follows. □

The next result follows immediately from Theorem 3.1 and Proposition 2.5.

COROLLARY 3.3. *Let $f : (X, 0) \longrightarrow (\mathbb{C}, 0)$ have an isolated singularity at 0. Choose a linear form $l \in \Omega_f$ as in Proposition* 2.5, *and $\lambda \in \mathbb{C}^*$ so that $f_\lambda := f + \lambda l$ is general at 0. Let $M_{f,X}$ and $M_{f_\lambda,X}$ be the Milnor fibres of f and f_λ, respectively, on X. Then*

$$\mathrm{Eu}_{f,X}(0) = \sum_i [\chi(V_i \cap M_{f_\lambda,X}) - \chi(V_i \cap M_{f,X})] \cdot \mathrm{Eu}_X(V_i).$$

REMARK 3.4. We notice that if $X = \mathbb{C}^d$ and $f : (\mathbb{C}^d, 0) \longrightarrow (\mathbb{C}, 0)$ is an analytic map with an isolated singularity at 0 with Milnor number μ, then Theorem 3.1 implies that

$$\mathrm{Eu}_{f,X}(0) = (-1)^d \mu,$$

because the Euler–Poincaré characteristic of the Milnor fibre of f is equal to $1 + (-1)^{d-1}\mu$ by [**23**].

This formula implies [**23**, Theorem 7.2]. In fact, since in this case the variety $X = \mathbb{C}^d$ is smooth, its Nash transform is $\mathbb{C}^d$ and the Nash bundle $\widetilde{T}$ is the tangent bundle of $\mathbb{C}^d$. Hence, by definition, $\mathrm{Eu}_{f,X}(0)$ is the Poincaré–Hopf index at 0 of the

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gradient vector field

$$\overline{\nabla} f = \left(\frac{\overline{\partial f}}{\partial x_1}, \ldots, \frac{\overline{\partial f}}{\partial x_N} \right).$$

This equals $(-1)^d$ times the Poincaré–Hopf index at 0 of the vector field

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_N} \right).$$

Thus we have that the Milnor number μ is the degree of ∇f, which is Milnor's [**23**, Theorem 7.2].

4. *Proof of Lemma* 3.2

The vector field w of Lemma 3.2 is defined as the sum of two vector fields that we construct in Lemmas 4.1 and 4.2. Let us choose $\eta' > 0$ sufficiently small with respect to η such that the disc $\mathbb{D}_{\eta'}(t_0)$ of radius η', centered at t_0, is contained in the interior of $\mathbb{D}_\eta$ and $f^{-1}(\mathbb{D}_{\eta'}(t_0))$ does not intersect the sphere $\mathbb{S}_{\varepsilon'}$.

LEMMA 4.1. *There exists a stratified vector field r on $\mathbb{B}_\varepsilon \cap f^{-1}(\mathbb{D}_\eta)$ satisfying the following.*

(i) *The restriction of r to $\mathbb{S}_\varepsilon \cap f^{-1}(\mathbb{D}_\eta)$ is tangent to all the fibers $f^{-1}(t)$, and is transversal to $\mathbb{S}_\varepsilon$, pointing outwards.*

