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When are all closed subsets recurrent?

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Abstract. In the paper we study relations of rigidity, equicontinuity and pointwise recurrence between an invertible topological dynamical system (t.d.s.) (X, T) and the t.d.s. $(K(X), T_K)$ induced on the hyperspace $K(X)$ of all compact subsets of X, and provide some characterizations. Among other examples, we construct a minimal, non-equicontinuous, distal and uniformly rigid t.d.s. and a weakly mixing t.d.s. which induces dense periodic points on the hyperspace $K(X)$ but itself does not have dense distal points, solving in that way a few open questions from earlier articles by Dong, and Li, Yan and Ye.

1. *Introduction*

A *topological dynamical system*, referred to more succinctly as just a *t.d.s.*, is a pair (X, T), where X is a compact metric space and $T : X \to X$ is a continuous surjective map from X into itself. When the t.d.s. considered is invertible, we use the acronym invertible topological dynamical system (*i.t.d.s.*) for simplicity. A well-known result of Birkhoff states that every t.d.s. has a recurrent point, i.e. there are $x \in X$ and a sequence $n_i \to \infty$ with $T^{n_i} x \to x$. In [**28**], the question of when all points are recurrent was discussed by Katznelson and Weiss, and transitive non-minimal systems with all points recurrent were

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constructed (see also [**1**]). Strengthening the notion of pointwise recurrence, Glasner and Maon [**19**] introduced the notions of n-rigidity, weak rigidity, rigidity and uniform rigidity, and showed among other things that every rigid system has zero topological entropy. Weiss in [**38**, Theorem 2] proved that in fact positive 2-rigidity implies zero topological entropy. Note that this result is also implied by a conclusion in [**9**], and it is an open question whether 2-rigidity implies zero topological entropy when (X, T) is a homeomorphism.

Sensitivity is a key notion in the definition of Devaney's chaos. When considering a system which is not sensitive, the notions of equicontinuous point and almost equicontinuous system appear naturally. Similarly to an equicontinuous system, each almost equicontinuous system is uniformly rigid; see [**1**, Theorem 3.6] or [**17**, Lemma 1.2]. A more striking result due to Hochman [**23**, Theorem 1.1] is that each zero-entropy ergodic measure-preserving transformation is isomorphic to a positively 2-rigid system. Thus, from the point of view of measure theory, the class of rigid systems is very large, making them an important object of study in the framework of theory of dynamical systems.

A t.d.s. (X, T) induces in a natural way the system $(K(X), T_K)$ on the hyperspace of all compact sets (more details on particular hyperspaces can be found in [**35**]). Bauer and Sigmund [**7**] initiated a systematic study of the connections between dynamical properties of (X, T) and $(K(X), T_K)$. Particularly, they showed that (X, T) is equicontinuous (respectively weakly mixing) if and only if so is $K(X)$, and provided an example which is distal but $(K(X), T_K)$ is not distal. Banks showed later in [**6**, Theorem 2] that the transitivity of $(K(X), T_K)$ coincides with its topological weak mixing. In [**34**], the authors further exploited these connections, and focused on periodic systems, P-systems, M-systems, E-systems and disjointness.

Following Bauer and Sigmund [**7**] and Katznelson and Weiss [**28**], we study the question of when all closed subsets are recurrent. It turns out that $(K(X), T_K)$ is pointwise minimal if and only if (X, T) is equicontinuous (see Theorem 3.4), and that $K(K(X))$ is pointwise recurrent if and only if $K(X)$ is weakly rigid if and only if (X, T) is uniformly rigid (see Theorem 4.2 and Corollary 5.3). Especially, $K(X)$ is pointwise recurrent if and only if (X, T) is uniformly rigid whenever X has a countable cardinality (see Theorem 5.15). It is also shown that the topological entropy of $(K(X), T_K)$ is zero when $K(X)$ is pointwise recurrent, i.e. 1-rigid (see Corollary 5.8). Moreover, for a class of minimal distal systems and $n \in \mathbb{N}$, equivalent conditions when $K(X)$ is n-rigid are given (see Corollary 5.13), and an example of a minimal distal non-equicontinuous uniformly rigid t.d.s. is constructed (see Example A.2).

Systems whose hyperspaces have dense recurrent points are considered in §6. Among other things, an example of a t.d.s. (X, T) such that $(K(X), T_K)$ has dense distal points and (X, T) does not have the property is displayed (see Theorem 6.4). It affirmatively answers a question left open in [**34**]. We note that it was shown in [**34**, Theorem 5.5] that a weakly mixing system (X, T) with $(K(X), T_K)$ having a dense set of distal points is disjoint from all minimal systems; and the fact that a t.d.s. (X, T) with a dense set of distal points is disjoint from all minimal systems was obtained previously in [**13**, Theorem 7.14] and [**37**, Theorem 4.4].

It is fair to mention at this point that when writing the final version of the paper we found a preprint [**3**], where the authors studied the dynamical properties on the induced space

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$K(X)$. Among other things, they showed that [**3**, Theorem 5.4] (X, T) is equicontinuous if and only if $K(X)$ is distal using a different method.

2. *Preliminaries*

Throughout this paper, denote by $\mathbb{N}$, $\mathbb{Z}_+$, $\mathbb{Z}$ and $\mathbb{R}$ the sets of positive integers, non-negative integers, integers and real numbers, respectively.

For a t.d.s. (X, T) and $x \in X$, let $\mathrm{orb}(x, T) = \{T^m x : m \in \mathbb{Z}_+\}$ be the (positive) orbit of x. Fix $n \in \mathbb{N}$, write $(X^n, T^{(n)})$ as the n-fold product system $(X \times X \times \cdots \times X, T \times T \times \cdots \times T)$ and set $\Delta_n = \{(x, x, \ldots, x) \in X^n : x \in X\}$.

2.1. *Recurrence and its stronger forms.*

Let (X, T) be an i.t.d.s. A point $x \in X$ is said to be *positively recurrent* (respectively *negatively recurrent*) if there exists a sequence $n_i \to +\infty$ (respectively $n_i \to -\infty$) such that $T^{n_i} x \to x$. Denote by $\mathrm{Rec}(T)$ (respectively $\mathrm{Rec}(T^{-1})$) the set of all positively recurrent (respectively negatively recurrent) points. We say that (X, T) is *pointwise positively recurrent* (respectively *pointwise negatively recurrent*) if $\mathrm{Rec}(T) = X$ (respectively $\mathrm{Rec}(T^{-1}) = X$). Note that there exists an example of a t.d.s. which is pointwise positively recurrent but not pointwise negatively recurrent (see [**5**, Section 2]). A point $x \in X$ is *recurrent* if x is either positively recurrent or negatively recurrent; and (X, T) is *pointwise recurrent* if each point in X is recurrent.

A t.d.s. (X, T) is *(topologically) transitive* if for any two non-empty open sets U and V, the set of transfer times $N_T(U, V) = \{n \in \mathbb{Z}_+ : U \cap T^{-n} V \neq \emptyset\}$ is infinite; is *weakly mixing* if $(X \times X, T \times T)$ is transitive; and is *mildly mixing* if $(X \times Y, T \times S)$ is transitive for any transitive t.d.s. (Y, S). We say that $x \in X$ is *a transitive point* if its orbit closure satisfies $\overline{\mathrm{orb}(x, T)} = X$. We denote by $\mathrm{Trans}(T)$ the set of transitive points. It is well known that the orbit closure of a recurrent point is transitive.

A t.d.s. (X, T) is *minimal* if $\mathrm{Trans}(T) = X$. Call $x \in X$ a *minimal point* or *almost periodic point* if the subsystem $(\overline{\mathrm{orb}(x, T)}, T)$ is minimal. We say that $x \in X$ is a *periodic point* if $T^n x = x$ for some $n \in \mathbb{N}$; and (X, T) is *pointwise periodic* if each point in X is periodic. The set of all periodic points (respectively minimal points) of (X, T) is denoted by $\mathrm{P}(T)$ (respectively $\mathrm{AP}(T)$). It is easy to see that for a transitive system, we have $\mathrm{P}(T) \subset \mathrm{AP}(T) \subset \mathrm{Trans}(T) \subset \mathrm{Rec}(T)$.

A pair $(x, y) \in X^2$ is said to be *proximal* if there is a sequence $n_i \to +\infty$ such that $d(T^{n_i} x, T^{n_i} y) \to 0$; and *regionally proximal* if for each $\varepsilon > 0$ there are $x', y' \in X$ and $k \in \mathbb{N}$ with $d(x, x') < \varepsilon$, $d(y, y') < \varepsilon$ and $d(T^k x', T^k y') < \varepsilon$. The subset consisting of all proximal (respectively regionally proximal) pairs is denoted by $P(X, T)$ (respectively $Q(X, T)$). A point $x \in X$ is said to be *distal* if x is only proximal to itself in its orbit closure; and be *an equicontinuous point* if for each $\varepsilon > 0$ there is $\delta > 0$ such that $\mathrm{diam}\, T^n(B(x, \delta)) < \varepsilon$ for each $n \geq 0$, where $B(x, \delta)$ is the open ball centered at x with radius δ. A t.d.s. (X, T) is *distal* if all points in X are distal or, equivalently, if $P(X, T) = \Delta_2$; and is *equicontinuous* if all points are equicontinuous.

We will also need a two-sided version of the relation $Q(X, T)$, which can be defined only for an i.t.d.s. A pair of points $x, y \in X$ in i.t.d.s. (X, T) belongs to the relation $(x, y) \in RP(X, T)$ if for each $\varepsilon > 0$ there are $x', y' \in X$ and $k \in \mathbb{Z}$ with $d(x, x') < \varepsilon$, $d(y, y') < \varepsilon$ and $d(T^k x', T^k y') < \varepsilon$.

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Since T is surjective, it is not hard to see that for every point x, if we take its backward extension $\{x_{-n}\}_{n\in\mathbb{N}}$, i.e. $x_0 = x$, $T(x_{-n}) = x_{-n+1}$, then there exists $y \in X$ such that $\liminf_{n\to\infty} d(x_{-n}, y) = 0$. But every point is proximal to a minimal point by result of Auslander and Ellis (see [**16**, Theorem 8.7]). Then equicontinuity of (X, T) immediately implies that x is contained in the closure of the orbit of y, that is, in a minimal set.

Remark 2.1. If (X, T) is equicontinuous, then X is a disjoint union of minimal systems. Additionally, (X, T) is an i.t.d.s. and (X, T^{-1}) is equicontinuous (see [**1**, Corollary 3.7]). This immediately implies that (X, T) and (X, T^{-1}) are distal.

It is known (see e.g. [**4**, Theorem 9.3]) that the maximal equicontinuous factor of an i.t.d.s. is induced by the smallest closed equivalence relation containing $RP(X, T)$; in particular, $RP(X, T) = \Delta_2$ implies that T (and at the same time also T^{-1}) is equicontinuous.

Another direction to strengthen the recurrence are various notions of rigidity (see e.g. [**19**]). Let $n \in \mathbb{N}$. An i.t.d.s. (X, T) is called *n-rigid* if each n-tuple $(x_1, \ldots, x_n) \in (X^n, T^{(n)})$ is a recurrent point; *weakly rigid* if (X, T) is n-rigid for each $n \in \mathbb{N}$; *rigid* if there is $m_i \to \infty$ such that $T^{m_i} \to \mathrm{id}$ pointwise, where id is the identity map; and *uniformly rigid* if there is $m_i \to \infty$ such that $T^{m_i} \to \mathrm{id}$ uniformly on X. It is equivalent to say that for each $\varepsilon > 0$ there is $m \in \mathbb{Z}$ such that $d(T^m x, x) < \varepsilon$ for each $x \in X$. In the same way (when the map is not necessarily invertible), we can define *positively n-rigid* and *positively weakly rigid* systems, replacing recurrence by positive recurrence in the definition. Also, when defining rigid and uniformly rigid, we can drop the assumption that (X, T) is invertible.

It is known that every minimal equicontinuous system is uniformly rigid, but there also exist minimal weakly mixing uniformly rigid systems [**19**]. A transitive system with an equicontinuous transitive point is called *almost equicontinuous*. Almost equicontinuous systems are uniformly rigid but at the same time they can also be proximal; see e.g. [**1**, **17**]. A minimal rigid but not uniformly rigid system is constructed in [**30**, Theorem 1.1]. Every minimal distal system is weakly rigid, and it is worth mentioning that the minimal distal system (X, T) defined by $T(x, y) = (x + \alpha, x + y)$ on $\mathbb{T}^2$ is not rigid (see [**19**]).

It is clear that a minimal system is 1-rigid, and it is easy to see that the Denjoy minimal system on the circle is 1-rigid but not 2-rigid. It is an open question if for each $n \geq 2$ there is a system which is n-rigid but not $n + 1$-rigid, though the general opinion is that such examples should exist.

2.2. *Factor and extension.* Let (X, T) and (Y, S) be two systems and $\pi : X \to Y$. We say that X is an *extension* of Y or Y is a *factor* of X if π is a continuous onto map which interwinds the actions, i.e. $\pi \circ T = S \circ \pi$. In this case we say that π is a *factor map*.

Let (X, T) be a t.d.s. A self homeomorphism ξ of X is an automorphism of (X, T) if it commutes with T, i.e. $\xi \circ T = T \circ \xi$. We denote by $\mathrm{Aut}(X, T)$ the collection of all automorphisms of (X, T). If K is a compact subgroup of $\mathrm{Aut}(X, T)$, then the map $x \mapsto Kx$ defines a factor map $\pi : (X, T) \to (Z, \widetilde{T})$ with $Z = X/K$ and $R_\pi = \{(x, kx) : x \in X, k \in K\}$. Such an extension is called a *group extension*. A special class of group extensions considered in this paper is so-called *skew products* (see [**16**]). Strictly speaking,

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(X, T) is a skew product over some t.d.s. (Y, S) and compact group G if $X = Y \times G$ and $T(y, g) = (Sy, \phi(y)g)$, where $\phi : Y \to G$ is a continuous map.

2.3. *Subset of integers.* Let S be a subset of $\mathbb{Z}$ (respectively $\mathbb{Z}_+$). We say that S is *syndetic* (respectively *positively syndetic*) if it has a bounded gap, i.e. there is $N \in \mathbb{N}$ with $\{i, i+1, \ldots, i+N\} \cap S \neq \emptyset$ for all $i \in \mathbb{Z}$ (respectively $i \in \mathbb{Z}_+$).

