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Absolute curvatures in integral geometry

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1. *Introduction.* Surface integrals of curvature arise naturally in integral geometry and geometrical probability, most often in connection with the *Quermassintegrale* or cross-section integrals of convex bodies. They enjoy many desirable properties, such as the ability to be determined by summing or averaging over lower-dimensional sections or projections. In fact the Quermassintegrale are the *only* functionals of convex bodies to meet certain, quite reasonable, requirements. The conclusion has often been drawn, especially in practical applications, that the Quermassintegrale and their associated curvature integrals have a canonical status to the exclusion of all other quantities.

Here, however, we construct different integrals of so-called *absolute* curvature which enjoy properties quite analogous to those mentioned above. They can be determined from sections or projections and they are characterized by reasonable requirements of invariance and additivity. Hence they provide an alternative to the classical scheme: the distinction concerns the natural domains of definition of the functionals used. Here there is no essential restriction of convexity of differentiability, although we ignore (or cannot discern) the orientation of a surface. Some applications have been found in medical pathology (**1**,**2**,**3**).

L. A. Santaló(**21**) introduced a family of integrals which includes the absolute curvatures. His interesting construction does not in general have the desired section and projection properties, and indeed no study of such properties was carried out. Independently R. T. DeHoff(**8**,**9**) recognized that a measurement called the tangent count, when summed over plane sections of surfaces in $\mathbb{R}^3$, yields a useful quantity. Here we identify that quantity as an absolute curvature integral and build the general theory of absolute curvatures.

After presenting the background, our first task is to construct absolute curvatures after Santaló and to write them as surface integrals (Section 4). Next we extract the section and projection properties (Section 5). All this is done within differential analysis, but extends to wider notions of surface integration. Thus we characterize the absolute curvatures as certain functionals of differential currents (Section 6). Much use is made of geometric measure theory (**11**), in particular of the coarea formula. Our results apply equally to the signed curvatures associated with the Quermass-integrale, giving a new characterization of signed curvatures as functionals *of surfaces*. The distinction between signed and absolute curvatures is a matter of their natural domain of definition, or rather, of the kind of additivity property they satisfy.

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2. *Background: integral geometry.* Classical geometrical probability and integral geometry in $\mathbb{R}^n$ (4,5,13,22) rely on the *Quermassintegrale* or cross-section integrals W_i^n $(0 \leqslant i \leqslant n)$, which are proportional to various fundamental characteristics of convex compact sets K, including the volume $\mathscr{L}(K) = W_0^n(K)$ and total surface 'area' $S(K) = nW_1^n(K)$.

Define the ith quermassintegral $W_i^n(K)$, $0 \leqslant i < n$, as the invariant measure of the set of all i-dimensional flats (affine subspaces) intersecting K, normalized so that the unit ball Q has all quermassintegrals equal to its volume ω_n. Conventionally

$$W_n^n(K) = \omega_n.$$

An equivalent definition is to set $W_i^n(K)$ proportional to the mean $(n-i)$-dimensional volume of the projection of K orthogonally onto an $(n-i)$-dimensional *subspace* of $\mathbb{R}^n$ chosen with the probability measure that is invariant under rotations of subspaces. Equivalently, W_i^n is proportional to the ith coefficient in Steiner's formula (13,22); see (10,15). If the surface ∂K is sufficiently smooth, then $W_i^n(K)$, $1 \leqslant i \leqslant n$, is the integral over ∂K of the $(i-1)$th elementary symmetric function of the $n-1$ principal curvatures of ∂K; and if K is polyhedral, $W_i^n(K)$ $(1 \leqslant i \leqslant n)$ is a weighted sum of the angles at $(n-i)$-dimensional edges of K (5,16). Significantly for geometrical probability, each W_i^n for $i < n$ can be determined by integration over lower-dimensional sections or projections: this is the basis of mathematical stereology (20,17,18).

All the above properties are equivalent, by virtue of Hadwiger's (13)

Characterization Theorem.

Let ψ be a real-valued function on the class $\mathscr{K}$ of convex bodies which is

(*a*) *additive* in the sense that $A, B \in \mathscr{K}$, $A \cup B \in \mathscr{K}$ implies

$$\psi(A \cup B) = \psi(A) + \psi(B) - \psi(A \cap B);$$

(*b*) *continuous* with respect to the Hausdorff metric ρ on $\mathscr{K}$ (13,22);

(*c*) *invariant* under rigid motions of $\mathbb{R}^n$.

Then $\psi = \Sigma_{i=0}^n a_i W_i^n$ for some unique real constants a_i.

Moreover, every convex body can be ρ-approximated by polyhedra, and also by sets with smooth boundary (4,5,13,22), so it is enough to construct and examine such continuous functionals on one or other of these simpler classes of sets. 'Direct' geometrical proofs can then be given (16 and section 5; see also 5,19) that, for example, the mean i-dimensional projection is equal to a certain curvature integral (or angle sum), essentially reducing the problem to a 'local' one over ∂K.

The requirement of convexity of K can be avoided in at least two ways. One might (5,7) define a quantity $V_i^n(S)$ for any smooth surface S in terms of integration over lower dimensional sections or projections, then identify it as a certain curvature integral over S and as a coefficient in the Steiner formula.

Alternatively, any additive continuous functional on $\mathscr{K}$ extends uniquely to an additive functional on the class $\cup\mathscr{K}$ of finite unions of convex bodies (not trivial; see theorem 1 of 12). The sectioning property carries over, and the projection property in a different form, but the Steiner formula does not, since the set will generally have zero *reach* (10). The characterization theorem holds *a fortiori*. If $A \in \cup\mathscr{K}$ has a smooth

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(or polyhedral) surface, analogous interpretations hold concerning curvatures; note that negative contributions to the overall sum W_i^n are possible. In particular

$$\Gamma(\tfrac{1}{2}n+1)\,\pi^{-\frac{1}{2}n}W_n^n,$$

identically 1 on $\mathscr{K}$, extends to the Euler–Poincaré characteristic χ (**4,12,13,14,22**), and $W_i^n(A)$ becomes the integral of $\chi(A \cap F)$ with respect to invariant measure on i-flats F.