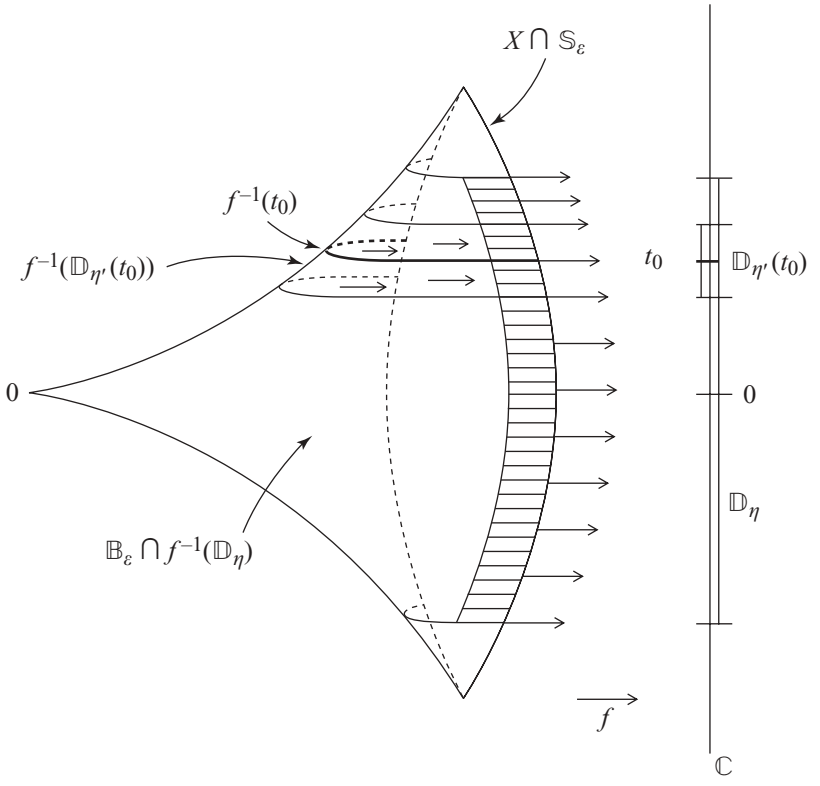


FIGURE 1. *The vector field r.*

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(ii) *r is tangent to the fiber $f^{-1}(t_0)$, where it has only isolated singularities (zeros). Furthermore, at each zero, r is transversally radial (in $f^{-1}(t_0)$) to the stratum which contains that zero.*

(iii) *r is tangent to the fiber $f^{-1}(t)$ for all $t \in \mathbb{D}_{\eta'}(t_0)$.*

Proof. We first construct r on $f^{-1}(t_0)$ satisfying (ii) and pointing out of $\mathbb{S}_\varepsilon$. This is done using the technique of Schwartz, as in the proof of [**3**, Lemma 3.4]. The proof is now similar to the proof of [**3**, Lemma 2.1]: by the fibration theorem of Lê (see [**14**, Theorem 1.3]), the function f determines a stratified locally trivial fibration of $\mathbb{B}_\varepsilon \cap f^{-1}(\mathbb{D}_\eta) \setminus f^{-1}(0)$ over $\mathbb{D}_\eta \setminus \{0\}$. This induces, in particular, a trivial fibration over $\mathbb{D}_{\eta'}(t_0)$. Hence r can be extended, as a product, to all the fibers over $\mathbb{D}_{\eta'}(t_0)$, satisfying statement (iii).

Let us choose ε'', with $\varepsilon' \ll \varepsilon'' < \varepsilon$, and such that the restriction of f to $f^{-1}(\mathbb{D}_\eta) \cap (\mathbb{B}_\varepsilon \setminus \mathrm{Int}(\mathbb{B}_{\varepsilon''}))$ (the cross-hatched part of Figure 1) is a trivial fibration. Since $\mathbb{D}_\eta$ retracts to $\mathbb{D}_{\eta'}(t_0)$, the vector field r can be extended to $\mathbb{S}_\varepsilon \cap f^{-1}(\mathbb{D}_\eta)$ being tangent to the fibers of f and transversal to the sphere $\mathbb{S}_\varepsilon$, pointing outwards.

Using a suitable partition of unity, we can extend r as zero in the complement of a neighborhood of $(f^{-1}(\mathbb{D}_\eta) \cap (\mathbb{B}_\varepsilon \setminus \mathrm{Int}(\mathbb{B}_{\varepsilon''}))) \cup f^{-1}(\mathbb{D}_{\eta'}(t_0))$ (see Figure 1). □

Notice that statements Lemma 4.1(i) and (ii) imply that for each stratum $Y_{t_0} \cap V_i \cap B_\varepsilon$, the sum of the Poincaré–Hopf indices of the restriction of r is $\chi(Y_{t_0} \cap V_i \cap B_\varepsilon)$.