Let $\{p_i\}_{i=1}^{\infty}$ be an infinite sequence in $\mathbb{N}$. Set

$$FS(\{p_i\}_{i=1}^{\infty}) = \{p_{i_1} + p_{i_2} + \cdots + p_{i_n} : 1 \le i_1 < i_2 < \cdots < i_n, n \in \mathbb{N}\}.$$

A subset $A \subset \mathbb{N}$ is said to be an *IP-set* if it contains some $FS(\{p_i\}_{i=1}^{\infty})$, and to be an *IP*-set* if it has non-empty intersection with any *IP*-set.

Let (X, T) be a t.d.s. For a point $x \in X$ and open subsets $U, V \subset X$, we put

$$N_T(x, U) = \{n \in \mathbb{Z}_+ : T^n x \in U\} \text{ and}$$
$$N_T(U, V) = \{n \in \mathbb{Z}_+ : U \cap T^{-n} V \neq \emptyset\}.$$

When the acting map T is clear from the context, we simply write $N(x, U)$ and $N(U, V)$.

The following characterizations are well known; see for instance [**16**].

LEMMA 2.2. *Let (X, T) be a t.d.s. If $x \in X$ and U is any neighborhood of x, then:*

(1) *x is positively recurrent if and only if $N(x, U)$ is an IP-set;*
(2) *x is minimal if and only if $N(x, U)$ is a positively syndetic set;*
(3) *x is distal if and only if $N(x, U)$ is an IP*-set.*

When related to the weak mixing property, we have the following results (see [**25**, Lemma 5.1] and [**16**, Theorem 9.12], respectively).

LEMMA 2.3. *Let (X, T) be a t.d.s. with infinite cardinality. Then:*

(1) *X is weakly mixing if and only if X is transitive and, for any non-empty open subset $U \subset X$, there is $n \in \mathbb{Z}_+$ such that $n, n+1 \in N(U, U)$;*
(2) *if (X, T) is minimal and weakly mixing, then no point of X is distal.*

Let J be a subset of $\mathbb{Z}_+$. The *density* and *upper Banach density* of J are defined by

$$d(J) = \lim_{n\to\infty} \frac{\#\{J \cap [0, n-1]\}}{n} \quad \text{and} \quad \mathrm{BD}^*(J) = \limsup_{N-M\to\infty} \frac{\#\{J \cap [M, N]\}}{N - M + 1},$$

where $\#\{\cdot\}$ denotes the cardinality of the set.

The following lemma is also well known (see e.g. [**16**] or [**39**, Proposition 2.3]), and we omit the simple proof.

LEMMA 2.4. *If J has positive upper Banach density and Q is an IP-set, then there exists $l \in Q$ such that*

$$\mathrm{BD}^*(J \cap (J - l)) > 0.$$

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2.4. *Hyperspace.* Let X be a compact and metrizable space with metric d. Define $K(X) = \{A \subset X : A \neq \emptyset,\ \overline{A} = A\}$, that is, the collection of non-empty closed subsets of X. Endow $K(X)$ with a *Hausdorff metric* d_H defined by

$$d_H(A, B) = \max\left\{\max_{x\in A}\min_{y\in B} d(x, y), \max_{y\in B}\min_{x\in A} d(x, y)\right\}$$

for $A, B \in K(X)$ or, equivalently, by

$$d_H(A, B) = \inf\{\varepsilon > 0 : B_\varepsilon(A) \supset B,\ B_\varepsilon(B) \supset A\},$$

where $B_\varepsilon(A) = \{x \in X : d(x, a) < \varepsilon \text{ for some } a \in A\}$ is an ε-neighborhood of A in X. Note that $K(X)$ with the metric d_H is compact, provided that X is compact (see [**27**, **35**]).

Fix $n \in \mathbb{N}$ and denote $K_n(X) = \{A \in K(X) : |A| \leq n\}$. It is easy to see that $K_n(X)$ is closed and $\bigcup_{n\geq 1} K_n(X)$ is dense in $K(X)$ (see [**7**, Lemma 2]).

For any non-empty open subsets $U_1, \ldots, U_n$ of X, $n \in \mathbb{N}$, let

$$\langle U_1, \ldots, U_n\rangle = \left\{A \in K(X) : A \subset \bigcup_{i=1}^{n} U_i \text{ and } A \cap U_i \neq \emptyset \text{ for each } i = 1, \ldots, n\right\}.$$

We can check that $\langle U_1, \ldots, U_n\rangle$ is a non-empty open subset of $K(X)$. Moreover, the following family:

$$\{\langle U_1, \ldots, U_n\rangle : U_1, \ldots, U_n \text{ are non-empty open subsets of } X,\ n \in \mathbb{N}\}$$

forms a basis for *the Vietoris topology*. It is known that for a compact metric space X, the Vietoris topology coincides with the topology defined by the Hausdorff metric (for more details, see [**35**] or [**27**]). Every t.d.s. (X, T) induces a natural continuous map on $K(X)$ by the formula

$$T_K : K(X) \ni A \mapsto T(A) \in K(X)$$

defining in that way a t.d.s. $(K(X), T_K)$.

Recall that (X, T) is a P-system if it is transitive with dense periodic points; and has *dense small periodic sets* [**25**] if for any non-empty open subset $U \subset X$, there exist a closed subset A of U and $n \in \mathbb{N}$ such that $T^n A \subset A$. Now we present some results we will use in the sequel. See [**6**, Theorem 2] and [**33**, Theorem 1.1], respectively, for details.

LEMMA 2.5. *Let (X, T) be a t.d.s. and $(K(X), T_K)$ be the hyperspace. Then:*

(1) *(X, T) is weakly mixing if and only if $(K(X), T_K)$ is transitive;*
(2) *(X, T) is weakly mixing and has dense small periodic sets if and only if $(K(X), T_K)$ is a P-system.*

3. *Pointwise minimality*

A t.d.s. (X, T) is *pointwise minimal* (respectively *distal*) if all points in X are minimal (respectively distal). Note that if $(K(X), T_K)$ is pointwise minimal (respectively distal), then so is (X, T), since it is a subsystem of $(K(X), T_K)$. But there are many examples showing that the reciprocal is not valid (see e.g. [**7**, Theorem 3]). Now we provide a simpler example.

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Example 3.1. Let $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$. Consider the map $T : \mathbb{T}^2 \to \mathbb{T}^2$, $(x, y) \mapsto (x + \alpha, x + y \pmod 1)$, where α is an irrational number. Then, for any $n \in \mathbb{N}$,

$$T^n(x, y) = (x + n\alpha, y + nx + a(n)\alpha \pmod 1)$$

with $a(n) = n(n-1)/2$. Note that $\mathbb{T}^2$ is distal, since it is a distal extension of the distal system $\mathbb{T}^1$. Now we show that $(K(\mathbb{T}^2), T_K)$ is not pointwise positively recurrent, and a fortiori neither distal nor pointwise minimal.

Let $y_0 \in \mathbb{T}^1$ and $A = \{(x, y_0) : x \in \mathbb{T}^1\} \in K(\mathbb{T}^2)$. Fix any $n \in \mathbb{N}$ and note that

$$d_H(T_K^n A, A) = d_H(\{(x + n\alpha, y_0 + nx + a(n)\alpha) : x \in \mathbb{T}^1\}, \mathbb{T}^1 \times \{y_0\}).$$

Put $\varepsilon_0 = 1/10$ and choose $x_n \in \mathbb{T}^1$ such that

$$\varepsilon_0 \le nx_n + a(n)\alpha \pmod 1 \le 1 - \varepsilon_0.$$

Thus, for any $(x, y_0) \in A$, by the definition of the Hausdorff metric, we get

$$\begin{aligned} d_H(T_K^n A, A) &\ge \min_{x \in \mathbb{T}^1} d((x_n + n\alpha, y_0 + nx_n + a(n)\alpha), (x, y_0)) \\ &\ge d(y_0 + nx_n + a(n)\alpha, y_0) \ge \varepsilon_0. \end{aligned}$$

This implies that $N(A, B_{\varepsilon_0}(A)) \setminus \{0\} = \emptyset$ and so A is not positively recurrent.

A natural question is when does the pointwise minimality (respectively distality) of $(K(X), T_K)$ hold? The above Example 3.1 suggests that $K(X)$ is pointwise minimal (respectively distal) if and only if it is equicontinuous in some sense. To give a strict proof, we need the notion of a *locally almost periodic system*, which was first introduced by Gottschalk in [**20**]. Here what we need is a particular case, only considering actions of the group $\mathbb{Z}$. So, we reformulate the definition to the following form.

Definition 3.2. Let (X, T) be an i.t.d.s. A point $x \in X$ is *locally almost periodic* if, for every neighborhood U of x, there exist a neighborhood V of x and a syndetic set $F \subset \mathbb{Z}$ such that $T^n V \subset U$ for all $n \in F$. A t.d.s. (X, T) is *locally almost periodic* if every point of X is locally almost periodic.

Next we need to invoke a useful lemma on the characterization of equicontinuity, and for completeness we provide a direct proof. A proof can be also derived from results in [**20**].

LEMMA 3.3. *Let (X, T) be a t.d.s. Then (X, T) is equicontinuous if and only if (X, T) is invertible, distal and locally almost periodic.*

Proof. By Remark 2.1, every equicontinuous t.d.s. (X, T) is distal and invertible. Hence, it remains to show that it is locally almost periodic. Fix any $x \in X$ and $\varepsilon > 0$. Since both (X, T) and (X, T^{-1}) are equicontinuous, there is $\delta = \delta(\varepsilon) > 0$ such that for any $y \in B(x, \delta)$, we have $d(T^n x, T^n y) < \varepsilon/2$ for all $n \in \mathbb{Z}$. But x is distal and hence x is minimal (for both T and T^{-1}) and so by Lemma 2.2(2) we can find a syndetic set $F \subset \mathbb{Z}$ such that $d(x, T^n x) < \varepsilon/2$ for each $n \in F$. This immediately implies that $T^n B(x, \delta) \subset B(x, \varepsilon)$ for all $n \in F$. Indeed, (X, T) is locally almost periodic.

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Now assume that (X, T) is invertible, distal and locally almost periodic. Every point is proximal to a minimal point (see [**16**, Theorem 8.7]) and hence X is a union of minimal distal systems. We claim that (X, T^{-1}) is also distal. Assume on the contrary that there are distinct $x, y \in X$ such that $\lim_{k\to\infty} d(T^{-n_k}(x), T^{-n_k}(y)) = 0$. By compactness of X, we may assume that $\lim_{k\to\infty} T^{-n_k}(x) = z$ for some $z \in X$. For every $\varepsilon > 0$, there are V, a neighborhood of z, and a syndetic set $F \subset \mathbb{Z}$ such that $T^n(V) \subset B(z, \varepsilon)$ for every $n \in F$. But, for sufficiently large k, we have $T^{-n_k}(x), T^{-n_k}(y) \in V$ and hence there is $n > 0$ such that $T^n(x), T^n(y) \in B(z, \varepsilon)$. Since ε can be arbitrarily small, the points x, y are proximal, which is a contradiction. Indeed, (X, T^{-1}) is distal.

Let $(x_1, x_2) \in X^2 \setminus \Delta_2$. We claim that there exist $\delta > 0$ and two positive real numbers r_1, r_2 such that $d(T^n x, T^n y) > \delta$ for any $(x, y) \in B(x_1, r_1) \times B(x_2, r_2)$ and $n \in \mathbb{Z}$. By the distality of (X, T) and (X, T^{-1}), there is $\varepsilon_0 > 0$ such that $d(T^n x_1, T^n x_2) \geq \varepsilon_0$ for all $n \in \mathbb{Z}$. By the assumptions, x_1 is locally almost periodic and hence we have a syndetic set $F_1 \subset \mathbb{Z}$ and $r_1 > 0$ such that $T^p B(x_1, r_1) \subset B(x_1, \varepsilon_0/3)$ for all $p \in F_1$. Take $m_1 \in \mathbb{N}$ and $\delta' > 0$ such that $F_1 + [0, m_1] \supset \mathbb{Z}$ and $d(T^j x, T^j y) < \varepsilon_0/3$ for $j = 0, 1, \ldots, m_1$, provided that $d(x, y) < \delta'$. Suppose that there are $x \in B(x_1, r_1)$ and $n_1 \in \mathbb{Z}$ such that $d(T^{n_1} x, T^{n_1} x_2) < \delta'$. There is $j \in [0, m_1]$ such that $n_1 + j \in F_1$. Denote $p_0 = n_1 + j$ and observe that $d(T^{p_0} x, T^{p_0} x_2) < \varepsilon_0/3$. Since $T^{p_0} B(x_1, r_1) \subset B(x_1, \varepsilon_0/3)$ and $x \in B(x_1, r_1)$, we have

$$d(T^{p_0} x_1, T^{p_0} x_2) \leq d(T^{p_0} x_1, x_1) + d(x_1, T^{p_0} x) + d(T^{p_0} x, T^{p_0} x_2) < \varepsilon_0,$$

which is a contradiction. This proves that there are $\delta' > 0$ and $r_1 > 0$ such that $\inf_{n\in\mathbb{Z}} d(T^n x, T^n x_2) \geq \delta'$ for each $x \in B(x_1, r_1)$.

Now, by the local almost periodicity of x_2, choose $r_2 > 0$ such that for some syndetic subset $F_2 \subset \mathbb{Z}$, we have $T^q B(x_2, r_2) \subset B(x_2, \delta'/4)$ for every $q \in F_2$. Take any $m_2 > 0$ with $F_2 + [0, m_2] \supset \mathbb{Z}$ and let $\delta > 0$ be such that if $d(x, y) < 2\delta$, then $d(T^k x, T^k y) < \delta'/2$ for $k = 0, 1, \ldots, m_2$. If there are $(x, y) \in B(x_1, r_1) \times B(x_2, r_2)$ and n_2 such that $d(T^{n_2} x, T^{n_2} y) \leq \delta$, then there is $0 \leq k \leq m_2$ with $q_0 = n_2 + k \in F_2$ such that $d(T^{q_0} x, T^{q_0} y) < \delta'/2$. But then

$$d(T^{q_0} x, T^{q_0} x_2) \leq d(T^{q_0} x, T^{q_0} y) + d(T^{q_0} x_2, T^{q_0} y) < \delta',$$

which again is a contradiction. This implies that $d(T^n x, T^n y) > \delta$ for every $(x, y) \in B(x_1, r_1) \times B(x_2, r_2)$ and $n \in \mathbb{Z}$. Indeed, the claim holds, which implies that $(x_1, x_2) \notin RP(X, T)$. We have just proved that $RP(X, T) = \Delta_2$ and so (X, T) is equicontinuous (see [**4**, Theorem 9.3]). □

Now we are ready to show the following.

THEOREM 3.4. *Let (X, T) be an i.t.d.s. The following statements are equivalent:*

(1) *(X, T) is equicontinuous;*
(2) *$(K(X), T_K)$ is equicontinuous;*
(3) *$(K(X), T_K)$ is distal;*
(4) *$(K(X), T_K)$ is pointwise minimal.*

Proof. The equivalence of (1) and (2) can be found in [**7**, Proposition 7]. Trivially, we have (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4), so it remains to show that (4) implies (1).