3. *Tangent counts.* Hadwiger's characterization theorem has been invoked in mathematical stereology to argue that the W_i^n are the only 'good' quantities estimable from random sections or projections of geometrical structures. Indeed, the curvature integrals can be associated with physical properties, say, in the growth of grain structures in metallurgy or the surface tension of membranes. More precisely, the W_i^n are (*signed*) surface integrals of functions whose absolute values are related to physical properties. However, integrals of these absolute values do not exhibit the desired stereological properties (of being calculable from sections or projections.)

DeHoff(**8**) established special cases of another stereological property of the W_i^n, that of *tangent counts*. Let L be a smooth curve in the plane and Ω a uniformly random direction on the unit circle. Write $T_L^+(\Omega)$ for the (a.s. finite) number of points of L at which the tangent to L is in the direction Ω, and $T_L^-(\Omega)$ for the number where that direction is $-\Omega$. In practice we move a test line t across the plane and note parallel positions of t giving a 'convex' (+) or 'concave' (−) tangent.

Then the expected value

$$E(T_L^{\mathrm{net}}) = E(T_L^+ - T_L^-) = \pi^{-1}C(L),$$

where $C(L)$ is the usual integral over L of its signed curvature κ. If L is *closed* then $T_L^{\mathrm{net}} = \pi^{-1}C(L)$ a.s.. When L is the random section of a surface in space, T_L^{net} then estimates the quermassintegral W_2^3, the integral mean curvature. In a later article (**9**) DeHoff argues that the *absolute tangent count*

$$T_L^{\mathrm{abs}}(\Omega) = T_L^+(\Omega) + T_L^-(\Omega),$$

also estimates a valuable stereological quantity. But such a quantity cannot be related to the quermassintegrals.

4. *Absolute curvature on smooth surfaces.* The *Grassmann manifold* $G(n,r)$ of all r-dimensional vector subspaces of $\mathbb{R}^n$ is an homogeneous space under the action of the orthogonal group $O(n)$, and so has a unique $O(n)$-invariant measure $\gamma_{n,r}$ of unit total mass. In integral geometry (of the classical style related to geometrical probability (**13,22**)) one instead defines a 'standard' invariant measure on $G(n,r)$ by means of a differential form dL_r^n whose total integral is

$$g_{n,r} = 2\binom{n}{r}\frac{\omega_n \dots \omega_{n-r+1}}{\omega_r \dots \omega_1}, \tag{1}$$

where $\omega_i = \pi^{\frac{1}{2}i}/\Gamma(1+\frac{1}{2}i)$ is the i-volume of the unit ball in $\mathbb{R}^i$, $i \geqslant 1$.

There is a natural embedding of $G(n,r)$ into a Euclidean space (**11**, 3.2.28) such that dL_r^n corresponds to Hausdorff measure $\mathscr{H}^{r(n-r)}$. The relevance of Hausdorff

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measure is that integrals over subsets of different Euclidean spaces can be related by the coarea formula (**11**, 3.2.22) when they are expressed relative to $\mathcal{H}^d$ (or to any like measure constructed from a Riemann metric by Carathéodory's technique).

We also let $F(n,r) = \{x+g : x \in \mathbb{R}^n,\ g \in G(n,r)\}$ be the set of *affine* r-spaces in $\mathbb{R}^n$, $0 \leqslant r \leqslant n$. If $g \in G(n,r)$, the set of affine subspaces obtained by parallel translation of g corresponds naturally to the points of the perpendicular set $g^\perp \in G(n, n-r)$. Given $y \in g^\perp$ we choose $h = y+g$, and given an affine space $h \perp g$ we recover $\{y\} = \pi_g^\perp(h)$ where $\pi_g^\perp : \mathbb{R}^n \to g^\perp$ is the orthogonal projection.

Let M be a compact $(n-1)$-manifold-with-boundary (∂M may be null) which is $\mathcal{C}^2$-embedded in $\mathbb{R}^n$. Write $\mathrm{Tan}(M,x)$ for the tangent space in $\mathbb{R}^n$ to M at x. Consider the 'Grassmann bundle' of tangent r-spaces of M:

$$\mathrm{Gra}_r M = \{(x,g) \in M \times G(n,r) : g \subset \mathrm{Tan}(m,x)\}$$

and the naturally associated bundle

$$\mathrm{Aff}_r M = \{(x, x+g) \in M \times F(n,r) : (x,g) \in \mathrm{Gra}_r M\}.$$

Fixing $g \in G(n,r)$ consider the set of projected points

$$\begin{aligned} \Pi(M,g) &= \{\pi_g^\perp(x+g) : (x,g) \in \mathrm{Gra}_r M\} \\ &= \{\pi_g^\perp\{x\} : (x,h) \in \mathrm{Aff}_r M, h \perp g\} \\ &\subset g^\perp. \end{aligned}$$

For almost all $g \in G(n,r)$ this set is a compact $(n-r-1)$ dimensional manifold. Define the *rth absolute curvature* of M

$$K_r^n(M) = \int_{G(n,r)} \mathcal{H}^{n-r-1}(\Pi(M,g))\, d\gamma_{n,r} g \tag{2}$$

as introduced in (**21**). For completeness we also consider the volume

$$K_0^n(M) = \mathcal{H}^{n-1}(M).$$

PROPOSITION 1. *The absolute curvatures $K_r^n(M)$ can be written as surface integrals over M:*

$$K_r^n(M) = \int_M m_r^{\mathrm{abs}}(x)\, d\mathcal{H}^{n-1}x \quad (r > 0).$$

The integrand is determined by the principal curvatures κ_i of M, through

$$m_r^{\mathrm{abs}}(x) = \int_{\mathrm{Gra}_r(M,x)} \left| \sum_{k(l)<\ldots<k(r)} \det\left(\sum_i \kappa_i u_j^i u_{k(l)}^i\right)_{j,l} \right|,$$

where $\mathrm{Gra}_r(M,x)$ is the fibre of $\mathrm{Gra}_r M$ at x. Given (x,g) in $\mathrm{Gra}_r(M,x)$ with any orthonormal basis $\{a_j\}$ for g, the real numbers $\{u_j^i\}$ are the coefficients in the representation of $\{a_j\}$ with respect to the orthonormal basis $\{e^i\}$ of principal directions.