LEMMA 4.2. *There exists a stratified vector field u defined on $(\mathbb{B}_\varepsilon \cap f^{-1}(\mathbb{D}_\eta)) \setminus \{0\}$ and satisfying the following.*

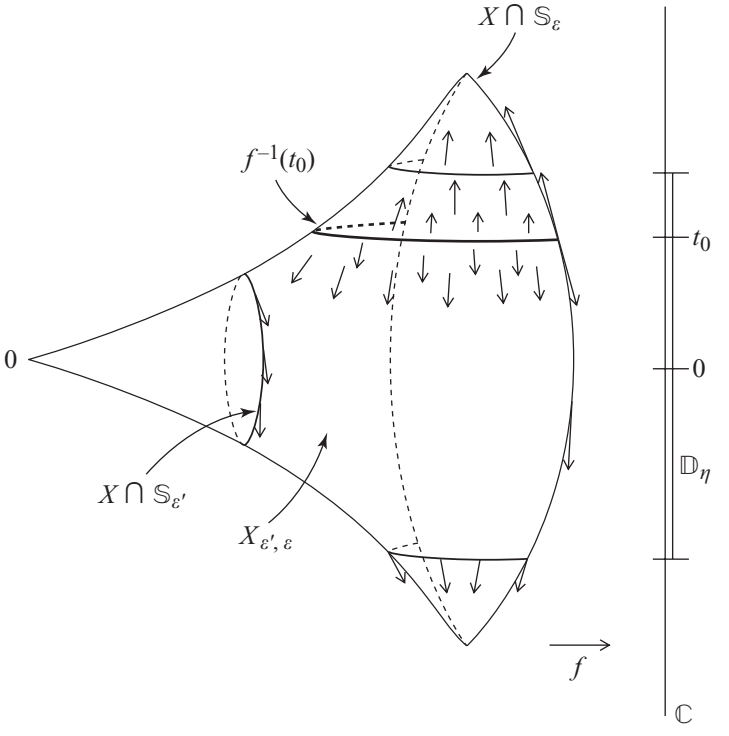


FIGURE 2. *The vector field u.*

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(i) *u is tangent to $\mathbb{S}_\varepsilon$.*
(ii) *Its zero set is $f^{-1}(t_0)$, and u is transversally radial to $f^{-1}(t_0)$.*
(iii) *u is transversal to $X \cap f^{-1}(\partial\mathbb{D}_\eta)$, pointing outwards.*

Proof. By hypothesis, the restriction of f to every stratum V_i (other than $\{0\}$) is regular; hence the kernel of df has codimension 1 in V_i. The Hermitian metric on $\mathbb{C}^N$ induces a metric on V_i and defines a splitting of the tangent bundle TV_i as the sum of the tangent bundle to the fiber and the normal bundle. The derivative df, restricted to the normal bundle, is an isomorphism. Therefore, we can lift every vector field on $\mathbb{D}_\eta$ as a vector field tangent to V_i and 'orthogonal' to the fibers $f^{-1}(t)$. Let us denote by ξ the vector field on $\mathbb{D}_\eta$, radial from t_0, and by u_i the lifting of ξ to V_i (see Figure 2).

Just as in the definition of the vector field $\overline{\nabla}_X f$, the Whitney conditions allow us to glue the different u_i in a stratified vector field u defined on $(\mathbb{B}_\varepsilon \cap f^{-1}(\mathbb{D}_\eta)) \setminus \{0\}$ and satisfying the conditions of the lemma. □

Proof of Lemma 3.2. We first define the vector field w on $(\mathbb{B}_\varepsilon \cap f^{-1}(\mathbb{D}_\eta)) \setminus \mathrm{Int}(\mathbb{B}_{\varepsilon'})$ as the sum (via a partition of unity) of the vector fields that we constructed in Lemmas 4.1 and 4.2. We obtain a stratified vector field satisfying the following.

(i) It is transverse to the sphere $\mathbb{S}_\varepsilon$, pointing outwards.
(ii) It is tangent to $Y_{t_0} = f^{-1}(t_0)$.
(iii) The singularities of w are all contained in Y_{t_0}.
(iv) On each stratum $Y_{t_0} \cap V_i \cap \mathbb{B}_\varepsilon$, w only vanishes at a finite number of points and the sum of the Poincaré–Hopf indices of the restriction of w is $\chi(Y_{t_0} \cap V_i \cap \mathbb{B}_\varepsilon)$.
(v) At each of its singular points, w is transversally radial to the stratum that contains this singular point.
(vi) It is transverse to the boundary $f^{-1}(\partial\mathbb{D}_\eta)$, pointing outwards.