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Assume that $(K(X), T_K)$ is pointwise minimal. First, we claim that (X, T) is distal. Let us assume on the contrary that there exists a pair $(x, y) \in P(X, T)$. Then there are $z \in X$ and $\{n_i\}_{i=1}^{\infty}$ such that $\lim_{i\to\infty} T^{n_i}x = \lim_{i\to\infty} T^{n_i}y = z$. Note that the sets $\{z\}$, $\{x, y\}$ are minimal points in $K(X)$. Hence, $\{x, y\} \in \overline{\mathrm{orb}(\{z\}, T_K)}$, which implies that $x = y$, so the claim holds. Therefore, X is invertible and (X, T^{-1}) is also distal.

Now we aim to show that (X, T) is locally almost periodic. Fix any $x \in X$ and any open neighborhood U of x. Take any open set V such that $x \in V \subset \overline{V} \subset U$. Then $\overline{V} \in \langle U\rangle$ and, since $(K(X), T_K)$ is pointwise minimal, there is a syndetic set $F \subset \mathbb{Z}$ such that $T_K^n \overline{V} \in \langle U\rangle$ for any $n \in F$. Equivalently, it means that $T^n V \subset T^n \overline{V} \subset U$ for all $n \in F$, showing that x is a locally almost periodic point, which, by Lemma 3.3, implies that (X, T) is equicontinuous, that is, (1) holds. The proof is completed. □

Remark 3.5. The notions of pointwise minimality, distality and equicontinuity are different in general. For example, the Denjoy extension of the irrational rotation (restricted to the non-wandering set, which is a Cantor set; see example from [**11**, Theorem 7.2.3]) is pointwise minimal but not distal. A disk rotated at different rates around a common center or Example 3.1 is distal but not equicontinuous. Theorem 3.4 shows that these three properties are the same for a map induced on the hyperspace.

4. *Levels of rigidity*

As mentioned before, the concepts of weak rigidity, rigidity and uniform rigidity were first introduced by Glasner and Maon [**19**]. It is known that these notions keep a strict inclusion relationship in general, but, for a minimal distal system, rigidity is equivalent to uniform rigidity and, for a minimal zero-dimensional system, weak rigidity is identical with equicontinuity [**19**]. A result due to Dong [**12**, Theorem 5.2.6] shows that if a minimal nilsystem is rigid then it is equicontinuous. Here we point out that the group extension of a rigid system is weakly rigid, which can be derived from a stronger result below. Note that in the following proof we strongly rely on the theory of Ellis semigroups. The reader not familiar with this topic is referred to [**2**, **4**].

We will denote by $E(X, T)$ the enveloping (or Ellis) semigroup associated with a t.d.s. (X, T), that is, the compact semigroup of X^X defined as the closure of $\{T^n : n \in \mathbb{Z}\}$ in X^X.

THEOREM 4.1. *Let (Y, S) be a weakly rigid i.t.d.s. and (X, T) a distal extension of (Y, S). Then (X, T) is weakly rigid.*

Proof. Fix any integer $n \geq 1$ and, for each $y \in Y^n$, put $F_y = \{p \in E(Y^n, S^{(n)}) : py = y\}$. Since Y^Y is endowed with the topology of pointwise convergence and (Y, S) is weakly rigid, we see that $\mathrm{id}^{(n)} \in E(Y^n, S^{(n)})$. Clearly, $\mathrm{id}^{(n)} \in F_y$ for every $y \in Y^n$ and hence $F = \bigcap_{y\in Y^n} F_y$ is a non-empty compact semigroup of $E(Y^n, S^{(n)})$. Then we can apply the Ellis–Numakura lemma (see [**2**, Lemma 2.1]; cf. [**36**]) to F, obtaining an idempotent element $u \in F$.

Let $\pi : X \to Y$ be the distal extension and then define π_n, the natural factor map given by $\pi_n(x_1, \ldots, x_n) = (\pi(x_1), \ldots, \pi(x_n))$. By [**4**, Theorem 7], there exists a unique continuous semigroup homomorphism $\theta : E(X^n, T^{(n)}) \to E(Y^n, S^{(n)})$ such that $\pi_n(xq) = \theta(q)\pi_n(x)$ for every $x \in X^n$ and $q \in E(X^n, T^{(n)})$. Clearly, $J = \theta^{-1}(u)$ is a closed semigroup in $E(X^n, T^{(n)})$, so again by the Ellis–Numakura lemma there is an

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idempotent element $v \in J$. Fix any $x \in X^n$ and observe that $\pi_n(xv) = u\pi_n(x) = \pi_n(x)$, which shows that x and vx are in the same fiber of π_n. By [**2**, Proposition 2.4], we immediately obtain that (x, vx) is a proximal pair and hence for each coordinate i we see that $(x_i, (vx)_i)$ is a proximal pair. But $\pi(x_i) = \pi((vx)_i)$; hence, $x_i = (vx)_i$ because π is distal. This shows that $x = vx$ and so again by [**2**, Proposition 2.4] we see that x is a recurrent point of $T^{(n)}$. This proves that T is n-rigid for every n, completing the proof. □

In the sequel, we will investigate the relations between various types of rigidity for (X, T) and $(K(X), T_K)$. It is easy to see that $(K(X), T_K)$ is weakly rigid; so is its subsystem (X, T). But the converse implication is not necessarily true. For example, set $\mathbb{T}^1 \times \{y\}$ in the t.d.s. (X, T) in Example 3.1 is not a positively recurrent point in $(K(X), T_K)$, but $(\mathbb{T}^2, T)$ is minimal and distal; therefore, it is weakly rigid by [**19**, Corollary 6.2].

Similarly, the rigidity of $(K(X), T_K)$ implies the same for (X, T), but the converse does not always hold. A rigid t.d.s. which is not uniformly rigid constructed in [**19**] can serve as an example. In this example, we take

$$X = \{re^{i\theta} : 0 \le \theta \le 2\pi, r = 1 - 2^{-n}, n = 1, 2, 3, \dots \text{ or } r = 1\}$$

and define: $Tz = z \exp(2\pi i \cdot 2^{-n})$ when $|z| = 1 - 2^{-n}$ and $Tz = z$ if $|z| = 1$. This map is rigid with respect to the sequence $n_k = 2^k$, but the set $R = \{r : r = 1 - 2^{-n}, n = 1, 2, 3, \dots \text{ or } r = 1\}$ is not a positively recurrent point of the hyperspace.

But the situation changes when we consider the property of uniform rigidity. It turns out that uniform rigidity holds always for both (X, T) and $(K(X), T_K)$, and furthermore on $(K(X), T_K)$ all the properties of rigidity considered so far coincide. Strictly speaking, we have the following.

THEOREM 4.2. *Let (X, T) be an i.t.d.s. Then the following statements are equivalent:*
(1) *(X, T) is uniformly rigid;*
(2) *$(K(X), T_K)$ is uniformly rigid;*
(3) *$(K(X), T_K)$ is rigid;*
(4) *$(K(X), T_K)$ is weakly rigid.*

Proof. First, we show that (1) $\Longrightarrow$ (2). Assume that (X, T) is uniformly rigid. That is, given $\varepsilon > 0$, there is $n \in \mathbb{N}$ such that $d(T^n x, x) < \varepsilon$ for all $x \in X$. Then, for each $A \in K(X)$, we have $d_H(A, T_K^n A) \le \max_{x \in A} d(T^n x, x) < \varepsilon$. This implies that $(K(X), T_K)$ is uniformly rigid.

Implications (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) follow by definition and hence it remains to show (4) $\Longrightarrow$ (1). Assume that $(K(X), T_K)$ is weakly rigid and fix any $\varepsilon > 0$. Let points $x_1, \dots, x_n \in X$ be such that $X = \bigcup_{i=1}^n B(x_i, \varepsilon/4)$. Denote $A_i = \overline{B(x_i, \varepsilon/4)} \in K(X)$ and observe that since $(K(X), T_K)$ is n-rigid for each $n \in \mathbb{N}$, it follows that $(A_1, \dots, A_n)$ is a recurrent point of $(K(X)^n, T_K^{(n)})$. Thus, there is $j \in \mathbb{Z}$ such that

$$(T_K^{(n)})^j (A_1, \dots, A_n) \in \langle B(x_1, \varepsilon/2)\rangle \times \cdots \times \langle B(x_n, \varepsilon/2)\rangle.$$

Therefore, $T_K^j A_i \in \langle B(x_i, \varepsilon/2)\rangle$, which equivalently means that $T^j A_i \subset B(x_i, \varepsilon/2)$, where $i = 1, \dots, n$. For every $x \in X$, we can find i such that $x \in A_i$ and so $d(T^j x, x) < \varepsilon$ for each $x \in X$. This shows that (X, T) is uniformly rigid, completing the proof. □

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5. *Pointwise recurrence*

In this section we focus on the pointwise recurrence on hyperspace. Firstly, we prove a few general facts on pointwise recurrence.

PROPOSITION 5.1. *Let (X, T) be a t.d.s. and $(K(X), T_K)$ be pointwise positively recurrent. Then we have:*

(1) *every non-trivial minimal subsystem of (X, T) is not mildly mixing. In particular, if (X, T) is minimal and $\#X > 1$, then (X, T) is not mildly mixing;*
(2) *(X, T) is positively weakly rigid.*

Proof. First, we prove (1). By [**24**, Theorem 5.8(2)], if (Y, T) is minimal and mildly mixing then for every pair of non-empty open subsets U and V, the set of transfer times $N(U, V)$ is an IP^*-set. Let (Y, T) be a mildly mixing minimal subsystem of (X, T) and $\#Y > 1$. Then we can find open sets U, V intersecting Y and such that $\overline{U} \cap \overline{V} = \emptyset$. Take any open set W such that $W \subset \overline{W} \subset U$ and $W \cap Y \neq \emptyset$. Denote $A = \overline{W} \cap Y \in K(X)$. Since $(K(X), T_K)$ is pointwise positively recurrent, $N(A, \langle U \rangle)$ is an IP-set by Lemma 2.2(1). But A has non-empty interior in Y and so $N(A, V)$ is an IP^*-set. In particular, $N(A, V) \cap N(A, \langle U \rangle) \neq \emptyset$, which implies that $T^j A \cap V \neq \emptyset$ and $T^j A \subset U$ for some $j \in \mathbb{N}$. This is a contradiction, because $U \cap V = \emptyset$.

Now, let us proceed with the proof of (2). It is known that if $\pi : Y \to Z$ is a factor map and $z \in Z$ is positively recurrent, then there is a positively recurrent point $y \in Y$ such that $\pi(y) = z$. Now assume that $(K(X), T_K)$ is pointwise positively recurrent. Then this property is shared by each subsystem $(K_n(X), T_K)$, where $n = 1, 2, \ldots$. Since $(X^n, T^{(n)})$ is an extension of $(K_n(X), T_K)$, for each $(x_1, \ldots, x_n) \in X^n$ we obtain some permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$ such that $(x_{i_1}, \ldots, x_{i_n})$ is positively recurrent. But dynamics of $(x_1, \ldots, x_n)$ under T^n is exactly the same as that of $(x_{i_1}, \ldots, x_{i_n})$, which shows that $(x_1, \ldots, x_n)$ is positively recurrent in $(X^n, T^{(n)})$. We obtain that $(X^n, T^{(n)})$ is pointwise positively recurrent, which shows that (X, T) is positively n-rigid for each $n \in \mathbb{N}$. The proof is finished. □

Remark 5.2. (1) In [**24**, Theorem 4.3], the authors showed that a non-trivial uniformly rigid system cannot be mildly mixing. So, comparing with Proposition 5.1(1), one can ask if there exists a mildly mixing system such that the hyperspace is positively 1-rigid. We do not know whether this is true.

(2) Note that if (X, T) is positively 2-rigid, then it has no asymptotic pairs and hence T is a homeomorphism. Hence, by Proposition 5.1(2), if $(K(X), T_K)$ is pointwise recurrent, then (X, T) is an i.t.d.s.

Combining Theorem 4.2 and Proposition 5.1, we obtain the following.

COROLLARY 5.3. *Let (X, T) be an i.t.d.s. The following are equivalent:*

(1) *(X, T) is uniformly rigid;*
(2) *$(K(X), T_K)$ is uniformly rigid;*
(3) *$(K(X), T_K)$ is rigid;*
(4) *$(K(X), T_K)$ is weakly rigid;*
(5) *$(K(K(X)), T_K)$ is pointwise recurrent.*

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By [**19**, Propostion 6.7], we know that uniform rigidity is equivalent to pointwise recurrence on $(K(X), T_K)$ when (X, T) is a zero-dimensional and minimal t.d.s. Later, we will show that this is also true for some class of minimal distal systems or the case of X being countable. By results we proved so far, it is easy to see that

$$X \text{ uniform rigidity} \implies K(X) \text{ pointwise recurrence} \implies X \text{ weak rigidity}$$

The t.d.s. (X, T) in Example 3.1 is weakly rigid and $(K(X), T_K)$ is not pointwise recurrent. The following problem remains unsolved.

PROBLEM 1. *Does there exist an i.t.d.s.* (X, T) *such that the induced i.t.d.s.* $(K(X), T_K)$ *is pointwise recurrent, but* (X, T) *is not uniformly rigid?*

We strongly believe that such an example exists, though we cannot provide one at this moment.

Now we consider the (topological) entropy of hyperspace. A remarkable result by Glasner and Weiss [**18**] is that there exists a minimal system (X, T) of zero entropy with a minimal subsystem (Y, T_K) of $(K(X), T_K)$ whose entropy is positive. From Remark 5.2(2) and the fact that a weakly rigid t.d.s. has zero entropy, we know that if $(K(X), T_K)$ is pointwise recurrent, then X has zero entropy. So, we ask if the entropy of $(K(X), T_K)$ is also zero in this case. We will answer this question affirmatively by showing a stronger result. For further considerations, we need the notion of a *locally recurrent system*, which can be found in Gottschalk and Hedlund's book [**21**].

Definition 5.4. Let (X, T) be an i.t.d.s. A point $x \in X$ is called *locally positively recurrent* if for each neighborhood U of x, there are a neighborhood V of x and an *IP*-set $Q \subset \mathbb{N}$ such that $T^l V \subset U$ for all $l \in Q \cup \{0\}$. We say that (X, T) is a *locally positively recurrent system* if all points in (X, T) are locally positively recurrent.

In a similar manner, we can define *locally negatively recurrent* and *locally recurrent* points and systems.

Remark 5.5.