Proof. At this point it is convenient to introduce a higher-dimensional manifold in which to calculate integrals using the coarea formula. This essentially is E. Cartan's 'method of moving frames' (**6**,**24**,**22**).

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An *orthonormal frame* in $\mathbb{R}^n$ is an $(n+1)$-tuple $z = (p, x_1, \ldots, x_n) \in (\mathbb{R}^n)^{n+1}$ where the set $\{x_i\}$ is orthonormal. In particular we consider right-handed frames, defined to satisfy $x_1 \wedge \ldots \wedge x_n = e_1 \wedge \ldots \wedge e_n$ where $\{e_i\}$ is the standard basis of $\mathbb{R}^n$.

The set $\mathcal{OF}^+$ of right hand orthonormal frames is a connected, $\binom{n+1}{2}$-dimensional manifold of class $\mathscr{C}^\infty$, with $\mathscr{C}^\infty$ coordinate mappings $P: \mathcal{OF}^+ \to \mathbb{R}^n$, $X_i: \mathcal{OF}^+ \to S^{n-1}$. As usual (**23, 24**) we construct $\mathscr{C}^\infty$ differential forms

$$\alpha_i = \langle DP, X_i \rangle$$
$$\omega_j^i = \langle DX_j, X_i \rangle. \tag{3}$$

Intuitively, α_i corresponds to motion of the frame in the direction X_i, while ω_j^i corresponds to rotation of X_j towards X_i. Equivalently we are specifying an orthonormal basis

$$a_i = x_i, \quad 1 \leqslant i \leqslant n,$$
$$w_{ji} = \wedge^1 x_j \otimes x_i, \quad 1 \leqslant i < j \leqslant n$$

for
$$\operatorname{Tan}(\mathcal{OF}^+, z) \subset \operatorname{Tan}(\mathbb{R}^n, p) \oplus \bigoplus_{i=1}^{n} \operatorname{Tan}(S^{n-1}, x_i)$$
$$\simeq \mathbb{R}^n \oplus (\Lambda^1 \mathbb{R}^n \otimes \mathbb{R}^n).$$

Cartan's method relies on the fact that $\mathcal{OF}^+$ is isotropic under the Euclidean group $\mathscr{E}(\mathbb{R}^n)$, i.e. the isotropy subgroup $\{g \in \mathscr{E}(\mathbb{R}^n): gz = z\}$ is trivial for all $z \in \mathcal{OF}^+$. Hence (3) corresponds to a basis $\{\alpha_i(g)\} \cup \{\omega_j^i(g)\}$ for $\Lambda^1 \operatorname{Tan}(\mathscr{E}(\mathbb{R}^n), g)$, $g \in \mathscr{E}(\mathbb{R}^n)$, such that the map $T: g \to h \circ g$ transforms $T^\# \alpha_i(g) = \alpha_i(h \circ g)$, $T^\# \omega_j^i(g) = \omega_j^i(h \circ g)$.

Now let $\nu: M \to S^{n-1}$ be a $\mathscr{C}^1$ unit normal, and let

$$\operatorname{Orth}^+ M = \{z \in \mathcal{OF}^+: P(z) \in M, X_n(z) = \nu(P(z))\}$$

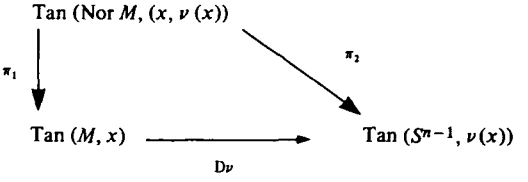
be the fibre bundle of 'frames tangent to M'.

Define $G_r: \mathcal{OF}^+ \to G(n, r)$ by $G_k(z) = \operatorname{span}\{X_1(z), \ldots, X_r(z)\}$. Define $\Pi^r: \mathcal{OF}^+ \to \mathbb{R}^n$ by $\Pi^r(z) = \pi_g^\perp(P(z))$ where $g = G_r(z)$. Let $\Gamma_r: \operatorname{Orth}^+ M \to \mathbb{R}^n \times G(n, r)$ be the map $\Gamma_r(z) = (\Pi^r(z), G_r(z))$. Using the coarea formula (**11**, 3.2.22) we obtain

$$K_n^r(M) = \int_{\operatorname{Orth}^+ M} J_{n-1} \Gamma_r \, d\mathscr{H}^{\frac{1}{2}n(n-1)} \tag{4}$$

where J_{n-1} denotes the $n-1$ dimensional Jacobian.