Then we extend w to all of $X_{\varepsilon',\varepsilon}$ using in [**3**, Theorem 2.3]. To complete the proof of Lemma 3.2, we notice that, restricted to $\mathbb{S}_{\varepsilon'}$, the vector field w is homotopic to the gradient vector field $\overline{\nabla}_X f$. This fact follows from Lemma 2.2 and the fact that, on $\mathbb{S}_{\varepsilon'}$, the vector field w is the vector field u of Lemma 4.2, which is homotopic to a constant vector field near 0. □

5. *Interpretation in terms of vanishing cycles*

Recall our notation from Section 2: X is a purely d-dimensional analytic subset of an open subset U of $\mathbb{C}^N$, f is the restriction to X of a complex analytic function $\widetilde{f} : (U, 0) \longrightarrow (\mathbb{C}, 0)$, and $\{V_i\}$ is a complex analytic Whitney stratification of X.

In Definition 2.1, we defined the local Euler obstruction of f in what we consider to be the most natural way, as a geometric obstruction. We then proved in Theorem 3.1 that when f has an isolated singularity, the local Euler obstruction of f measures the defect in f satisfying the local Euler condition.

In this section, we use a formal, derived category argument to prove that the defect

$$D_{f,X}(0) := \mathrm{Eu}_X(0) - \left(\sum_i \chi(V_i \cap B_\varepsilon \cap f^{-1}(t_0)) \cdot \mathrm{Eu}_X(V_i) \right)$$

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has a nice interpretation in terms of vanishing cycles, even when the singularity is non-isolated.

Let $\mathbf{A}^\bullet$ be a bounded complex of sheaves of complex vector spaces on X which is constructible with respect to $\{V_i\}$. For each V_i, we denote the closure of the conormal variety of V_i in U by $\overline{T^*_{V_i}U}$. We let $\psi_f\mathbf{A}^\bullet$ and $\phi_f\mathbf{A}^\bullet$ denote the nearby and vanishing cycles of $\mathbf{A}^\bullet$ along f, respectively.

We let

$$\mathrm{Ch}(\mathbf{A}^\bullet) = \sum_i m_i[\overline{T^*_{V_i}U}]$$

denote the characteristic cycle of $\mathbf{A}^\bullet$ in U. As there are two or three different definitions of the characteristic cycle, differing by multiples of -1, we need to be clear; we mean that

$$m_i := (-1)^d\chi(\phi_{L_{|X}}[-1]\mathbf{A}^\bullet)_p = (-1)^{d-d_i}\chi\big(\phi_{L_{|N_i}}[-1]\mathbf{A}^\bullet{}_{|N_i}\big)_p$$

for any point p in the d_i-dimensional stratum V_i, with normal slice N_i to V_i at p, and any $L:(U,p)\longrightarrow(\mathbb{C},0)$ such that d_pL is a non-degenerate covector at p (with respect to $\{V_i\}_i$), and $L_{|V_i}$ has a Morse singularity at p.

The advantages of this definition of $\mathrm{Ch}(\mathbf{A}^\bullet)$ are as follows.

(i) The m_i are intrinsic to X and $\mathbf{A}^\bullet$; they do *not* depend on the dimension of the ambient space U.

(ii) If X is a closed submanifold of U, then $\mathrm{Ch}(\mathbb{C}^\bullet_X) = T^*_XU$.

The disadvantage of our definition of $\mathrm{Ch}(\mathbf{A}^\bullet)$ is that, if we extend $\mathbf{A}^\bullet$ by zero to all of U, the characteristic cycle changes; it is multiplied by $(-1)^{N-d}$.

With our definition of $\mathrm{Ch}(\mathbf{A}^\bullet)$, the index formula of Brylinski, Dubson and Kashiwara [**5**] becomes

$$\chi(\mathbf{A}^\bullet)_x = \sum_i (-1)^{d-d_i} m_i\,\mathrm{Eu}_{\overline{V}_i}(x). \tag{1}$$

Furthermore, the vanishing cycle index theorem of Dubson [**7**], Ginsburg [**9**], Lê [**13**, **15**] and Sabbah [**26**] becomes as follows. If 0 is an isolated point in the support of $\phi_f[-1]\mathbf{A}^\bullet$, then

$$\chi(\phi_f[-1]\mathbf{A}^\bullet)_0 = (-1)^d(\mathrm{Ch}(\mathbf{A}^\bullet)\cdot \mathrm{im}\, d\widetilde{f})_{(0,d_0\widetilde{f})}. \tag{2}$$

LEMMA 5.1. *There exists a complex $\mathbf{A}^\bullet$ on X, constructible with respect to $\{V_i\}$, such that*

$$\mathrm{Ch}(\mathbf{A}^\bullet) = [\overline{T^*_{X_{\mathrm{reg}}}U}].$$

Proof. This is a trivial induction. See, for instance, [**18**, Lemma 3.1]. □

DEFINITION 5.2. We call a complex such as that of Lemma 5.1 a *characteristic complex for X* (constructible with respect to $\{V_i\}$).