(1) It is not hard to check the following examples:
   (a) the Denjoy minimal system (X, T) is locally recurrent but not 2-rigid. To see this, it is enough to note that the factor map $\pi : X \to \mathbb{T}^1$ defining a Denjoy minimal system is semi-open;
   (b) $(\mathbb{T}^2, T)$ in Example 3.1 is weakly rigid but not locally recurrent. Simply, for any $\varepsilon > 0$, the diameter of the set $(-\varepsilon, \varepsilon) \times \{y\}$ starts to exceed $1/2$ after sufficiently many iterations of T;
   (c) the rigid but not uniformly rigid system described in [**19**] is also not locally recurrent (it is sufficient to check the point $(1, 0)$ and its neighborhoods).

(2) From the definition, we know that the local recurrence of X implies its 1-rigidity. Also, we observe that if $K(X)$ is 1-rigid, then X is locally recurrent. Since the proof is similar to the one in Theorem 3.4, we omit the simple verification.

Topological entropy of a t.d.s. (X, T), denoted by $h_{\text{top}}(X, T)$, measures the complexity of the system. The notion of an entropy pair was introduced by Blanchard in [**8**]. Among other things, Blanchard showed that a t.d.s. has positive entropy if and only if there exists

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an entropy pair. In [**26**], a characterization of an entropy pair was obtained using an interpolating set. This approach was further extended in [**29**] after reformulation using the notion of an independence set. The following fact is a special case of [**29**, Lemma 3.4].

PROPOSITION 5.6. *Let (X, T) be a t.d.s. Then $(x_1, x_2) \in X^2 \setminus \Delta_2$ is an entropy pair if and only if for each neighborhood U_i of x_i for $i = 1, 2$, there is an independence set of positive density for (U_1, U_2), i.e. there is a subset $J \subset \mathbb{Z}_+$ with positive density such that for any non-empty finite subset $I \subset J$, we have*

$$\bigcap_{i \in I} T^{-i} U_{s(i)} \neq \emptyset$$

for any $s \in \{1, 2\}^I$.

After this brief introduction into the theory of entropy pairs, we are ready to prove the following.

THEOREM 5.7. *If (X, T) is a locally recurrent i.t.d.s., then $h_{\mathrm{top}}(K(X), T_K) = 0$ and hence $h_{\mathrm{top}}(X, T) = 0$.*

Proof. Assume that (X, T) is locally recurrent. We aim to prove that $h_{\mathrm{top}}(K(X), T_K) = 0$. Assume on the contrary that $h_{\mathrm{top}}(K(X), T_K) > 0$. Then there exists a non-diagonal entropy pair $(K_1, K_2) \in K(X)^2$ (see e.g. [**8**, Proposition 2]). Since $K_1 \neq K_2$, without loss of generality we may assume that there is $z \in K_2 \setminus K_1$. Then there exists an open set such that $V_1 \ni z$ and $K_1 \cap \overline{V_1} = \emptyset$. Then there are also open sets $U_1, \ldots, U_m, V_2, \ldots, V_n$ such that $V_1 \cap U_i = \emptyset$ for every $i = 1, \ldots, m$ and $K_1 \in \mathscr{U} = \langle U_1, U_2, \ldots, U_m \rangle$ and $K_2 \in \mathscr{V} = \langle V_1, V_2, \ldots, V_n \rangle$. By assumption, (X, T) is locally recurrent, so z is positively locally recurrent or negatively locally recurrent. But, by [**29**, Lemma 3.2], we see that (K_1, K_2) is an entropy pair also for T^{-1} and hence without loss of generality we may assume that z is positively locally recurrent. Then there are an open neighborhood $W_0 \subset V_1$ of z and an *IP*-set $Q \subset \mathbb{N}$ with

$$T^l W_0 \subset V_1 \quad \text{for all } l \in Q \cup \{0\}.$$

Then we can also find non-empty open sets $W_1, \ldots, W_s$ such that

$$\mathscr{W} = \langle W_0, W_1, \ldots, W_s, V_2, \ldots, V_n \rangle$$

forms another neighborhood of K_2. Let J be a positive-density subset of $\mathbb{N}$ associated to $(\mathscr{U}, \mathscr{W})$ and T_K by Proposition 5.6.

By Lemma 2.4, there is $l \in Q$ such that $\mathrm{BD}^*(J \cap (J - l)) > 0$, so there exists at least one $q \in J \cap (J - l)$. Then $q \in J$, $q + l \in J$ and, since J is an independence set for $(\mathscr{U}, \mathscr{W})$, we have $T^{-q}\mathscr{W} \cap T^{-(q+l)}\mathscr{U} \neq \emptyset$. If $K \in T^{-q}\mathscr{W} \cap T^{-(q+l)}\mathscr{U}$, then $T^q K \in \mathscr{W}$, $T^{q+l} K \in \mathscr{U}$, which, in other words, means that

$$T^q K \cap W_0 \neq \emptyset \quad \text{and} \quad T^{q+l} K \subset \bigcup_{i=1}^{m} U_i.$$

Observe that this leads to a contradiction, because

$$\emptyset = \bigcup_{i=1}^{m} U_i \cap V_1 \supset T^{q+l} K \cap V_1 \supset T^{q+l} K \cap T^l W_0 = T^l (T^q K \cap W_0) \neq \emptyset.$$

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We have just proved that $(K(X), T_K)$ does not have an entropy pair and hence its entropy is zero. □

As a direct consequence, we have the following.

COROLLARY 5.8. *Let (X, T) be an i.t.d.s. If $(K(X), T_K)$ is pointwise recurrent, then $h_{\text{top}}(K(X), T_K) = 0$.*

Remark 5.9. Clearly, there are numerous examples of pointwise recurrent dynamical systems (X, T) with positive topological entropy (e.g. minimal systems with positive entropy). By Corollary 5.8, for such systems the dynamics of T_K will always contain a non-recurrent point.

Remark 5.10. It is a long open question [**38**] whether 2-rigidity implies zero entropy. Note that under the additional assumption that X is positively 2-rigid, the answer is affirmative [**9**, **23**, **38**]. Theorem 5.7 indicates that this question has a positive answer whenever $X \times X$ is locally recurrent (stronger than $X \times X$ is pointwise recurrent, i.e. X is 2-rigid).

Related to entropy, we have the following questions.

PROBLEM 2. *Let (X, T) be an i.t.d.s. Denote $K^{n+1}(X) = K^n(K(X)),\ n \geq 0$.*

(1) *Is it true that $h_{\text{top}}(K^n(X), T_K) = 0$ for all $n \geq 2$ if $(K(X), T_K)$ is pointwise recurrent?*
(2) *Is it true that $h_{\text{top}}(K(X), T_K) = 0$ if (X, T) is weakly rigid?*

5.1. *Minimal distal systems.* This subsection is devoted to the problem of the equivalence between uniform rigidity and pointwise recurrence on hyperspace. A partial positive answer is given below for a special class of minimal distal systems.

THEOREM 5.11. *Assume that (X, T) is a skew product of a compact metric group G over a minimal equicontinuous system (Y, S). Then the following statements are equivalent:*

(1) *$Y \times \{e\}$ is recurrent for T_K, where e is the unit of G;*
(2) *$Y \times \{g\}$ is recurrent under action of T_K for some $g \in G$;*
(3) *$(K(X), T_K)$ is pointwise recurrent;*
(4) *(X, T) is uniformly rigid.*

Proof. By Theorem 4.2, we obtain (4) $\Longrightarrow$ (3) $\Longrightarrow$ (1) $\Longrightarrow$ (2), so it remains to show (2) $\Longrightarrow$ (4).

By definition, $X = Y \times G$, $T(y, g) = (Sy, \phi(y)g)$ for any $(y, g) \in Y \times G$, where $\phi : Y \to G$ is continuous, (Y, S) is a minimal equicontinuous t.d.s. and G is a compact metric group. Note that for any $(y, g) \in Y \times G$ and $n \in \mathbb{N}$, we have

$$T^n(y, g) = (S^n y, \phi(S^{n-1}y) \cdots \phi(y)g).$$

Fix any $\varepsilon > 0$. Since $G \times G \to G$, $(g_1, g_2) \mapsto g_1 g_2$ is continuous, there is $\delta > 0$ such that if

$$d((g_1, g_2), (g_1', g_2')) < \delta, \text{ then } d(g_1 g_2, g_1' g_2') < \varepsilon.$$

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Since (Y, S^{-1}) is also a minimal equicontinuous t.d.s. and the inverse of T is also a skew product defined by $T^{-1}(y, g) = (S^{-1}y, (\phi(S^{-1}y))^{-1}g)$, without loss of generality we may assume that $Y \times \{g\}$ is positively recurrent. Let V be a neighborhood of g in G with $\operatorname{diam}(Vg^{-1}) < \delta$. It is clear that Vg^{-1} is a neighborhood of e. Observe that

$$F_1 = \{n \in \mathbb{N} : T_K^n(Y \times \{g\}) \in \langle Y \times V \rangle\}$$

is an *IP*-set, which implies that for $n \in F_1$ we have $T^n(Y \times \{g\}) \subset Y \times V$. Since (Y, S) is minimal and equicontinuous, we can find an equivalent metric ρ on Y such that $\rho(Sx, Sy) = \rho(x, y)$ for every $x, y \in Y$ (due to the famous Halmos–von Neumann theorem [**22**, Theorem 6]). Take any $y_0 \in Y$ and note that

$$F_2 = \{n \in \mathbb{N} : \rho(S^n y_0, y_0) < \varepsilon/2\}$$

is an IP^* set (see Lemma 2.2(3)). Take any $n \in F_1 \cap F_2$ and observe that:

(a) $\rho(S^n y, y) < \varepsilon$ for each $y \in Y$, because we have $\rho(S^{n+k}y_0, S^k y_0) < \varepsilon/2$ for each $k \in \mathbb{N}$ and (Y, S) is minimal;

(b) $\{\phi(S^{n-1}y) \cdots \phi(y)g : y \in Y\} \subset V$.

By (b), we get that

$$\begin{aligned}\operatorname{diam}\{\phi(S^{n-1}y) \cdots \phi(y) : y \in Y\} &\le \operatorname{diam}(\{\phi(S^{n-1}y) \cdots \phi(y)g : y \in Y\}g^{-1}) \\ &\le \operatorname{diam}(Vg^{-1}) < \delta.\end{aligned}$$

For any $(y, h) \in X$, we have $T^n(y, h) = (S^n y, \phi(S^{n-1}y) \cdots \phi(y)h)$. So,

$$d(T^n(y, h), (y, h)) < \varepsilon + d(\phi(S^{n-1}y) \cdots \phi(y)h, h) < 2\varepsilon$$

and this implies that (X, T) is uniformly rigid. □

Remark 5.12. We remark here that there exists a minimal distal t.d.s. which is not equicontinuous and meets all the requirements of Theorem 5.11. We provide a (slightly technical) verification of this statement in Example A.2 of Appendix A.

Let $n \in \mathbb{N}$. We say that (X, T) is a *minimal distal system of class n* if X can be realized as $n + 1$ consecutive skew products, i.e. there are a compact metric Abelian group G_0 with minimal rotation $S = S_0$, compact metric groups $G_1, G_2, \ldots, G_n$ and skew products

$$S_i(g_0, \ldots, g_i) = (S_{i-1}(g_0, \ldots, g_{i-1}), \phi_i(g_0, \ldots, g_{i-1})g_i)$$

acting on $G_0 \times G_1 \times \cdots \times G_i$ for $i = 1, \ldots, n$ such that $T = S_n$ and $X = G_0 \times \cdots \times G_n$. The following result is an extension of Theorem 5.11.

THEOREM 5.13. *Let $n \ge 2$ and assume that (X, T) is a minimal distal system of class n. The following statements are equivalent:*

(1) $(G_0 \times \cdots \times G_{n-1} \times \{e_n\}, \ldots, G_0 \times \{e_1\} \times \cdots \times \{e_n\})$ *is recurrent under* $T_K^{(n)}$;

(2) $(G_0 \times \cdots \times G_{n-1} \times \{g_n^1\}, \ldots, G_0 \times \{g_1^1\} \times \cdots \times \{g_n^n\})$ *is recurrent under* $T_K^{(n)}$ *for some* $g_j^i \in G_j$, $i = 1, \ldots, j$;

(3) $(K(X), T_K)$ *is n-rigid;*

(4) (X, T) *is uniformly rigid.*

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Proof. By Theorem 4.2, it is clear that (4) $\Longrightarrow$ (3) $\Longrightarrow$ (1) $\Longrightarrow$ (2), so it remains to show (2) $\Longrightarrow$ (4). For simplicity, we present the proof only for the case $n = 2$ since the proof for the general case follows the same lines, but however is much more technical in detail.

Let $X = G_0 \times G_1 \times G_2$. By assumption, there are continuous maps $\phi_1 : G_0 \to G_1$, $\phi_2 : G_0 \times G_1 \to G_2$. Let $S_0 : G_0 \to G_0$, $S_1 : G_0 \times G_1 \to G_0 \times G_1$. Then, for $(g_0, g_1, g_2) \in X$, we have

$$T(g_0, g_1, g_2) = (S_1(g_0, g_1), \phi_2(g_0, g_1)g_2).$$

Thus,

$$T^n(g_0, g_1, g_2) = (S_1^n(g_0, g_1), \phi_2(S_1^{n-1}(g_0, g_1)) \cdots \phi_2(S_1(g_0, g_1))\phi_2(g_0, g_1)g_2).$$

Note that

$$S_1^k(g_0, g_1) = (S_0^k\, g_0, \phi_1(S_0^{k-1}g_0) \cdots \phi_1(g_0)g_1).$$

Let $\varepsilon > 0$. We choose $\delta > 0$ such that if $(h_1, h_2), (h_1', h_2') \in G_1 \times G_1$(respectively $G_2 \times G_2$) with $d((h_1, h_2), (h_1', h_2')) < \delta$, then $d(h_1h_2, h_1'h_2') < \varepsilon$. Let $p = (G_0 \times \{g_1^1\} \times \{g_2^1\}, G_0 \times G_1 \times \{g_2^2\})$, V_1, V_2, V_2' be neighborhoods of g_1^1, g_2^1, g_2^2 respectively with $\operatorname{diam}(V_1(g_1^1)^{-1}) < \delta$ and $\operatorname{diam}(V_2'(g_2^2)^{-1}) < \delta$. Put $V_p = (\langle G_0 \times V_1 \times V_2\rangle, \langle G_0 \times G_1 \times V_2'\rangle)$. Then, without loss of generality,

$$F_1 = \{n \in \mathbb{N} : (T \times T)^n(p) \in V_p\}$$

is an *IP*-set. Since (G_0, S_0) is a minimal group rotation, then, by the Halmos–von Neumann theorem, it is equicontinuous. So, for some fixed $g_0' \in G_0$, we have that

$$F_2 = \{n \in \mathbb{N} : \rho(S^n g_0', g_0') < \varepsilon/2\}$$

is an IP^*-set, where, as in the proof of Theorem 5.11, ρ is an equivalent metric on Y which does not increase distance under iteration of S.