Now recall from classical differential geometry (**24**) that the diagram



commutes, where $\operatorname{Nor} M \subset M \times S^{n-1}$ is the normal bundle of M, with coordinate projections π_1 and π_2, and $D\nu: \operatorname{Tan}(M, x) \to \operatorname{Tan}(S^{n-1}, \nu(x)) = \operatorname{Tan}(M, x)$ is a linear symmetric map. Define a map $\rho: \operatorname{Tan}(M, x) \to \operatorname{Tan}(\mathcal{OF}^+, z)$ for $z \in \operatorname{Orth}^+ M$ by $\rho(x_i) = \omega_n^i(z)$, $x_i = X_i(z)$. Then elements of $\operatorname{Tan}(\operatorname{Orth}^+ M, z)$ are of the form

$$v + (\rho \circ D\nu)\, v + s,$$

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where $v \in \mathrm{Tan}\,(M, x)$ and $s \in S = \mathrm{span}\,\{\omega_j^i : 1 \leqslant i < j \leqslant n\}$. Now clearly

$$(D\Gamma_r(z))\,(\omega_j^i(z)) = 0$$

if both $i, j \leqslant r$ or if both $i, j > r$. Thus, decomposing

$$\mathrm{Tan}\,(\mathscr{O}\mathscr{F}^+, z) \subset \mathbb{R}^n \oplus R_r \oplus S_r \oplus K_r,$$

where

$$R_r = \mathrm{span}\,\{\omega_n^i : 1 \leqslant i \leqslant r\},$$
$$S_r = \mathrm{span}\,\{\omega_j^i : 1 \leqslant i \leqslant r < j < n\},$$
$$K_r = \mathrm{span}\,\{\omega_j^i : 1 \leqslant i < j \leqslant r \quad \text{or} \quad r < i < j \leqslant n\},$$

we find that

$$\begin{aligned} &D\Gamma_r(z)\,(v \oplus (\rho \circ D\nu)\,v \oplus s \oplus t) \\ &= D\Pi^r(z)\,(v) \oplus (\pi_R \circ \rho \circ D\nu)\,v \oplus s \\ &= \pi_T(v) \oplus (\pi_R \circ \rho \circ D\nu)\,v \oplus s, \end{aligned}$$

where π_R, π_T denote orthogonal projections onto $R = R_r$ and

$$T = \mathrm{span}\,\{X_{r+1}(z), \ldots, X_{n-1}(z)\}.$$

In case $r = n-1$, S_r is trivial and

$$D\Gamma_r(z)\,(v \oplus (\rho \circ D\nu)\,v \oplus t) = (\rho \circ D\nu)\,v.$$

But $\pi_R \circ \rho \circ D\nu = \rho \circ \pi_G \circ D\nu$ where $G = \mathrm{span}\,\{X_1(z), \ldots, X_r(z)\}$ so that

$$\begin{aligned} J_{n-1}\,\Gamma_r(z) &= \|\Lambda_{n-1}\,\Gamma_r(z)\| \\ &= \|\Lambda_{n-1}(\pi_T \oplus \pi_G)\|\,\|\Lambda_r(\rho \circ \pi_G \circ D\nu | G)\| \\ &= J_{n-1}P\,.\,\|\Lambda_r(\pi_G \circ D\nu | G)\| \end{aligned} \tag{5}$$

where $P\colon \mathrm{Orth}^+ M \to M$ is again the projection.

To calculate the last term in (5) it suffices (**11**, 1.7.6) to find $|\pi_G \circ D\nu(\xi)|$ for any simple $\xi \in \Lambda_r G$ with $|\xi| = 1$. Let $a_1, \ldots, a_r$ be orthonormal in G. Since $D\nu$ is symmetric it has $n-1$ orthonormal eigenvectors e_i and eigenvalues κ_i, the principal curvatures, with respect to which

$$a_j = \sum_{i=1}^{n-1} u_j^i e_i, \quad u_j^i \in \mathbb{R}$$

and conversely

$$\pi_G e_i = \sum_{j=1}^{r} u_j^i a_j.$$

Hence

$$\begin{aligned} \pi_G \circ D\nu(a_j) &= \pi_G(\Sigma\, \kappa_i u_j^i e_i) \\ &= \sum_i \kappa_i u_j^i \left(\sum_{l=1}^{r} u_l^i a_l \right) \\ &= \sum_{l=1}^{r} \left(\sum_i \kappa_i u_j^i u_l^i \right) a_l, \end{aligned}$$

and so

$$\|\Lambda_r(\pi_G \circ D\nu | G)\| = \Big| \sum_{\mathrm{k}(1) < \ldots < \mathrm{k}(r)} \det \left(\sum_i \kappa_i u_j^i u_{\mathrm{k}(l)}^i \right)_{j,l} \Big|. \tag{6}$$

In case $r = n-1$ this is simply

$$\left| \det \left(\sum_i \kappa_i u_j^i u_l^i \right)_{j,l} \right|,$$

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i.e. the absolute value of the Gaussian curvature of M at x with respect to ν. Clearly for general r, (6) is the absolute value of the Gaussian curvature of the r dimensional section $M \cap N$, where N is the $r+1$ dimensional affine space containing x whose tangent space contains $G_r(z)$ and νx. Writing

$$\lambda_r(x, g) = \sum_{k(1)<\ldots<k(r)} \det\left(\sum_i \kappa_i u_j^i u_{k(l)}^i\right)_{j,l}$$

for $(x, g) \in \mathrm{Gra}_r M$, we apply the coarea formula to (5)–(6) to get

$$K_r^n(M) = \int_M m_r^{\mathrm{abs}}(x)\, d\mathscr{H}^{n-1}x. \tag{8}$$

where

$$m_r^{\mathrm{abs}}(x) = \int_{P^{-1}\{x\}} |\lambda_r(P(z), G_r(z))|\, d\mathscr{H}^{\frac{1}{2}n(n-1)}z$$

$$= \int_{\mathrm{Gra}_r(M,x)} |\lambda_r(x, g)|\, d\gamma_{n-1,r}g. \tag{9}$$

This completes the proof of proposition 1. ∎

Compare (7–9) with the more familiar integral of *signed* curvature

$$V_r^n(M) = \int_M m_r(x)\, d\mathscr{H}^{n-1}x, \tag{10}$$

where

$$m_r(x) = \int_{\mathrm{Gra}_r(M,x)} \lambda_r(x, g)\, d\gamma_{n-1,r}g$$

$$= \binom{n-1}{r}^{-1} \sum_{i(1)<\ldots<i(r)} \kappa_{i(1)} \ldots \kappa_{i(r)} \tag{11}$$

is the rth elementary symmetric function of the principal curvatures. If $M = \partial K$ is the $\mathscr{C}^2$ surface of a convex body K then $V_r^n(M) = W_{r+1}^n(K)$.