While the setting is different, the proof of the following theorem is essentially identical to the proof of [**28**, Theorem 0.2].

THEOREM 5.3. *Let $\mathbf{A}^\bullet$ be a characteristic complex for X. Then $D_{f,X}(0)$ equals the negative of the Euler characteristic of the stalk cohomology of $\phi_f\mathbf{A}^\bullet$ at the*

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origin, that is,

$$\mathrm{Eu}_X(0) = \left(\sum_i \chi(V_i \cap B_\varepsilon \cap f^{-1}(t_0)) \cdot \mathrm{Eu}_X(V_i)\right) - \chi(\phi_f \mathbf{A}^\bullet)_0.$$

Proof. As the local Euler obstruction is additive over the irreducible components of X, formula (1) implies that, for any point $x \in X$, $\chi(\mathbf{A}^\bullet)_x$ is equal to the local Euler obstruction $\mathrm{Eu}_X(x)$.

By the distinguished triangle relating the nearby cycles $\psi_f \mathbf{A}^\bullet$ and the vanishing cycles $\phi_f \mathbf{A}^\bullet$, we obtain $\chi(\mathbf{A}^\bullet)_0 = \chi(\psi_f \mathbf{A}^\bullet)_0 - \chi(\phi_f \mathbf{A}^\bullet)_0$.

Now, using the notation from Theorem 3.1,

$$\chi(\psi_f \mathbf{A}^\bullet)_0 = \chi(\mathbb{H}^*(\mathbb{B}_\varepsilon \cap f^{-1}(t_0); \mathbf{A}^\bullet)) = \sum_i \chi(V_i \cap \mathbb{B}_\varepsilon \cap f^{-1}(t_0)) \cdot \chi(\mathbf{A}^\bullet)_{p_i},$$

where p_i is any point in V_i. As in Remark 1.4, in the expression $\chi(V_i \cap \mathbb{B}_\varepsilon \cap f^{-1}(t_0))$, we may use the open ball $\mathbb{B}_\varepsilon^\circ$ and/or the compactly supported Euler characteristic χ_c. In fact, the additivity property expressed in the second equality is a global property if one uses χ_c (and there are a finite number of strata).

Combining the above steps, we obtain the desired formula. □

In order for Theorem 5.3 not to contradict Theorem 1.3, we need to be able to show that $\chi(\phi_f \mathbf{A}^\bullet)_0 = 0$ if f is a generic linear form. Certainly, a generic linear form l will have an isolated stratified critical point at the origin, and

$$(0, d_0 l) \notin \overline{T^*_{X_{\mathrm{reg}}} U},$$

so the following corollary tells us that, in fact, $\chi(\phi_l \mathbf{A}^\bullet)_0 = 0$.

COROLLARY 5.4. *Let $\mathbf{A}^\bullet$ be a characteristic complex for X.*

If f has an isolated stratified singularity at 0, then $(0, d_0\widetilde{f})$ is an isolated point of the intersection

$$\overline{T^*_{X_{\mathrm{reg}}} U} \cap \mathrm{im}\, d\widetilde{f},$$

and

$$\mathrm{Eu}_{f,X}(0) = -\chi(\phi_f \mathbf{A}^\bullet)_0 = (-1)^d \big(\overline{T^*_{X_{\mathrm{reg}}} U} \cdot \mathrm{im}\, d\widetilde{f}\big)_{(0,d_0\widetilde{f})},$$

where this intersection number is equal to the Milnor number of f at the origin in the case where X is affine space.

Proof. The first equality follows immediately from Theorem 5.3 and Theorem 3.1. The second equality follows from formula (2). That the intersection number equals the Milnor number in the affine case is well known; see, for instance, [**20**, Proposition I.2.19]. □

REMARK 5.5. Our proof above that $\mathrm{Eu}_{f,X}(0) = -\chi(\phi_f \mathbf{A}^\bullet)_0$, if f has an isolated singularity, is indirect. The referee of this paper pointed out that a more direct proof is possible.