For each $n \in F_1 \cap F_2$, we have:

(a) $\rho(S_0^n\, g_0, g_0) < \varepsilon$ for each $g_0 \in G_0$, because we have $\rho(S_0^{n+k}g_0', S_0^k g_0') < \varepsilon/2$ for each $k \in \mathbb{N}$ and (G_0, S_0) is minimal;

(b) $\{\phi_1(S_0^{n-1}g_0) \cdots \phi_1(g_0)g_1^1 : g_0 \in G_0\} \subset V_1$;

(c) $\{\phi_2(S_1^{n-1}(g_0, g_1)) \cdots \phi_2(g_0, g_1)g_2^2 : (g_0, g_1) \in G_0 \times G_1\} \subset V_2'$.

From (b) and (c), we know that

$$\operatorname{diam}\{\phi_1(S_0^{n-1}g_0) \cdots \phi_1(g_0) : g_0 \in G_0\} \le \operatorname{diam}(V_1(g_1^1)^{-1}) < \delta,$$
$$\operatorname{diam}\{\phi_2(S_1^{n-1}(g_0, g_1)) \cdots \phi_2(g_0, g_1) : (g_0, g_1) \in G_0 \times G_1\} \le \operatorname{diam}(V_2'(g_2^2)^{-1}) < \delta.$$

It implies that

$$d(T^n(g_0, g_1, g_2), (g_0, g_1, g_2)) < 3\varepsilon$$

and then (X, T) is uniformly rigid. □

PROBLEM 3. *The following questions arise naturally.*

(1) *Does the equivalence of (3) and (4) in Theorem 5.13 still hold when (X, T) is a general minimal distal system (not necessarily a group skew product)?*

(2) *Is there a minimal distal system (X, T) of class n such that $(K(X), T_K)$ is $(n-1)$-rigid but $(G_0 \times \cdots \times G_{n-1} \times \{e_n\}, \ldots, G_0 \times \{e_1\} \times \cdots \times \{e_n\})$ is not recurrent?*

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We remark that if Problem 3(2) has a positive answer, then $(K(X), T_K)$ is $(n-1)$-rigid, and not n-rigid. Particularly, if Problem 3(2) has a positive answer for $n=2$, then Problem 1 will be solved, i.e. there is a t.d.s. (X, T) such that $(K(X), T_K)$ is pointwise recurrent and at the same time (X, T) is not uniformly rigid.

5.2. *The countable case.* In this section we discuss pointwise recurrence induced on the hyperspace $K(X)$ when X is countable.

First, we recall the notion of the derived set of X of order α. A point x of X is an *accumulation point* of the set X if $x \in \overline{X \setminus \{x\}}$. The set of accumulation points of X is said to be the *derived set* of X, denoted X^*. The derived set of X of order α is defined by the conditions: $X^{(0)} = X$, $X^{(1)} = X^*$, $X^{(\alpha+1)} = (X^{(\alpha)})^*$ and $X^{(\lambda)} = \bigcap_{\alpha<\lambda} X^{(\alpha)}$ if λ is a limit ordinal number. We put $d(X) = \alpha$ if $X^{(\alpha)} \neq \emptyset$ and $X^{(\alpha+1)} = \emptyset$. Here $d(X)$ is called the *derived degree* of X.

It is well known that a compact metric space X is a countable set if and only if $d(X)$ exists and it is a countable ordinal number. In this case, if $d(X) = \alpha$, then $X^{(\alpha)}$ is a finite set. More details can be found in the book by Kuratowski [**31**, p. 261]. In the proof of the next result, we will need the following auxiliary result.

LEMMA 5.14. *Let (X, T) be a countable i.t.d.s. consisting of periodic points and with $d(X) = \alpha$. Assume that $A = X^{(\alpha)} = \{a_1, a_2, \ldots, a_m\}$ consists of fixed points of T. Let $\varepsilon > 0$ be such that $B_\varepsilon(A) = \overline{B_\varepsilon(A)}$, $B_\varepsilon(a_j) = \overline{B_\varepsilon(a_j)}$ for each $a_j \in A$ and $T(B_\varepsilon(a_i)) \cap B_\varepsilon(a_j) = \emptyset$ when $i \neq j$. Denote:*

(i) $X_1 = \{x \in X : x \notin B_\varepsilon(A)\}$;

(ii) $X_2 = \bigcup_{j=1}^m X_{2,j}$, *where* $X_{2,j} = \{x \in X : x \in B_\varepsilon(a_j)$ *and* $\operatorname{orb}(x, T) \not\subset B_\varepsilon(a_j)\}$;

(iii) $X_3 = \bigcup_{j=1}^m X_{3,j}$, *where* $X_{3,j} = \{x \in X : x \in B_\varepsilon(a_j)$ *and* $\operatorname{orb}(x, T) \subset B_\varepsilon(a_j)\}$.

Then there exists an integer n such that if we denote $\widetilde{T} = T^n$ and:

(I) $\widetilde{X}_1 = \{x \in X : x \notin B_\varepsilon(A)\}$;

(II) $\widetilde{X}_2 = \bigcup_{j=1}^m \widetilde{X}_{2,j}$, *where* $\widetilde{X}_{2,j} = \{x \in X : x \in B_\varepsilon(a_j)$ *and* $\operatorname{orb}(x, \widetilde{T}) \not\subset B_\varepsilon(a_j)\}$;

(III) $\widetilde{X}_3 = \bigcup_{j=1}^m \widetilde{X}_{3,j}$, *where* $\widetilde{X}_{3,j} = \{x \in X : x \in B_\varepsilon(a_j)$ *and* $\operatorname{orb}(x, \widetilde{T}) \subset B_\varepsilon(a_j)\}$,

then $\widetilde{X}_2 = \emptyset$.

The proof of Lemma 5.14 is highly technical. For clarity of presentation, we postpone its details to Appendix B.

THEOREM 5.15. *Let (X, T) be an i.t.d.s. acting on a countable space X. The following statements are equivalent:*

(1) *$(K(X), T_K)$ is pointwise recurrent;*

(2) *$(K(X), T_K)$ is uniformly rigid;*

(3) *$(K(X), T_K)$ is rigid;*

(4) *$(K(X), T_K)$ is weakly rigid;*

(5) *(X, T) is uniformly rigid;*

(6) *(X, T) is pointwise periodic.*

Proof. We have already proved the equivalence of (2)–(5) in Theorem 4.2, and clearly (5) implies (1). To see that (1) implies (6), take any $x \in X$ and put $A = \overline{\operatorname{orb}(x, T)}$ or

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$A = \overline{\mathrm{orb}(x, T^{-1})}$ if x is positively or negatively recurrent, respectively. Since A is compact and countable, it is not a perfect set and hence has at least one isolated point. This shows that x must be a periodic point.

It remains to show (6) $\Longrightarrow$ (5). If X is finite, (5) is obvious. When X is infinite, there is a countable ordinal number α such that $d(X) = \alpha$. We will prove that (X, T) is uniformly rigid by induction on α.

We denote $A = X^{(\alpha)} = \{a_1, a_2, \ldots, a_m\}$ and may further assume that each $a_i \in A$ is a fixed point, since, in the proof of uniform rigidity, we can always replace T by its higher iterate.

First, assume that $\alpha = 1$. Fix any $\varepsilon > 0$. We can decrease ε if necessary in such a way that $B_\varepsilon(A) = \overline{B_\varepsilon(A)}$, $B_\varepsilon(a_j) = \overline{B_\varepsilon(a_j)}$ for each $a_j \in A$ and $T(B_\varepsilon(a_i)) \cap B_\varepsilon(a_j) = \emptyset$ when $i \neq j$. Now we divide X into three subsets:

(I) $X_1 = \{x \in X : x \notin B_\varepsilon(A)\}$;

(II) $X_2 = \bigcup_{j=1}^m X_{2,j}$, where $X_{2,j} = \{x \in X : x \in B_\varepsilon(a_j) \text{ and } \mathrm{orb}(x, T) \not\subset B_\varepsilon(a_j)\}$;

(III) $X_3 = \bigcup_{j=1}^m X_{3,j}$, where $X_{3,j} = \{x \in X : x \in B_\varepsilon(a_j) \text{ and } \mathrm{orb}(x, T) \subset B_\varepsilon(a_j)\}$.

Note that X_1 is closed and $X_1 \cup X_2$ is invariant under T. By Lemma 5.14, we can find $n \in \mathbb{N}$ such that $\widetilde{X}_2 = \emptyset$, $(\widetilde{X}_1, \widetilde{T}|_{\widetilde{X}_1})$ is a subsystem and $\widetilde{X}_1$ is finite, where:

(I)′ $\widetilde{X}_1 = \{x \in X : x \notin B_\varepsilon(A)\}$;

(II)′ $\widetilde{X}_2 = \bigcup_{j=1}^m \widetilde{X}_{2,j}$, where $\widetilde{X}_{2,j} = \{x \in X : x \in B_\varepsilon(a_j) \text{ and } \mathrm{orb}(x, \widetilde{T}) \not\subset B_\varepsilon(a_j)\}$;

(III)′ $\widetilde{X}_3 = \bigcup_{j=1}^m \widetilde{X}_{3,j}$, where $\widetilde{X}_{3,j} = \{x \in X : x \in B_\varepsilon(a_j) \text{ and } \mathrm{orb}(x, \widetilde{T}) \subset B_\varepsilon(a_j)\}$.

Let r be the common period of each point in $\widetilde{X}_1$ under $\widetilde{T}$. Then we have $d(x, T^{nr}x) = d(x, \widetilde{T}^r x) = 0 < 2\varepsilon$ for every $x \in \widetilde{X}_1$. For any $t \in \{1, 2, \ldots, m\}$, it is clear that $d(x, T^{nr}x) = d(x, \widetilde{T}^r x) < 2\varepsilon$ for each $x \in \widetilde{X}_{3,t} = B_\varepsilon(a_t)$. Hence, $d(x, T^{nr}x) < 2\varepsilon$ for each $x \in X$. Since ε was arbitrary, we obtain that T is uniformly rigid.

Next we assume that (5) holds for all cases with $d(X) < \alpha$; the goal is to prove that (5) is still true for $d(X) = \alpha$. Fix $\varepsilon > 0$ and define sets X_1, X_2, X_3 as above. By Lemma 5.14 (after decreasing ε if necessary), we can find $n \in \mathbb{N}$ such that $\widetilde{X}_2 = \emptyset$, $(\widetilde{X}_1, \widetilde{T}|_{\widetilde{X}_1})$ is a subsystem and $d(\widetilde{X}_1) < \alpha$, where as before sets $\tilde{X}_1$, $\tilde{X}_2$, $\tilde{X}_3$ are defined by conditions (I)′–(III)′. By our inductive assumption, i.t.d.s. $(\widetilde{X}_1, \widetilde{T}|_{\widetilde{X}_1})$ is uniformly rigid. Moreover, note that for any $t \in \{1, 2, \ldots, m\}$, we have $d(x, \widetilde{T}^r x) < 2\varepsilon$ for each $x \in \widetilde{X}_{3,t} = B_\varepsilon(a_t)$ and $r \in \mathbb{N}$. Hence, there is an integer $r > 0$ such that $d(x, T^{nr}x) < 2\varepsilon$ for each $x \in X$. Since ε was arbitrary, this shows that T is uniformly rigid, completing the induction and finishing the proof. □

6. *Dense recurrent points in the hyperspace*

In this section we discuss the situation when $(K(X), T_K)$ has a dense set of recurrent points.

THEOREM 6.1. *The following assertions hold.*

(1) *If (X, T) is an i.t.d.s. and $(K(X), T_K)$ has a dense set of recurrent points, then so does (X, T).*

(2) *If (X, T) is a transitive t.d.s., then $(K(X), T_K)$ has a dense set of positively recurrent points.*

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Proof. Assume that $(K(X), T_K)$ has dense recurrent points. Let $U, V \subset X$ be two non-empty open subsets of X with $\overline{V} \subset U$. Then $\langle V \rangle$ is a non-empty open set in $K(X)$. By assumption, without loss of generality, there exist a positively recurrent point $A \in \langle V \rangle$ and an *IP*-subset Q of $\mathbb{N}$ such that $T_K^l A \in \langle V \rangle$, i.e. $T^l A \subset V$, for all $l \in Q$. By [**13**, Lemma 2.3], $\overline{V} \cap \mathrm{Rec}(T) \neq \emptyset$, so $U \cap \mathrm{Rec}(T) \neq \emptyset$ and (1) holds.

Now assume that (X, T) is transitive and fix any non-empty open sets $V_1, \ldots, V_n \subset X$. Let $x \in X$ be a point with dense orbit and let $k_1, \ldots, k_n \in \mathbb{N}$ be such that $T^{k_i} x \in V_i$. If we put $A = \{T^{k_i} x : i = 1, \ldots, n\}$, then $A \in \langle V_1, \ldots, V_n \rangle$ and, since x is positively recurrent,

$$\liminf_{j \to \infty} d_H(A, T_K^j A) = 0,$$

completing the proof. □

From the above proof, we obtain that the self-product of a transitive t.d.s. (X, T) has dense recurrent points. But we note that there are transitive t.d.s.'s (X, T) and (Y, S) such that $X \times Y$ does not have a dense set of positively recurrent points (for example, see [**13**, Theorem 6.4]). These dynamical systems can be used to produce the following example.

Example 6.2. If we denote by Z the disjoint union $Z = X \cup Y$ and define the t.d.s. (Z, F) by $F|_X = T$, $F|_Y = S$, then clearly (Z, F) has dense positively recurrent points, but there are open sets $U \subset X$, $V \subset Y$ such that $(F^{(2)})^j (U \times V) \cap U \times V = \emptyset$. Then $\langle U, V \rangle$ does not contain positively recurrent points of F_K; in particular, $(K(Z), F_K)$ does not have dense positively recurrent points.

6.1. *Dense distal closed subsets.* It is easy to check that if (X, T) has dense distal points, then so does $(K(X), T_K)$. In this subsection we shall show that the converse implication is not true, which answers a question left open in [**34**].

Before we proceed with the example, let us first recall some basic notions on symbolic dynamics. Let $\Sigma_2^+ = \{0, 1\}^{\mathbb{N}}$ equipped with the product topology. Then Σ_2^+ is compact, and the *shift map* $\sigma : \Sigma_2^+ \to \Sigma_2^+$ defined by $\sigma(x)_n = x_{n+1}$ for $n \in \mathbb{N}$ is continuous. Any non-empty, closed and σ-invariant subset $X \subset \Sigma_2^+$ is called a *subshift* and is identified with the subsystem (X, σ).