Take the example of $\mathbb{R}^3$. Equation (7) reads

$$\lambda_2(g) = \kappa_1 \kappa_2,$$
$$\lambda_1(g) = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta,$$

where for $r = 1$ the line g has inclination θ to the first principal direction e_1. The average signed curvatures are the familiar mean curvature $m_1 = \frac{1}{2}(\kappa_1 + \kappa_2)$ and Gauss curvature $m_2 = \kappa_1 \kappa_2$, whereas

$$m_2^{\mathrm{abs}}(x) = |\kappa_1 \kappa_2|$$

$$m_1^{\mathrm{abs}}(x) = \frac{1}{2\pi} \int_{[0,\, 2\pi)} |\kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta| d\theta$$

$$= \begin{cases} \frac{1}{2}|\kappa_1 + \kappa_2| & \text{if } \kappa_1, \kappa_2 \text{ have the same sign} \\ \frac{1}{2}(\kappa_1 + \kappa_2)\left(\frac{4\theta_0}{\pi} - 1\right) + \frac{\kappa_1 - \kappa_2}{\pi} \sin 2\theta_0 & \text{otherwise} \end{cases} \tag{12}$$

where $0 \leqslant \theta_0 \leqslant \frac{1}{2}\pi$, $\tan\theta_0 = \sqrt{(-\kappa_1/\kappa_2)}$.

In particular, m_1^{abs} is the absolute value of the mean curvature m_1, *provided M has no saddle points* (and not in general, as asserted in (21), equation (35)). Contrary to (9), footnote to p. 110, m_1^{abs} is never zero, except of course at points x where $\kappa_1 = \kappa_2 = 0$.

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5. *Stereological properties.* The ability to determine curvature integrals from lower-dimensional sections or projections is a crucial property in applications of geometrical probability. On these lower-dimensional affine spaces A we interpret K_r^s through any isometric isomorphism of A with $\mathbb{R}^s$. Similarly, natural Grassmann manifolds $G(A,r)$ and frame spaces $\mathcal{OF}^+(A)$ are constructed through isomorphisms when needed.

PROPOSITION 2. *Absolute curvatures satisfy the section formula* $(1 \leqslant r < s < n)$

$$K_r^n(M) = \frac{\omega_{n-r+1}\omega_{s-1}}{\omega_{s-r}\omega_{n-s+1}} \int_{F(n,s)} K_r^s(M \cap f)\, d\mu_{n,s} f \tag{13}$$

and the projection formula $(1 \leqslant s < r < n)$

$$K_r^n(M) = \frac{\omega_s}{\omega_{n-s+1}} \int_{G(n,s)} K_{s-r}^{n-s}(\Pi(M,g))\, d\gamma_{n,s} g. \tag{14}$$

Here $\mu_{n,s}$ is the invariant measure on $F(n,s)$ (11) *naturally associated with $\gamma_{n,s}$ (described fully below).*

Proof. Consider first the projection formula. Applying (2) to K_{s-r}^{n-s} we expand the integral on the right-hand side of (14) as

$$\int_{G(n,s)} \int_{G(g^\perp, r-s)} \mathcal{H}^{n-s-1}(\Pi(M,g),h)\; d\gamma_{n-s,r-s} h\; d\gamma_{n,s} g.$$

But
$$h \in G(g^\perp, r-s) \Leftrightarrow h \in G(n, r-s), \quad h \perp g$$
$$\Rightarrow \Pi(\Pi(M,g),h) = \Pi(M, g+h).$$

Also, under the map $\eta\colon (g,h) \to g+h$ we get an invariant measure

$$\eta^*(\gamma_{n,s} \times \gamma_{n-s,r-s}) = (\omega_{n-s+1}/\omega_s)\, \gamma_{n,r}$$

so the expression becomes

$$\omega_{n-s+1}/\omega_s \int_{G(n,r)} \mathcal{H}^{n-r}\, \Pi(M,h)\, d\gamma_{n,r} h.$$

Comparison with (2) gives (14).

Similarly if $1 \leqslant r < s < n$ then the section integral

$$\int_{F(n,s)} K_r^s(M \cap f)\, d\mu_{n,s} f \tag{15}$$

can be expanded according to (2) (or (8)–(9)). Here $\mu_{n,s}$ is the invariant measure on $F(n,s)$ (11) pulled back from $\mathcal{H}^{n-s} \times \gamma_{n,s}$ under the map $f \to (\pi_g^\perp f, g) \in \mathbb{R}^{n-s} \times G(n,s)$ where $g = (f^\perp)^\perp$.

Consider the larger set

$$\mathrm{Fra}_{s-1} M = \{z \in \mathcal{OF}^+ \colon P(z) = x \in M, \quad G_{s-1}(z) \subset \mathrm{Tan}\,(M,x)\}$$

of frames having only the first $s-1$ axes in $\mathrm{Tan}\,(M,x), x \in M$. Define maps

$$f_s\colon \mathcal{OF}^+ \to F(n,s) \quad \text{by} \quad f_s(z) = P(z) + G_s(z)$$

and
$$F_r^s\colon \mathcal{OF}^+ \to \{(f,x,g) \in F(n,s) \times \mathbb{R}^n \times G(n,r) \colon x \in f, g \subset f\}$$

by
$$F_r^s(z) = (f_s(z), \Gamma_r(z)).$$

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The coarea formula converts (15) to

$$\int_{\mathrm{Fra}_{s-1}M} J_{d_1} F_r^s \, d\mathscr{H}^d, \tag{16}$$

$$d = \binom{s-1}{2} + \binom{n-s+1}{2} + n - 1, \quad d_1 = \binom{s+1}{2} + (s+1)(n-s-1).$$