When $(0, d_0\widetilde{f})$ is an isolated point of the intersection

$$\overline{T^*_{X_{\mathrm{reg}}} U} \cap \mathrm{im}\, d\widetilde{f},$$

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then, over a neighborhood of 0, $\operatorname{im} d\widetilde{f}$ induces a section of the dual Nash bundle $\widetilde{T^*} \cong \widetilde{T}$, which is nowhere-zero outside of $\nu^{-1}(0)$. Then, as in Definition 2.1, one can use this section to define an obstruction. In the case where f has a stratified isolated singularity, this new obstruction agrees with $\mathrm{Eu}_{f,X}(0)$.

However, there now remains a substantial argument to be made in order for us to conclude that the obstruction defined via $\operatorname{im} d\widetilde{f}$ is equal to

$$(-1)^d \big(\overline{T^*_{X_{\mathrm{reg}}} U} \cdot \operatorname{im} d\widetilde{f}\big)_{(0,d_0\tilde{f})}.$$

Instead of using the roundabout method of comparing the formulas of Theorems 3.1 and 5.3, one can use the more general and more direct argument given by Schürmann in [**29**, Section 5.2]. Schürmann's dr is replaced by our $d\widetilde{f}$.

We remark that if one creates the same construction as above for the real distance function, one gets MacPherson's definition of the local Euler obstruction [**17**]. The agreement corresponding with our Proposition–Definition 1.2 is then contained in [**2**, Proposition 10.1].

In particular, Theorem 1.3 and Proposition 2.4 are true under the weaker assumption that $\operatorname{im}(d\widetilde{f})$ does not intersect

$$\overline{T^*_{X_{\mathrm{reg}}} U},$$

that is, $d\widetilde{f}$ does not vanish on any generalized tangent space coming only from points in X_{reg}, as in [**28**].

Finally, one can show that this new definition of $\mathrm{Eu}_{f,X}(0)$ gives the 'micro-local intersection number' of Corollary 5.4. The corresponding result for the Euler obstruction of MacPherson comes from [**7**, Theorem 1, Proposition 2, p. 114].

6. *Calculation in terms of Lê numbers*

In this section we describe an algebraic method for calculating $\chi(\phi_f \mathbf{A}^\bullet)_0$, when $\mathbf{A}^\bullet$ is a characteristic complex for X, even when the critical locus of f is non-isolated. In the affine case, this method uses Lê cycles and numbers, as described in [**19**]. When X is arbitrary, one must use the more general Lê–Vogel cycles and numbers, as described in [**20**].

In either case, our method of calculation requires a 'generic' choice of coordinates for the ambient, affine space U. This choice of coordinates is made as follows. Refine the stratification $\{V_i\}$ to a stratification $\mathcal{W} := \{W_j\}$ such that $\mathcal{W}$ satisfies Thom's a_f condition and such that $f^{-1}(0)$ is a union of strata of $\mathcal{W}$. While this refinement certainly produces non-isolated $\mathcal{W}$-stratified critical points of f, this will cause us no difficulty as we are considering the non-isolated case. Notice that by the results of [**4**, **24**], a Whitney refinement of $\{V_i\}$ such that $f^{-1}(0)$ is a union of strata would satisfy Thom's a_f condition; however, we are specifically *not* assuming that $\mathcal{W}$ is still a Whitney stratification. That one does not require the Whitney conditions on the refined stratification makes it easier to produce such refinements.

Choose the first coordinate z_1 so that the hyperplane $z_1^{-1}(0)$ transversely intersects, in some neighborhood of the origin, all positive-dimensional strata of $\{W_j\}$. Then there is an induced stratification (on the germ at the origin) of $X \cap z_1^{-1}(0)$ given by $\{W_j \cap z_1^{-1}(0)\}$. We choose z_2 so that $z_2^{-1}(0)$ transversely intersects, in some neighborhood of the origin, all positive-dimensional strata of $\{W_j \cap z_1^{-1}(0)\}$. We continue in this inductive manner to choose $z_1, \ldots, z_N$. We call such a coordinate choice *prepolar* (at the origin).