Fix $n \in \mathbb{N}$; we call $w \in \{0, 1\}^n$ a *word of length* n. We denote by $|w| = n$ the length of w and $|w|_a = \#\{i \in \mathbb{N} : w_i = a\}$ denotes the number of occurrences of the symbol a in the word w. For any two words $u = u_1 u_2 \ldots u_n$ and $v = v_1 v_2 \ldots v_m$, the *concatenation* of u, v is defined by $uv = u_1 u_2 \ldots u_n v_1 v_2 \ldots v_m$. Analogously, u^m is defined by the concatenation of m copies of u for some $m \in \mathbb{N}$, and u^∞ the infinite concatenation of u. Let X be a subshift of Σ_2^+ and $x = x_1 x_2 \ldots \in X$. We say that a word $w = w_1 w_2 \ldots w_n$ appears in x at position t if $x_{t+j-1} = w_j$ for $j = 1, 2, \ldots, n$. By $\mathscr{L}(X)$, we denote the *language* of subshift X, that is, the set consisting of all words that can appear in some $x \in X$, and we write $\mathscr{L}_n(X)$ as the set of all words of length n in $\mathscr{L}(X)$. For any word $u \in \mathscr{L}_n(X)$, its *cylinder set* is defined by $[u] = \{x \in X : x_1 x_2 \ldots x_n = u\}$. Note that all cylinder sets $\{[u] : u \in \mathscr{L}(X)\}$ form a basis of the topology of X.

The basis for our construction will be a sequence of weakly mixing minimal subshifts provided by the following lemma.

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LEMMA 6.3. *Fix an $\varepsilon > 0$ and a non-empty word w with at least one occurrence of* 1. *Then there exists a subshift $X = X(w, \varepsilon) \subset \Sigma_2^+$ such that (for some integer $n > 0$) the following conditions are satisfied:*

(i) *X is minimal and $\#X = \infty$;*
(ii) *there are weakly mixing minimal systems $(D_0, \sigma^n), \ldots, (D_{n-1}, \sigma^n)$ (not necessarily distinct) such that $X = \bigcup_{i=0}^{n-1} D_i$ and $\sigma(D_i) = D_{i+1(\mathrm{mod}\, n)}$;*
(iii) *$D_0 \subset [w]$;*
(iv) *for some integer $k > 0$, we have $k, k+1 \in N_\sigma([w], [w])$ in X, i.e. there are words u, v such that $|u| = |v| + 1$ and wuw, $wvw \in \mathscr{L}(X)$;*
(v) *there is $N \le n$ such that $|v|_1/m < \varepsilon$ for every $v \in \mathscr{L}_m(X)$ and $m \ge N$;*
(vi) *for every $v \in \mathscr{L}(X)$ with $|v| \ge |w|$, we have $|v|_1/|v| \le |w|_1/|w|$.*

Proof. First, we will present a construction of the subshift $X = X(w, \varepsilon)$. Let $s = |w|$ and let $t > 3s$ be such that $2s/(s+t) < \varepsilon$. We also assume that $2s/t < |w|_1/|w|$. Put $u_0 = w0^t w0^{t+1} w0^t$ and $v_0 = w0^{3t+2s+1}$ and note that $|u_0| = |v_0|$. Denote $n = |u_0|$. Set $u_1 = u_0 v_0 u_0$ and $v_1 = u_0 v_0$. Then we recursively define

$$u_{k+1} = u_k v_k u_k u_k \quad \text{and} \quad v_{k+1} = u_k v_k v_k u_k$$

for all $k \ge 1$. Observe that $|u_1| - |v_1| = n$ and therefore $|u_{k+1}| - |v_{k+1}| = |u_k| - |v_k| = n$ for every $k \ge 1$. Moreover, n divides $|u_k|$ and $|v_k|$ for each $k \ge 1$.

Let $z = \lim_{k \to +\infty} u_k$ and $X = \overline{\mathrm{orb}(z, \sigma)}$, i.e. each u_k is a prefix of z. Note that each u_{k+1} and v_{k+1} is a concatenation of u_k and v_k and both words appear at least once. Therefore, since z can be presented as an infinite concatenation of u_{k+1} and v_{k+1}, we immediately obtain that $N_\sigma(z, [u_k])$ is positively syndetic for every k. This shows that z is a minimal point and hence X is minimal.

Put $D_0 = \overline{\mathrm{orb}(z, \sigma^n)}$ and $D_i = \sigma^i(D_0)$ for $i = 1, \ldots, n-1$. It is clear that each (D_i, σ^n) is minimal and σ permutes sets $D_0, \ldots, D_{n-1}$ periodically. Note that z is an infinite concatenation of u_0 and v_0. Furthermore, both u_0, v_0 have length n and both have word w as a prefix. Therefore, $\sigma^{ni}(z) \in [w]$ for every $i \ge 0$; in particular, $D_0 \subset [w]$ and so (iii) is satisfied. Furthermore, u_k is both prefix and suffix of $u_{k+1} = u_k v_k u_k u_k$ and $v_{k+1} = u_k v_k v_k u_k$. This shows that for some m, we have $m, m+1 \in N_{\sigma^n}([u_k], [u_k])$ since $|u_k| - |v_k| = n$. As each D_i is minimal, by Lemma 2.3(1), we have that (D_i, σ^n) is weakly mixing for all i. This proves (ii). By the definition of u_0, we also have $t + s, t + s + 1 \in N_\sigma([w], [w])$, which gives (iv). Additionally, observe that D_0 has at least two points (starting with u_0 and v_0) and hence by weak mixing (D_0, σ^n) is infinite; in particular, X is infinite, which proves (i).

Finally, put $N = s + t$ and fix any $v \in \mathscr{L}_m(X)$ for some $m \ge N$. We can write $m = jN + r$ with $r < N$. Clearly, v must appear at some position in z. By the definition of u_0, v_0, every word of N consecutive symbols in z can have at most $|w|_1 \le s$ occurrences of symbol 1. Then

$$\frac{|v|_1}{|v|} \le \frac{(j+1)|w|_1}{j(s+t)} \le \frac{2s}{s+t} < \varepsilon$$

and so (v) holds. This also shows that if $|v| \ge N$, then $|v|_1/|v| < (2s/t) \le |w|_1/|w|$. But, if $|v| \le N$, then $|v|_1 \le |w|_1$ and, since in (vi) we need only to consider $|v| \ge |w|$, then

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also in this case $|v|_1/|v| \le |w|_1/|w|$. This shows (vi), that is, for every $v \in \mathscr{L}(X)$ with $|v| \ge |w|$, we have $|w|_1/|w| \ge |v|_1/|v|$. The proof is completed. □

Now we are ready to perform the main construction.

Start with $w_1 = 010^{10}$ and $\varepsilon_1 = 1/9$ and let $M_1 = X(w_1, \varepsilon_1)$ be a minimal system provided by Lemma 6.3. For each $n \ge 1$, we put $\varepsilon_n = 9^{-n}$. Now we will present how to construct subshifts M_n inductively. For induction, assume that we already constructed subshifts $M_1, \dots, M_n$ by Lemma 6.3 with words $w_1, \dots, w_n$ and $\varepsilon_1, \dots, \varepsilon_n$. We also assume that $|w_i| < |w_{i+1}|$ for each i and $|w_n| \ge (n-1)9^{n-1}$. Enumerate words $\mathscr{L}_n(\bigcup_{i=1}^n M_i) = \{u_1, \dots, u_{s_n}\}$. Let

$$w_{n+1} = u_1 0^{n9^n} u_2 0^{n9^n} \dots u_{s_n} 0^{n9^n}$$

and let M_{n+1} be the minimal subshift constructed by Lemma 6.3 for word w_{n+1} and ε_{n+1}, i.e. $M_{n+1} = X(w_{n+1}, \varepsilon_{n+1})$. Denote $\mathbb{X} = \overline{\bigcup_{n=1}^{+\infty} M_n}$. Clearly, $\mathbb{X}$ is closed and σ-invariant and therefore it is a subshift.

THEOREM 6.4. *The subshift $\mathbb{X}$ has the following properties:*

(1) *if $M \subset \mathbb{X}$ is minimal, then either $M = \{0^\infty\}$ or $M = M_n$ for some $n \ge 1$;*
(2) *$(\mathbb{X}, \sigma)$ is weakly mixing; and*
(3) *$(K(\mathbb{X}), \sigma_K)$ has dense periodic points.*

Proof. Fix any minimal set $M \subset \mathbb{X}$ and assume that $M \ne \{0^\infty\}$. Fix $z \in M$ and observe that the set $\{i : z_i = 1\}$ is positively syndetic. We additionally assume that $z_0 = 1$. For $k = 2, 3, \dots$, let v_k denote the prefix of z ending with the symbol 1 and such that $|v_k|_1 = k$. Since the symbol 1 appears in z syndetically, there is $\delta > 0$ such that $|v_k|_1/|v_k| > \delta$ for every k. Let $m \in \mathbb{N}$ with $1/m < \delta$. Assume on the contrary that $M \ne M_n$ for every n. Then there exists K such that $v_k \notin \mathscr{L}(M_n)$ for every $n \le m$ and $k > K$. Take any $k > K$ and let j be the minimal integer such that $v_k \in \mathscr{L}(M_j)$. If $|v_k| > |w_j|$, then by (vi) we obtain

$$\frac{|v_k|_1}{|v_k|} \le \frac{|w_j|_1}{|w_j|} \le \frac{1}{9^j} < \frac{1}{9^m} < \frac{1}{m},$$

which is a contradiction. But, if $|v_k| \le |w_j|$, then, by the method of construction of $X(w_j, \varepsilon_j)$ (in particular, the definition of s, t in the proof of Lemma 6.3) and the fact that v_k starts and ends with the symbol 1, we obtain that v_k is a subword of

$$w_j = u_1 0^{(j-1)9^{(j-1)}} u_2 0^{(j-1)9^{(j-1)}} \dots u_{s_{j-1}} 0^{(j-1)9^{(j-1)}},$$

where by the definition each word $u_i \in \mathscr{L}_{j-1}(M_r)$ for some $r < j$. By the minimality of j, we see that v_k cannot be a subword of any u_i and, since it starts and ends with the symbol 1, it must be placed in w_j in such a way that it contains at least one word $0^{(j-1)9^{(j-1)}}$. Let p be a minimal number such that for some i, the word v_k is a subword of $u_i 0^{(j-1)9^{(j-1)}} u_{i+1} 0^{(j-1)9^{(j-1)}} \dots u_{i+p}$. Then simple calculations yield that

$$\frac{|v_k|_1}{|v_k|} \le \frac{(p+1)(j-1)}{p(j-1)9^{(j-1)}} \le \frac{2}{9^{j-1}} \le \frac{2}{9^m} < \frac{1}{m} < \delta,$$

which is again a contradiction, proving that $M = M_n$ for some $n \ge 1$.

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In order to prove that $(\mathbb{X}, \sigma)$ is weakly mixing, it is enough to show that for any $u, v \in \mathscr{L}(\mathbb{X})$, there is k such that $k, k+1 \in N_\sigma([u], [v])$. Without loss of generality, we may assume that $|u| = |v|$. Let m be such that $u, v \in \mathscr{L}(\bigcup_{i=1}^{m} M_i)$. We can extend words u, v if necessary, obtaining that $|u| = |v| = m$. Then u, v are subwords of w_{m+1} and, then, by the definition of M_{m+1} and (iv), we obtain an integer t such that $t, t+1 \in N_\sigma([w_{m+1}], [w_{m+1}])$. But, since both u, v are subwords of w_{m+1}, there clearly exists k such that $k, k+1 \in N_\sigma([u], [v])$. This shows that $(\mathbb{X}, \sigma)$ is weakly mixing by Lemma 2.3(1).

By the same argument, if we fix any word $u \in \mathscr{L}(\mathbb{X})$, then there is m such that u is a subword of w_{m+1} and by (iii) there are n and $D \subset M_{m+1} \cap [w_{m+1}]$ such that $\sigma^n(D) = D$. But there is also $j \geq 0$ such that $\sigma^j([w_{m+1}]) \subset [u]$, which shows that there is a periodic set $\sigma^j(D) \subset [u]$. Indeed, $(\mathbb{X}, \sigma)$ has dense periodic sets. □

Let (X, T) and (Y, S) be two t.d.s.'s. A non-empty, closed and invariant subset $J \subset X \times Y$ is a *joining* of X and Y if J projects onto X and Y, respectively. We say that (X, T) and (Y, S) are *disjoint* if $X \times Y$ is the only joining.

The question of which t.d.s. is disjoint from all minimal systems was asked in [**15**] and Furstenberg showed that every weakly mixing P-system has this property. A systematic study of the question was carried out in [**25**], where the condition of a P-system was weakened to a system with dense small periodic sets. In [**13**, Theorem 7.14] and [**37**, Theorem 4.4], it was shown that each weakly mixing system with dense distal points is disjoint from any minimal systems; and in [**34**, Theorem 5.5] the authors proved that if (X, T) is a weakly mixing t.d.s. and $K(X)$ has dense distal points, then (X, T) is disjoint from all minimal systems. The t.d.s. constructed in Theorem 6.4 does not have dense distal points but $(K(X), T_K)$ has the property, as explained below. It gives a positive answer to a question left open in [**34**].

COROLLARY 6.5. *The above t.d.s.* $(\mathbb{X}, \sigma)$ *does not have dense distal points; however,* $(K(\mathbb{X}), \sigma_K)$ *is a* P*-system. In particular, both* $(\mathbb{X}, \sigma)$ *and* $(K(\mathbb{X}), \sigma_K)$ *are disjoint from any minimal system.*

Proof. Note that closure of the orbit of a distal point is a minimal set. By Theorem 6.4, the minimal set in $\mathbb{X}$ is either $\{0^\infty\}$ or M_n for some $n \geq 1$. But Lemma 6.3 shows that each M_n can be divided into several infinite, relatively weakly mixing and minimal subsystems. Applying Lemma 2.3(2), we see that $(\mathbb{X}, \sigma)$ does not have dense distal points (in fact, 0^∞ is the unique distal point in $\mathbb{X}$). By Lemma 2.5, Theorem 6.4 and [**34**, Theorem 5.5], the result follows. □

Remark 6.6. The construction in Theorem 6.4 was inspired by the technique developed in [**14**] to prove the existence of a mixing shift space with a dense set of periodic points but without ergodic measure with full support.