Suppose that $z \in \mathrm{Fra}_{s-1} M$, $w \in \mathrm{Orth}^+ M$ satisfy $P(z) = P(w)$ and $X_i(z) = X_i(w)$, $i \leqslant s-1$. Let $\theta: \mathbb{R}^n \to \mathbb{R}^n$ be the change of basis from $\{X_i(z)\}$ to $\{X_i(w)\}$. Then

$$J_{d_1} F_r^s(z) = |\det \Theta| \, J_{n-1} \Gamma_r(w).$$

This leads us to define $H: \mathrm{Fra}_{s-1} M \to \mathrm{Orth}^+ M$ by

$$H(p, x_1, \ldots, x_n) = (p, x_1, \ldots, x_{s-1}, a_s, \ldots, a_{n-1}, \nu)$$

where $\nu = \nu(p)$ and the sequence $(x_1, \ldots, x_{s-1}, \nu, a_s, \ldots, a_{n-1})$ is obtained from

$$(x_1, \ldots, x_{s-1}, \nu, x_s, \ldots, x_n)$$

by the Gram–Schmidt process.

Then $J_{\frac{1}{2}n(n-1)} H = |\det \Theta|$ as above, and (16) becomes

$$\int_{\mathrm{Orth}^+ M} J_{n-1} \Gamma_r \left(\int_{H^{-1}\{w\}} |\det \Theta| \, d\mathscr{H}^{\frac{1}{2}(n-s)(n-s-1)} \right) d\mathscr{H}^{\frac{1}{2}n(n-1)}.$$

The inner integral is a constant so we finally obtain (13). ∎

The constants in (13) and (14) can be determined directly, but it is also possible to exploit the relationship with signed curvatures. If $M = \partial K$ is the $\mathscr{C}^2$ surface of a convex set, then $K_r^n(M) = V_r^n(M) = W_{r+1}^n(K)$. Hence, given that section and projection formulae hold for $K_r^n(M)$, the constants are identical to those in the corresponding formulae for W_{r+1}^n (13).

The same analogy suggests from (2) the *signed tangent count formula*

$$V_r^n(M) = \int_{G(n,r)} (\mathscr{H}^{n-r-1} \Pi^+(M, g) - \mathscr{H}^{n-r-1} \Pi^-(M, g)) \, d\gamma_{n,r} g,$$

where

$$\left.\begin{aligned} \Pi^+(M, g) &= \{\pi_g^\perp x \in \Pi(M, g) : \lambda_r(x, g) > 0\}, \\ \Pi^-(M, g) &= \{\pi_g^\perp x \in \Pi(M, g) : \lambda_r(x, g) < 0\}. \end{aligned}\right\} \tag{17}$$

This was developed for convex bodies in ((13), p. 239) and for smooth curves in $\mathbb{R}^2$ in (9).

6. *Characterizations.* One expects that the above calculations for $\mathscr{C}^2$ manifolds should apply in a wider context. In fact the coarea formula (11, 3.2.22) on which they rely applies to Hausdorff–rectifiable sets in the frame space $\mathscr{OF}^+$, using Hausdorff-approximate differentials to calculate the Jacobians (11, 3.1). Thus the preceding formulae apply more generally to the absolute curvature integral for $(\mathscr{H}^m, m)$ rectifiable sets in $\mathscr{OF}^+$, $(m = \frac{1}{2}n(n-1))$

$$K_r^n(S) = \int_S \mathrm{ap}\, J_{n-1} \Gamma_r \, d\mathscr{H}^m$$

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and to $$V_r^n(S) = \int_S \sigma \operatorname{ap} J_{n-1} \Gamma_r \, d\mathcal{H}^m,$$

where $$\sigma = \operatorname{sign} \det (\pi_G \circ \operatorname{ap} Dv | G).$$

Notice that the integration does not simply depend on a rectifiable set in M in $\mathbb{R}^n$, unless M is smooth and $P(S) = M$. For example, a polyhedral surface is a union of affine simplices A, each having $K_r^n(A) = 0$.

Nevertheless, when $S \subset \mathcal{OF}^+$ is rectifiable, then so are $P(S) = M$, almost every r-dimensional affine section of M, and almost every r-dimensional orthogonal projection of M.

The role of rectifiable sets here is the operation of integration over them. This leads us to characterize the curvature integrals as functionals of differential currents. Briefly (**11**, 4.1.1, 4.1.7), if $U \subset \mathbb{R}^n$ is open let $\mathcal{D}^m(U)$ be the vectorspace of $\mathcal{C}^\infty$ differential m-forms on U with compact support, topologized by the family of seminorms ν_K^i corresponding to norms of differentials D^i on compact sets K. The class of *currents* $\mathcal{D}_m(U)$ is the vectorspace of continuous linear functions $T : \mathcal{D}^m(U) \to \mathbb{R}$. In particular the operation of integrating $\xi \in \mathcal{D}^m(U)$ over a compact oriented manifold M is such a function, and these currents generate (**11**, 4.1.24-4.1.28) the vectorspace $R_m(U)$ of *rectifiable currents*.

For each compact $K \subset U$ one defines a seminorm F_K on $\mathcal{D}_m(U)$ which is finite on all rectifiable currents T whose support $\operatorname{spt} T \subset K$. Indeed F_K is finite on the larger subspace $F_{m,K}(U)$ defined (**11**, p. 368) as the F_K-closure of $\{T : \operatorname{spt} T \subset K;\ T$ and (if $m > 0$) ∂T are representable by integration of vector fields$\}$. The union over all compact K of $F_{m,K}(U)$ forms the vector space $F_m(U)$ of *flat currrents*.

Exactly analogous definitions will be made for currents on the manifold $\mathcal{OF}^+$. Consider currents of degree $m = \binom{n}{2}$ on $\mathcal{OF}^+$ and define the m-forms

$$\eta_r = \bigwedge_{i=r+1}^{n-1} \alpha_i \wedge \bigwedge_{i \leqslant r} \bigwedge_{j>r} \omega_j^i \wedge \Omega_{r,n-r-1}, \tag{19}$$

where $$\Omega_{r,n-r-1} = \bigwedge_{i<j\leqslant r} \omega_j^i \wedge \bigwedge_{r<i<j\leqslant n-1} \omega_j^i,$$

with α_i and ω_j^i as in (3).