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Prepolar coordinates are not as generic as possible, but they are generic enough for our purposes. Being prepolar at the origin implies that the coordinates are also prepolar at each point in a neighborhood of the origin, and we assume that we are in such a neighborhood throughout the remainder of Section 6. Note that if we began (before we refined $\{V_i\}$) with an X that had a one-dimensional singular set at the origin and an f with a one-dimensional stratified critical locus, then the choice of prepolar coordinates is trivial.

Fix a choice of prepolar coordinates $(z_1, \ldots, z_N)$.

6.1. *The affine case*

We now describe the case where $X = U$ is an open subset of affine space. This case will be a good introduction to the general case.

In this case, we may take $\mathbf{A}^\bullet$ to be the constant sheaf $\mathbb{C}^\bullet_U$. We write Σf for the ordinary critical locus of $f : U \longrightarrow \mathbb{C}$. Our problem is to calculate $\chi(\phi_f \mathbb{C}^\bullet_U)_0$.

As described in [**19**], this Euler characteristic is an alternating sum of Lê numbers λ^k_f, which are defined as intersection numbers of linear subspaces with the Lê cycles. We need to define these devices.

Consider the analytic cycle $[U]$. This cycle can be written as a sum of purely N-dimensional cycles

$$[U] = \Gamma^N_f + \Lambda^N_f,$$

where no component of Γ^N_f is contained in Σf and every component of Λ^N_f is contained in Σf. Of course, if f is not constant on a connected component of U, we have $[U] = \Gamma^N_f$.

Now we define Γ^k_f and Λ^k_f by downward induction. If we have defined the purely $(k+1)$-dimensional cycle Γ^{k+1}_f, then the hypersurface (cycle)

$$\left[V\left(\frac{\partial f}{\partial z_{k+1}}\right)\right]$$

properly intersects Γ^{k+1}_f inside U (this is a result of the coordinates being prepolar), and therefore there is a well defined, purely k-dimensional intersection cycle

$$\Gamma^{k+1}_f \cdot \left[V\left(\frac{\partial f}{\partial z_{k+1}}\right)\right],$$

which we can decompose as

$$\Gamma^{k+1}_f \cdot \left[V\left(\frac{\partial f}{\partial z_{k+1}}\right)\right] =: \Gamma^k_f + \Lambda^k_f,$$

where no component of Γ^k_f is contained in Σf and every component of Λ^k_f is contained in Σf.

The cycle Λ^k_f is the *k-dimensional Lê cycle of f*. As our coordinates are prepolar, Λ^k_f properly intersects the linear subspace $V(z_1, \ldots, z_k)$ at the origin, and we define the *k-dimensional Lê number of f at* 0 to be the intersection number

$$\lambda^k_f(0) := \left(\Lambda^k_f \cdot V(z_1, \ldots, z_k)\right)_0.$$

Note that Λ^0_f is purely 0-dimensional; thus $\lambda^0_f(0)$ is simply the coefficient of $[0]$ in Λ^0_f.

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PROPOSITION 6.1. *If we let* $s := \dim_0 \Sigma f$, *then the Lê numbers* $\lambda_f^k(0)$ *are zero if* $k > s$, *and the Euler characteristic of the vanishing cycles is given by*

$$D_{f,X}(0) = -\chi(\phi_f \mathbb{C}_U^\bullet)_0 = \sum_{k=0}^{s} (-1)^{N-k} \lambda_f^k(0).$$

In the isolated case, $s = 0$ and $\lambda_f^0(0)$ is simply the Milnor number of f at the origin. Hence we recover the result stated in Remark 3.4.

6.2. *The general case*

We redo the Lê cycle and number construction, but we work in the cotangent bundle to U and with conormal varieties.

The cotangent bundle $T^*U \xrightarrow{\pi} U$ is isomorphic to the trivial bundle $U \times \mathbb{C}^N \longrightarrow U$. The choice of the prepolar coordinates $(z_1, \ldots, z_N)$ determines a basis $dz_1, \ldots, dz_N$ for the cotangent vectors. We use $w_1, \ldots, w_N$ for coordinates with respect to this basis. We carry out our calculations with coordinates $(z_1, \ldots, z_N, w_1, \ldots, w_N)$ on $U \times \mathbb{C}^N$.