A. *Appendix. Some examples*

In [**12**, Theorem 5.2.6], Dong showed that a minimal nilsystem (X, T) is uniformly rigid if and only if (X, T) is equicontinuous. So, one may ask if this holds for minimal distal systems. In this section we consider a special class of distal systems and show that in this class uniform rigidity and equicontinuity are different properties.

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Let $X = \mathbb{T}^2$ and $T : X \to X$ be a group extension over an irrational rotation on $\mathbb{T}^1$, i.e. for any $(x, y) \in [0, 1) \times [0, 1)$,

$$T(x, y) = (x + \alpha \pmod 1, \phi(x) + y \pmod 1), \tag{A.1}$$

where $\phi : \mathbb{T}^1 \to \mathbb{T}^1$ is continuous. Then, for any $(x, y) \in X$ and $n \in \mathbb{N}$, we have

$$T^n(x, y) = \left(x + n\alpha \pmod 1, \sum_{i=0}^{n-1} \phi(x + i\alpha) + y \pmod 1\right).$$

It is easy to see that (X, T) is distal.

We will choose a suitable ϕ to induce desired properties. Our construction will rely on the degree of ϕ. Recall that [**10**, Proposition 1] an integer $d = \deg(\phi)$ is called the *degree* of ϕ if it is the unique number satisfying that $F(x + 1) = F(x) + d$ for some continuous map $F : \mathbb{R} \to \mathbb{R}$ and each $x \in \mathbb{R}$ satisfying $\Pi \circ F = \phi \circ \Pi$ for the surjective map $\Pi : \mathbb{R} \to \mathbb{T}^1$ defined by $\Pi(t) = (\cos 2\pi t, \sin 2\pi t)$ for $0 \le t < 1$. It is easy to check that additivity holds, i.e. if there are two continuous maps ϕ_1, ϕ_2 on $\mathbb{T}^1$ with $\deg(\phi_1) = d_1$ and $\deg(\phi_2) = d_2$, then $\deg(\phi_1 + \phi_2) = d_1 + d_2$. One can refer to [**10**, Ch. IX] for more properties of degree.

PROPOSITION A.1. *If* $|\deg(\phi)| \ge 1$, *then* $(K(X), T_K)$ *is not positively recurrent. In particular,* (X, T) *is not uniformly rigid.*

Proof. Set

$$A_n(y) = \left\{\sum_{i=0}^{n-1} \phi(x + i\alpha) + y \pmod 1 : x \in [0, 1)\right\}.$$

Let $\phi_0 = \phi$ and $\phi_n : \mathbb{T}^1 \to \mathbb{T}^1$ with $\phi_n(x) = \phi(x + n\alpha)$. If $|\deg(\phi)| = d \ge 1$, then $|\deg(\phi_{n-1} + \cdots + \phi_0)| = nd$. Thus, for $n \ge 2$, $A_n(y) = \mathbb{T}^1$ for any $y \in [0, 1)$. This implies that $\mathbb{T}^1 \times \{y\}$ is not positively recurrent in $K(X)$ and hence (X, T) is not uniformly rigid. □

Now we consider the case when $\deg(\phi) = 0$. If there are f and c such that $\phi(x) = f(x + \alpha) - f(x) + c$, then

$$\sum_{j=0}^{n-1} \phi(x + j\alpha) = f(x + n\alpha) - f(x) + nc.$$

It is easy to see that (X, T) is equicontinuous.

Example A.2. There is ϕ with $\deg(\phi) = 0$ such that (X, T) is minimal, distal, non-equicontinuous and uniformly rigid.

Proof. Consider (X, T) given by (A.1) with suitable $\phi : \mathbb{R} \to \mathbb{R}$, which we define below. Choose a sequence $\{n_j\}_{j=1}^{\infty} \subset \mathbb{N}$ such that $n_1 = 100$, $n_{j+1} = (n_1 n_2 \dots n_j)^3$ for $j \ge 1$. Denote $\alpha = \sum_{j=1}^{\infty} 1/n_j$; clearly, α is an irrational number (since it is an infinite non-repeating decimal). We claim that such α also satisfies the following two conditions:

(a) $|e^{2\pi i n_k \alpha} - 1| < 14/n_k^2$ for every $k \in \mathbb{N}$;

(b) $|e^{2\pi i n_k n_l \alpha} - 1| < 14/n_k n_l$ for every $k \in \mathbb{N}$ and $l \in \mathbb{N} \setminus \{1\}$.

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Indeed, for (a) we notice that $n_k\alpha = \sum_{j=1}^{k} n_k/n_j + \sum_{j=k+1}^{\infty} n_k/n_j$, $\sum_{j=k+1}^{\infty} n_k/n_j < 2/n_k^2$ and $n_{k+1} = (n_1 n_2 \dots n_k)^3$; then

$$|e^{2\pi i n_k \alpha} - 1| = |e^{2\pi i \sum_{j=k+1}^{\infty} \frac{n_k}{n_j}} - 1| < 2\pi \sum_{j=k+1}^{\infty} \frac{n_k}{n_j} < \frac{14}{n_k^2}$$

for every $k \in \mathbb{N}$. We will use similar arguments to check that (b) holds. Simply note that if we assume that $l \geq k$, then $n_l n_k \alpha = \sum_{j=1}^{l} n_l n_k/n_j + \sum_{j=l+1}^{+\infty} n_l n_k/n_j$ and $\sum_{j=l+1}^{+\infty} n_l n_k/n_j < n_l n_k(2/n_{l+1}) = 2n_l n_k/(n_1 n_2 \dots n_l)^3 < 2/n_k n_l$, which gives

$$|e^{2\pi i n_k n_l \alpha} - 1| < 2\pi \sum_{j=l+1}^{\infty} \frac{n_k n_l}{n_j} < 2\pi \frac{2}{n_k n_l} < \frac{14}{n_k n_l}.$$

Let $n_{-k} = -n_k$, $n_0 = 0$ and, for each $x \in \mathbb{T}^1$, put

$$\phi(x) = \sum_{k=-\infty}^{+\infty} (e^{2\pi i n_k \alpha} - 1) e^{2\pi i n_k x}.$$

Observe that for every $x \in \mathbb{T}^1$, we have $\phi(x) \in \mathbb{T}^1$ and hence the function $\phi : \mathbb{T}^1 \to \mathbb{T}^1$ is well defined. By (a), we have $|\phi(x)| \leq 28 \sum_{k=1}^{\infty} 1/n_k^2 < \infty$, so the series converges uniformly on [0, 1] and then ϕ is continuous. It follows directly from the definition that $\deg(\phi) = 0$.

Similarly to the discussion in [**4**, pp. 73–75], we obtain that the t.d.s. $(\mathbb{T}^2, T)$ is a minimal distal system. We claim that it is also non-equicontinuous. To see this, consider $(x_{n_l}, 0) \to (0, 0)$, $(l \to \infty)$ with $x_{n_l} = 1/(n_1 n_l)$, and choose $m_l = n_l^3/n_1$, $\delta = 1/1000$. It suffices to show that

$$d(T^{m_l}(x_{n_l}, 0), T^{m_l}(0, 0)) > \delta. \tag{A.2}$$

To validate this, we first note that for each $x \in \mathbb{T}^1$ and $n \in \mathbb{N}$,

$$\begin{aligned}
\sum_{t=0}^{n-1} \phi(x + t\alpha) &= \sum_{k=-\infty}^{+\infty} (e^{2\pi i n_k \alpha} - 1) \sum_{t=0}^{n-1} e^{2\pi i n_k (x + t\alpha)} \\
&= \sum_{k=-\infty}^{+\infty} e^{2\pi i n_k x} \sum_{t=0}^{n-1} (e^{2\pi i n_k (1+t)\alpha} - e^{2\pi i n_k t \alpha}) \\
&= \sum_{k=-\infty}^{+\infty} (e^{2\pi i n_k n \alpha} - 1) e^{2\pi i n_k x}.
\end{aligned}$$

If $k > l$, then $n_k x_{n_l}$ is an integer. Also, for each $1 \leq k \leq l$, we have

$$n_k m_l \alpha = \sum_{j=1}^{l} \frac{n_k m_l}{n_j} + \sum_{j=l+1}^{\infty} \frac{n_k m_l}{n_j}.$$

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With the help of the famous Euler's and trigonometric formulae (product-to-sum and sum-to-product identities), we can compute

$$\begin{aligned}
&\sum_{t=0}^{m_l-1}\phi(0+t\alpha)-\sum_{t=0}^{m_l-1}\phi(x_{n_l}+t\alpha)\\
&\quad=\sum_{k=-l}^{l}(e^{2\pi i n_k m_l\alpha}-1)-\sum_{k=-l}^{l}(e^{2\pi i n_k m_l\alpha}-1)e^{2\pi i n_k x_{n_l}}\\
&\quad=2\sum_{k=1}^{l}\{(\cos(2\pi n_k m_l\alpha)-\cos 0)-[\cos(2\pi n_k(m_l\alpha+x_{n_l}))-\cos(2\pi n_k x_{n_l})]\}\\
&\quad=8\sum_{k=1}^{l}\sin(\pi n_k m_l\alpha)\ \sin(\pi n_k x_{n_l})\ \cos(\pi n_k(m_l\alpha+x_{n_l}))\\
&\quad=8\sum_{k=1}^{l}\sin\left(\pi\sum_{j=l+1}^{\infty}\frac{n_k m_l}{n_j}\right)\ \sin(\pi n_k x_{n_l})\ \cos\left(\pi\sum_{j=l+1}^{\infty}\frac{n_k m_l}{n_j}+\pi n_k x_{n_l}\right).
\end{aligned}$$

Now fix $k\in[1,l]$. Since $n_k/n_1n_l<\sum_{j=l+1}^{\infty}n_km_l/n_j<2/n_1$ and $x>\sin x>(2/\pi)x$, $x\in(0,\pi/2)$,

$$\frac{2\pi}{n_1}>\sin\left(\pi\sum_{j=l+1}^{\infty}\frac{n_km_l}{n_j}\right)>\frac{2}{\pi}\cdot\pi\sum_{j=l+1}^{\infty}\frac{n_km_l}{n_j}>\frac{2n_k}{n_1n_l}.$$

Similarly, we can obtain

$$\frac{\pi n_k}{n_1n_l}>\sin(\pi n_kx_{n_l})>\frac{2}{\pi}\cdot\pi n_kx_{n_l}=\frac{2n_k}{n_1n_l}.$$

Moreover, since $2n_k/n_1n_l<\sum_{j=l+1}^{\infty}(n_km_l)/n_j+n_kx_{n_l}<3/n_1$ and $1>\cos x>-(2/\pi)x+1$, $x\in(0,\pi/2)$, we have

$$\cos\left(\pi\sum_{j=l+1}^{\infty}\frac{n_km_l}{n_j}+\pi n_kx_{n_l}\right)>1-\frac{6}{n_1}.$$

Hence,

$$\frac{1}{10}>\frac{16\pi^2}{n_1^2}\left(1+\frac{l-1}{n_{l-1}^2}\right)>\sum_{t=0}^{m_l-1}\phi(0+t\alpha)-\sum_{t=0}^{m_l-1}\phi(x_{n_l}+t\alpha)>\frac{32}{n_1^2}\left(1-\frac{6}{n_1}\right)>\frac{1}{1000}.$$

Set $\delta=1/1000$; we immediately have that (A.2) follows.

Now it remains to check the uniform rigidity of $(\mathbb{T}^2,T)$. Let $\varepsilon>0$ and choose $s>1$ with $\alpha/n_s<\varepsilon/28$ and $2/n_s^2<\varepsilon$. Then, by (b), we obtain that

$$\begin{aligned}
\left|\sum_{t=0}^{n_s-1}\phi(x+t\alpha)\right|&=\left|\sum_{k=-\infty}^{+\infty}(e^{2\pi i n_k n_s\alpha}-1)\right|\le\sum_{k=-\infty}^{+\infty}|e^{2\pi i n_k n_s\alpha}-1|\\
&=2\sum_{k=1}^{+\infty}|e^{2\pi i n_k n_s\alpha}-1|<2\sum_{k=1}^{+\infty}\frac{14}{n_sn_k}<\frac{28\alpha}{n_s}<\varepsilon
\end{aligned}$$

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for any $x \in \mathbb{T}^1$. Clearly, we also have

$$n_s\alpha \pmod 1 \le \sum_{j=s+1}^{\infty} \frac{n_s}{n_j} \le \frac{2}{n_s^2} < \varepsilon.$$

It follows that for every $(x, y) \in \mathbb{T}^2$,

$$d(T^{n_s}(x, y), (x, y)) = \max\left\{d(x + n_s\alpha, x), d\left(\sum_{t=0}^{n_s-1} \phi(x + t\alpha) + y, y\right)\right\} < \varepsilon,$$

completing the proof. □

B. *Appendix. The proof of Lemma 5.14*

Before we can prove Lemma 5.14, we need the following two lemmas.

LEMMA B.1. *Let (X, T) be a countable i.t.d.s. consisting of periodic points and with $d(X) = \gamma + \alpha + \xi$, $\alpha < \omega$, where ω is the first limit ordinal number and ξ, γ are ordinal numbers. Assume that $A = X^{(\gamma+\xi+\alpha)} = \{a_1, a_2, \ldots, a_m\}$ consists of fixed points of T. Let $\varepsilon > 0$ be such that $B_\varepsilon(A) = \overline{B_\varepsilon(A)}$, $B_\varepsilon(a_j) = \overline{B_\varepsilon(a_j)}$ for each $a_j \in A$ and $T(B_\varepsilon(a_i)) \cap B_\varepsilon(a_j) = \emptyset$ when $i \neq j$. Denote*

(i) $X_1 = \{x \in X : x \notin B_\varepsilon(A)\}$,

(ii) $X_2 = \bigcup_{j=1}^m X_{2,j}$, *where* $X_{2,j} = \{x \in X : x \in B_\varepsilon(a_j)$ *and* $\mathrm{orb}(x, T) \not\subset B_\varepsilon(a_j)\}$,

(iii) $X_3 = \bigcup_{j=1}^m X_{3,j}$, *where* $X_{3,j} = \{x \in X : x \in B_\varepsilon(a_j)$ *and* $\mathrm{orb}(x, T) \subset B_\varepsilon(a_j)\}$

and assume that $X_2 \cap X^{(\eta)} = \emptyset$ for each ordinal $\xi + \alpha \le \eta \le \xi + \gamma + \alpha$. Then there exists n such that if we denote $\widetilde{T} = T^n$ and

(I) $\widetilde{X}_1 = \{x \in X : x \notin B_\varepsilon(A)\}$;

(II) $\widetilde{X}_2 = \bigcup_{j=1}^m \widetilde{X}_{2,j}$, *where* $\widetilde{X}_{2,j} = \{x \in X : x \in B_\varepsilon(a_j)$ *and* $\mathrm{orb}(x, \widetilde{T}) \not\subset B_\varepsilon(a_j)\}$;

(III) $\widetilde{X}_3 = \bigcup_{j=1}^m \widetilde{X}_{3,j}$, *where* $\widetilde{X}_{3,j} = \{x \in X : x \in B_\varepsilon(a_j)$ *and* $\mathrm{orb}(x, \widetilde{T}) \subset B_\varepsilon(a_j)\}$,

then $\widetilde{X}_2 \cap X^{(\eta)} = \emptyset$ for each ordinal $\xi \le \eta \le \xi + \gamma + \alpha$.