The equation

$$V_r^n(T) = T(\eta_r) = T(\beta_T \eta_r), \quad T \in \mathcal{D}_m(\mathcal{OF}^+)$$

defines a linear function on $\mathcal{D}_m(\mathcal{OF}^+)$. Here β_T is any $\mathcal{C}^\infty$ scalar function with compact support and which is identically 1 on the support $(\operatorname{spt} T)$ of T. The definition agrees with the earlier definition for smooth manifolds M if we consider the current

$$T(\xi) = \int_{\operatorname{Orth}^+ M} \xi.$$

Note that V_r^n is invariant under transformations of $\mathcal{D}_m(\mathcal{OF}^+)$ induced by Euclidean motion of $\mathbb{R}^n$. It is also F_K-continuous on

$$R_{m,K}(\mathcal{OF}^+) = R_m(\mathcal{OF}^+) \cap \{T : \operatorname{spt} T \subset K\}$$

for all compact $K \subset \mathcal{OF}^+$. We shall see below that these properties characterize the V_r^n.

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Notice however that K_r^n is more properly a function of rectifiable *sets*, rather than oriented rectifiable sets corresponding to rectifiable currents. Any extension to $R_m(\mathcal{OF}^+)$, i.e. any function $\Phi: R_m(\mathcal{OF}^+) \to \mathbb{R}$ such that $\Phi(T) = K_r^n(M)$ whenever M rectifies T, *cannot be linear*. To see this, let $\Xi: \mathcal{OF}^+ \to \mathcal{OF}^+$ be the map

$$\Xi(p, x_1, \ldots, x_n) = \begin{cases} (p, -x_1, \ldots, -x_n) & \text{if } n \text{ is odd,} \\ (p, x_1, -x_2, \ldots, -x_n) & \text{if } n \text{ is even,} \end{cases}$$

which has the property that if M_+, M_- are two different orientations of the same C^2 manifold then $z \in \mathrm{Orth}^+ M_+ \Leftrightarrow \Xi z \in \mathrm{Orth}^+ M_-$, and $\Xi \circ \Xi = id_{\mathcal{OF}}$. Now $\det D\Xi = -1$ so $L \circ \Xi^\# = -L$ for any linear function L, including V_r^n. Nevertheless $K_r^n(M_+) = K_r^n(M_-)$ entails $\Phi \circ \Xi^\# = \Phi$.

Define a function $K_r^n: R_m(\mathcal{OF}^+) \to \mathbb{R}$ by

$$K_r^n(T) = \int_B \mathrm{ap}\, J_{n-1} \Gamma_r \, d\mathcal{H}^{\frac{1}{2}n(n-1)}, \tag{20}$$

where B rectifies T, i.e. B is a manifold associated with T as in (**11**, 4.1.28). This extends to a function on $F_m(\mathcal{OF}^+)$ by continuity (applying lemma 1 below to each $F_{m,K}(\mathcal{OF}^+)$).

Now K_r^n satisfies a 'weak' form of linearity:

(WL): A function $\Phi: V \to \mathbb{R}$ (where V is a subspace of $\mathcal{D}_m(\mathcal{OF}^+)$) is *weakly linear* if

$\Phi(\alpha T) = \alpha \Phi(T), \quad \alpha > 0, \quad T \in V$

and $\Phi(T+S) = \Phi(T) + \Phi(S)$ whenever

$T, S \in V$ are such that

$\mathrm{spt}\, T, \mathrm{spt}\, S, \Xi(\mathrm{spt}\, T), \Xi(\mathrm{spt}\, S)$ are all mutually disjoint.

THEOREM. *Let* $\Phi: F_m(\mathcal{OF}^+) \to \mathbb{R}$, $m = \binom{n}{2}$ *be a weakly-linear, Euclidean-motion-invariant, function which is* F_K*-continuous on* $F_{m,K}(\mathcal{OF}^+)$ *for all compact* $K \subset \mathcal{OF}^+$. *Then there exist constants* $c_0, \ldots, c_{n-1}, d_1, \ldots, d_{n-1} \in \mathbb{R}$ *such that*

$$\Phi = \sum_{r=0}^{n-1} c_r K_r^n + \sum_{r=1}^{n-1} d_r V_r^n + L, \tag{21}$$

where $L: F_m(\mathcal{OF}^+) \to \mathbb{R}$ *satisfies* $L(\mathrm{Orth}^+ M) = 0$ *for smooth manifolds* M *in* $\mathbb{R}^n$.

The proof relies on polyhedral approximation. First for $U \subset \mathbb{R}^N$, $N = \binom{n+1}{2}$ consider the vector subspace $P_m(U)$ in $R_m(U)$ generated by currents corresponding to compact oriented *affine simplices* $[u_0, \ldots, u_m]$, $u_i \in U$.

LEMMA 1. (**11**, 4.1.23). *Whenever* $C, K \subset \mathbb{R}^N$ *are compact with* $C \subset \mathrm{int}\, K$, *the* F_K*-closure of* $P_{m,K}(\mathbb{R}^N) = P_m(\mathbb{R}^N) \cap \{T: \mathrm{spt}\, T \subset K\}$ *contains* $F_{m,C}(\mathbb{R}^N)$.

To carry this into $\mathcal{OF}^+$, let $\phi: V \to U$ be a $\mathcal{C}^\infty$ chart, $U \subset \mathbb{R}^N$, $N = \binom{n+1}{2}$. If $K \subset V$ is compact and $Q \in P_{m,\phi(K)}(\mathbb{R}^N)$, then the induced current $\phi_\# Q \in R_{m,K}(\mathcal{OF}^+)$.