Recall that $\widetilde{f} : U \longrightarrow \mathbb{C}$ is our extension of f to all of U. Note that in our coordinates, the image of $d\widetilde{f}$ is given by

$$\operatorname{im} d\widetilde{f} := V\left(w_1 - \frac{\partial \widetilde{f}}{\partial z_1}, \ldots, w_N - \frac{\partial \widetilde{f}}{\partial z_N}\right).$$

It will be important to us that the projection π, restricted to $\operatorname{im} d\widetilde{f}$, induces an isomorphism onto its image; the inverse map sends a point p to

$$\left(p, \frac{\partial \widetilde{f}}{\partial z_1}_{|_p}, \ldots, \frac{\partial \widetilde{f}}{\partial z_N}_{|_p}\right).$$

The cycle

$$[\overline{T^*_{X_{\mathrm{reg}}} U}]$$

can be written as a sum of purely N-dimensional cycles

$$[\overline{T^*_{X_{\mathrm{reg}}} U}] = \widehat{\Gamma}_f^N + \widehat{\Lambda}_f^N,$$

where no component of $\widehat{\Gamma}_f^N$ is contained in $\operatorname{im} d\widetilde{f}$ and every component of $\widehat{\Lambda}_f^N$ is contained in $\operatorname{im} d\widetilde{f}$.

Now we define $\widehat{\Gamma}_f^k$ and $\widehat{\Lambda}_f^k$ by downward induction. If we have defined the purely $(k+1)$-dimensional cycle $\widehat{\Gamma}_f^{k+1}$, then the hypersurface

$$\left[V\left(w_{k+1} - \frac{\partial \widetilde{f}}{\partial z_{k+1}}\right)\right]$$

properly intersects $\widehat{\Gamma}_f^{k+1}$ inside U, and therefore there is a well defined, purely k-dimensional intersection cycle

$$\widehat{\Gamma}_f^{k+1} \cdot \left[V\left(w_{k+1} - \frac{\partial \widetilde{f}}{\partial z_{k+1}}\right)\right],$$

which we can decompose as

$$\widehat{\Gamma}_f^{k+1} \cdot \left[V\left(w_{k+1} - \frac{\partial \widetilde{f}}{\partial z_{k+1}}\right)\right] =: \widehat{\Gamma}_f^k + \widehat{\Lambda}_f^k,$$

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where no component of $\widehat{\Gamma}_f^k$ is contained in $\operatorname{im} d\widetilde{f}$ and every component of $\widehat{\Lambda}_f^k$ is contained in $\operatorname{im} d\widetilde{f}$.

As the $\widehat{\Lambda}_f^k$ are contained in $\operatorname{im} d\widetilde{f}$, the projection, π, maps each $\widehat{\Lambda}_f^k$ isomorphically onto a cycle in U; we let

$$\Lambda_f^k := \pi\big(\widehat{\Lambda}_f^k\big)$$

(this is the proper projection of a cycle, and is frequently denoted by π_*). We refer to Λ_f^k as the *k-dimensional Lê–Vogel cycle*.

Note that in the affine case,

$$\big[\,\overline{T^*_{X_{\mathrm{reg}}}U}\,\big] = U \times \{0\},$$

and the Lê–Vogel cycles coincide with the Lê cycles.

Now, exactly as before, Λ_f^k properly intersects the linear subspace $V(z_1, \ldots, z_k)$ at the origin, and we define the *k-dimensional Lê–Vogel number of f at* 0 to be the intersection number

$$\lambda_f^k(0) := \big(\Lambda_f^k \cdot V(z_1, \ldots, z_k)\big)_0.$$

THEOREM 6.2. *If we let*

$$s := \dim_0 \pi\big(\overline{T^*_{X_{\mathrm{reg}}}U} \cap \operatorname{im} d\widetilde{f}\big),$$

the Lê–Vogel numbers $\lambda_f^k(0)$ are zero if $k > s$, and the Euler characteristic of the vanishing cycles is given by

$$D_{f,X}(0) = -\chi(\phi_f \mathbf{A}^\bullet)_0 = \sum_{k=0}^{s} (-1)^{d-k} \lambda_f^k(0).$$

Note that when $s = 0$, the only Lê–Vogel number which is possibly non-zero is $\lambda_f^0(0)$, and

$$\lambda_f^0(0) = \big(\overline{T^*_{X_{\mathrm{reg}}}U} \cdot \operatorname{im} d\widetilde{f}\big)_{(0, d_0\widetilde{f})}.$$

Therefore we recover the result of Corollary 5.4.

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