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The chart $\phi: V \to U$ is *standard* if the ϕ pullback of the standard basis

$$\{\Lambda^1 e_i, 1 \leqslant i \leqslant N\} \quad \text{for} \quad \Lambda^1 \mathbb{R}^N$$

is the standard basis (3). We let $P_m(\mathscr{OF}^+)$ be the vectorspace generated by currents $\phi_\# Q$ where $\phi: V \to U$ is standard and $Q \in P_{m,\phi(K)}(U)$ for compact $K \subset V$.

LEMMA 2. *Whenever C, $K \subset \mathscr{OF}^+$ are compact sets with $C \subset \operatorname{int} K$, the F_K-closure of $P_{m,K}(\mathscr{OF}^+)$ contains $F_{m,C}(\mathscr{OF}^+)$.*

Proof. Since C is compact it is covered by a finite family of relatively compact open sets W_i, $1 \leqslant i \leqslant I$, such that there exist standard charts $\phi_i: V_i \to U_i \subset \mathbb{R}^N$, $\overline{W}_i \subset V_i$. Let $\{\beta_i\}$ be a C^∞ partition of unity on $\cup W_i$ subordinate to $\{W_i\}$. Then any $T \in R_{m,C}(\mathscr{OF}^+)$ can be decomposed as

$$T = \Sigma T_i, \quad T_i(\xi) = T(\beta_i \xi)$$

and each component T_i has support $\operatorname{spt} T_i \subset \overline{W}_i$.

Now there exist compact sets $K_i \subset V_i \cap \operatorname{int} K$ such that $\overline{W}_i \subset \operatorname{int} K_i$. Applying the chart ϕ_i to T_i we find there exists $Q_i \in P_{m,K_i}(\mathscr{OF}^+)$ such that

$$F_{K_i}(Q_i - T_i) < \epsilon/I \quad \text{for given} \quad \epsilon > 0.$$

But $\operatorname{spt}(Q_i - T_i) \subset K_i \subset K$ so this is $F_K(Q_i - T_i) < \epsilon/I$. Summing over i produces the result. ∎

An *aligned* simplex $u_0, \ldots, u_m$ is one satisfying

$$(u_1 - u_0) \wedge \ldots \wedge (u_m - u_0) = c\ \eta_r \circ \pi$$

for some $c \in \mathbb{R}$, where π is some cyclic permutation of $\{X_i\}$. Let $P^\perp_{m,K}(\mathscr{OF}^+)$ be the vector subspace of $P_{m,K}(\mathscr{OF}^+)$ generated by aligned simplices, and their union over all compact K be $P^\perp_m(\mathscr{OF}^+)$. Further let $P^\square_{m,K}(\mathscr{OF}^+)$ be the sub*set* consisting of sums $\Sigma_i r_i A_i$ such that the family $\{\operatorname{spt} A_i\} \cup \{\Xi(\operatorname{spt} A_i)\}$ is pairwise disjoint. A compactness argument replaces $P_{m,K}(\mathscr{OF}^+)$ by $P^\square_{m,K}(\mathscr{OF}^+)$ in the previous lemma.

Proof of Theorem. By continuity and the above results, it suffices to consider a function Ψ on $P^\square_{m,K}(\mathscr{OF}^+)$ with the stated properties. Since Ψ is invariant under Euclidean transformations, its restriction to

$$\{[u_0, \ldots, u_m]: \Lambda(u_i - u_0) = c\eta_r \circ \pi \quad \text{for some} \quad c \in \mathbb{R}\}$$

is of the form $f \circ \mu_r$ where $\mu_r: \operatorname{span}\{\eta_r \circ \pi\} \to \mathbb{R}$ is the linear function mapping $\eta_r \circ \pi$ to 1, and $f: \mathbb{R} \to \mathbb{R}$ satisfies $f(\alpha x) = \alpha f(x)$, $\alpha > 0$. Write $f = g + h$ where $g(x) = f(x) - f(-x)$, $h(x) = f(x) + f(-x)$ so that $g(\alpha x) = \alpha g(x)$, $\alpha \in \mathbb{R}$, and $h(\alpha x) = |\alpha| h(x)$, $\alpha \in \mathbb{R}$. Hence $f(x) = xg(1) + |x|h(1) = d_r x + c_r|x|$, say. The constants are independent of π because ψ is Euclidean-invariant.

Thus $f \circ \mu_r = d_r . \mu_r + c_r \operatorname{mod} \circ \mu_r$ where $\operatorname{mod}: \mathbb{R} \to \mathbb{R}$ maps $x \to |x|$. But this is (21) for $P^\square_m(\mathscr{OF}^+)$. By ($WL$) and continuity the unique extension to $R_m(\mathscr{OF}^+)$ and hence to $F_m(\mathscr{OF}^+)$ is (21). ∎

COROLLARY. *Let $\Phi: F_m(\mathscr{OF}^+) \to \mathbb{R}$ be a linear function satisfying the conditions of the theorem. Then there exists a Euclidean-invariant differential form ξ on $\mathscr{OF}^+$ such that*

$$\Phi(T) = T(\beta_T \xi), \quad T \in F_m(\mathscr{OF}^+)$$

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with β_T as before. In particular, there exist constants $d_1, \ldots, d_{n-1}$ such that

$$\Phi(T) = \Sigma\, d_r\, V_r^n(T)$$

whenever Orth^+M *rectifies* T, *some smooth* $M \subset \mathbb{R}^n$.

COROLLARY. *Let Φ be as in the theorem and suppose Φ is Ξ-invariant. Then*

$$\Phi = \Sigma\, c_r\, K_r^n + L$$

for constants $c_r \in \mathbb{R}$ and Ξ-invariant weakly-linear map L as in the theorem.

Remark. Given these characterizations of the curvature integrals, one could prove the section and projection formulae indirectly by showing that the projection average and section integral have the required properties. To determine the constants it suffices to consider the unit sphere.

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