Proof. First, we want to prove two special cases, i.e. $\xi = 0$, $\gamma = 0$ and $\alpha = 1$ or $\alpha = 2$. The ideas in the proof will help the reader to have a better understanding of the remaining proofs of this section.

If $\xi = 0$, $\gamma = 0$ and $\alpha = 1$, then X_1, X_2, X_3 may look as presented in Figure B.1.

We claim that in this case X_2 is finite. To prove the claim, we simply need to show that for each $a_t \in A$, the set $X_{2,t}$ is finite. Assume on the contrary that there exists an $a_t \in A$ such that

$$X_{2,t} = \{x \in X : x \in B_\varepsilon(a_t) \text{ and } \mathrm{orb}(x, T) \not\subset B_\varepsilon(a_t)\}$$

is infinite. Since every point in X is a periodic point, by choosing only one point in each orbit, there are $x_i \in X_{2,t}$ for each $i \in \mathbb{N}$ such that the orbits of points x_i are pairwise disjoint. Note that by the definition, if $x_i \in X_{2,t}$, then

$$\mathrm{orb}(x_i, T) \cap (X \setminus B_\varepsilon(A)) \neq \emptyset$$

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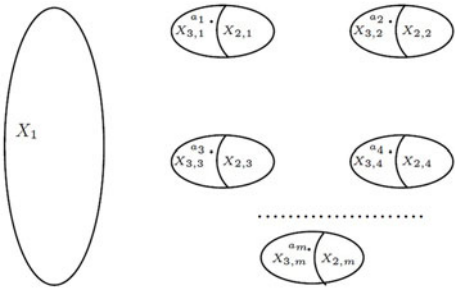


FIGURE B.1. X_1, X_2, X_3.

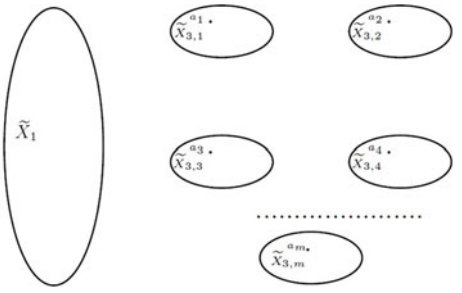


FIGURE B.2. $\widetilde{X}_1$, $\widetilde{X}_2$, $\widetilde{X}_3$.

and $X \setminus B_\varepsilon(A)$ is finite (otherwise we can find a point outside A with degree 1), which is a contradiction. This means that $X_{2,t}$ is finite for any $t = 1, 2, \ldots, m$ and hence X_2 is finite. Let n be the common period of each point in X_2 and denote $\widetilde{T} = T^n$. Then each point in X_2 under $\widetilde{T}$ is a fixed point, which means that $\widetilde{X}_2 \cap X^{(0)} = \emptyset$, completing the proof of this case (see Figure B.2).

If $\xi = 0$, $\gamma = 0$ and $\alpha = 2$, similarly we know that $X_2 \cap X^{(1)}$ is a finite set. Let n_1 be the common period of each point in $X_2 \cap X^{(1)}$ and denote $X_2^1 = \bigcup_{t=1}^m X_{2,t}^1$ with

$$X_{2,t}^1 = \{x \in X : x \in B_\epsilon(a_t) \text{ and } \mathrm{orb}(x, T^{n_1}) \not\subset B_\varepsilon(a_t)\}.$$

Clearly, $X_2^1 \cap X^{(1)} = \emptyset$. We claim that $X_2^1 \cap X^{(0)}$ is a finite set. Assume on the contrary that there exists $a_t \in A$ such that $X_{2,t}^1 \cap X^{(0)}$ is an infinite set, i.e. there are infinitely many points $x_i \in B_\varepsilon(a_t)$ with derived degree 0 such that $\mathrm{orb}(x_i, T^{n_1}) \not\subset B_\varepsilon(a_t)$ for each $i \in \mathbb{N}$. We may assume that the orbits of the x_i are pairwise disjoint, because all elements of X are periodic points. In each set $\mathrm{orb}(x_i, T^{n_1})$, fix a point $x_i' \in B_\varepsilon(a_t)$ such that $T^{n_1}(x_i') \notin B_\varepsilon(a_t)$. Observe that since x_i is periodic, each point in $\mathrm{orb}(x_i, T^{n_1})$ has the same derived degree as x_i. We obtain a sequence of pairwise distinct points with derived degree 0 and, by compactness of X, we may assume that $z = \lim_{i\to\infty} x_i'$ exists (see Figure B.3).

Then z has derived degree one, $z \in B_\varepsilon(a_t)$ and $T^{n_1}(z) \notin B_\varepsilon(a_t)$; thus, $\mathrm{orb}(z, T^{n_1}) \not\subset B_\varepsilon(a_t)$. Therefore, $z \in X_2^1$, but $X_2^1 \cap X^{(1)} = \emptyset$, which is a contradiction. Indeed, $X_2^1 \cap X^{(0)}$ is a finite set.

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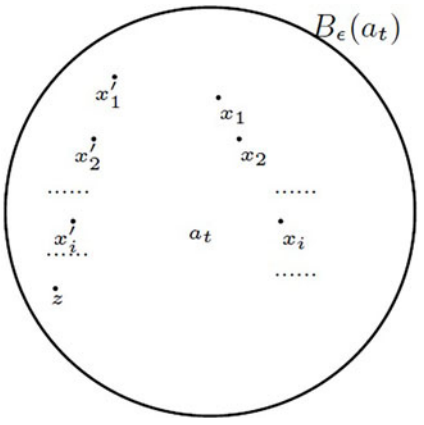


FIGURE B.3. $z = \lim_{i\to\infty} x_i'$ and $T^{n_1}x_i' \notin B_\varepsilon(a_t)$.

Let $n = ln_1$ be the common period of each point in $X_2^1 \cap X^{(0)}$. If we denote $\widetilde{T} = T^n = T^{ln_1}$, then $\widetilde{X}_2 \cap X^{(1)} = \emptyset$ (since $X_2^1 \cap X^{(1)} = \emptyset$) and $\widetilde{X}_2 \cap X^{(0)} = \emptyset$ (since each point in $X_2^1 \cap X^{(0)}$ under $\widetilde{T}$ is a fixed point).

Now we can continue with the general cases.

If $\alpha = 0$, then there is nothing to prove; hence, we may assume that $1 \le \alpha < \omega$ and denote $\beta = \xi + \alpha - 1$.

Observe that $X_2 \cap X^{(\beta)}$ is a finite set, since otherwise we can find an index t and a sequence of points from pairwise different (periodic) orbits $y_i \in B_\varepsilon(a_t)$ of derived degree β and such that $T(y_i) \notin B_\varepsilon(a_t)$. By continuity of T and compactness of X, there is a point $y \in X_2$ of derived degree $\beta + 1$, which is a contradiction with the assumptions of the lemma. Let $n_1 \in \mathbb{N}$ be the common period of points in $X_2 \cap X^{(\beta)}$, that is, each point in $X_2 \cap X^{(\beta)}$ under T^{n_1} is a fixed point.

Denote $X_2^\beta = \bigcup_{j=1}^m X_{2,j}^\beta \subset X_2$, where

$$X_{2,j}^\beta = \{x \in X : x \in B_\varepsilon(a_j) \text{ and } \mathrm{orb}(x, T^{n_1}) \not\subset B_\varepsilon(a_j)\},$$

and observe that $X_2^\beta \cap X^{(\beta)} = \emptyset$, since all points in $X_2 \cap X^{(\beta)}$ are fixed points for T^{n_1}. Clearly, $X_2^\beta \subset X_2$ and hence $X_2^\beta \cap X^{(\eta)} = \emptyset$ for each ordinal $\beta \le \eta \le \xi + \gamma + \alpha$.

We claim that $X_2^\beta \cap X^{(\beta-1)}$ is a finite set. Assume on the contrary there exists $a_t \in A$ such that $X_{2,t}^\beta \cap X^{(\beta-1)}$ is an infinite set, i.e. there are infinitely many points $x_i \in B_\varepsilon(a_t)$ with derived degree $\beta - 1$ such that $\mathrm{orb}(x_i, T^{n_1}) \not\subset B_\varepsilon(a_t)$ for each $i \in \mathbb{N}$. We may assume that the orbits of the x_i are pairwise disjoint, because all elements of X are periodic points. In each set $\mathrm{orb}(x_i, T^{n_1})$, fix a point $x_i' \in B_\varepsilon(a_t)$ such that $T^{n_1}(x_i') \notin B_\varepsilon(a_t)$. Observe that since x_i is periodic, each point in $\mathrm{orb}(x_i, T^{n_1})$ has the same derived degree as x_i. We obtain a sequence of pairwise distinct points with derived degree $\beta - 1$ and, by compactness of X, we may assume that $z = \lim_{i\to\infty} x_i'$ exists. Then z has derived degree β, $z \in B_\varepsilon(a_t)$ and $T^{n_1}(z) \notin B_\varepsilon(a_t)$; thus, $\mathrm{orb}(z, T^{n_1}) \not\subset B_\varepsilon(a_t)$. Therefore, $z \in X_2^\beta$, but $X_2^\beta \cap X^{(\beta)} = \emptyset$, which is a contradiction. Indeed, $X_2^\beta \cap X^{(\beta-1)}$ is a finite set.

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Let $n_2 \in \mathbb{N}$ be a multiple of n_1 which is a common period of points in $X_2^{\beta} \cap X^{(\beta-1)}$. Similarly to the previous step, we denote

$$X_2^{\beta-1} = \bigcup_{j=1}^{m} X_{2,j}^{\beta-1}, \quad \text{where } X_{2,j}^{\beta-1} = \{x \in X : x \in B_\varepsilon(a_j) \text{ and } \operatorname{orb}(x, T^{n_2}) \not\subset B_\varepsilon(a_j)\}.$$

Observe that $X_2^{\beta-1} \cap X^{(\beta-1)} = \emptyset$ and $X_2^{\beta-1} \cap X^{(\beta)} = \emptyset$. Repeating the previous argument, we see that each $X_{2,t}^{\beta-1} \cap X^{(\beta-2)}$ is finite because otherwise we can find an index t and a point $z \in B_\varepsilon(a_t)$ with derived degree $\beta - 1$ and such that $T^{n_2}(z) \notin B_\varepsilon(a_t)$. This implies that $z \in X_{2,t}^{\beta-1} \cap X^{(\beta-1)} = \emptyset$, which is impossible. We can then find an integer $n_3 > 0$ which is a multiple of n_2 such that each point in $X_2^{\beta-1} \cap X^{(\beta-2)}$ is a fixed point of T^{n_3}. Putting

$$X_2^{\beta-2} = \bigcup_{j=1}^{m} X_{2,j}^{\beta-2}, \quad \text{where } X_{2,j}^{\beta-2} = \{x \in X : x \in B_\varepsilon(a_j) \text{ and } \operatorname{orb}(x, T^{n_3}) \not\subset B_\varepsilon(a_j)\},$$

we obtain a set such that $X_2^{\beta-2} \cap X^{(\beta-k)} = \emptyset$ for $k = 0, 1, 2$.

Repeating the above arguments a finite number of times, we can find $n_\beta \in \mathbb{N}$ such that if we put

$$X_2^{\xi} = \bigcup_{j=1}^{m} X_{2,j}^{\xi}, \quad \text{where } X_{2,j}^{\xi} = \{x \in X : x \in B_\varepsilon(a_j) \text{ and } \operatorname{orb}(x, T^{n_\beta}) \not\subset B_\varepsilon(a_j)\},$$

then $X_2^{\xi} \cap X^{(\beta-k)} = \emptyset$ for $k = 0, 1, 2, \ldots, \alpha - 1$. Clearly, by the definition, also $X_2^{\xi} \subset X_2$ and hence $X_2^{\xi} \cap X^{(\eta)} = \emptyset$ for each ordinal $\xi \le \eta \le \xi + \gamma + \alpha$. Observe that if we denote $n = n_\beta$ and $\widetilde{T} = T^n$, then $\widetilde{X}_2 = X_2^{\xi}$, which completes the proof. □

LEMMA B.2. *Let (X, T) be a countable i.t.d.s. consisting of periodic points and with $d(X) = \gamma + \xi$, where ξ is a limit ordinal number and γ is an ordinal number. Assume that $A = X^{(\gamma+\xi)} = \{a_1, a_2, \ldots, a_m\}$ consists of fixed points of T. Let $\varepsilon > 0$ be such that $B_\varepsilon(A) = \overline{B_\varepsilon(A)}$, $B_\varepsilon(a_j) = \overline{B_\varepsilon(a_j)}$ for each $a_j \in A$ and $T(B_\varepsilon(a_i)) \cap B_\varepsilon(a_j) = \emptyset$ when $i \neq j$. Denote:*

(i) $X_1 = \{x \in X : x \notin B_\varepsilon(A)\}$;

(ii) $X_2 = \bigcup_{j=1}^{m} X_{2,j}$, *where* $X_{2,j} = \{x \in X : x \in B_\varepsilon(a_j)$ *and* $\operatorname{orb}(x, T) \not\subset B_\varepsilon(a_j)\}$;

(iii) $X_3 = \bigcup_{j=1}^{m} X_{3,j}$, *where* $X_{3,j} = \{x \in X : x \in B_\varepsilon(a_j)$ *and* $\operatorname{orb}(x, T) \subset B_\varepsilon(a_j)\}$

and assume that $X_2 \cap X^{(\eta)} = \emptyset$ for each ordinal $\xi \le \eta \le \xi + \gamma$. Then there exists an ordinal number $\beta < \xi$ such that $X_2 \cap X^{(\eta)} = \emptyset$ for each ordinal $\beta \le \eta \le \xi + \gamma$.

Proof. Assume on the contrary that there is an increasing sequence of ordinals $\eta_n < \xi$ such that $\lim_{n\to\infty} \eta_n = \xi$ (e.g. [**32**, p. 235]), that is, for every $\eta < \xi$, there is an n such that $\eta < \eta_n < \xi$ and $X_2 \cap X^{(\eta_n)} \neq \emptyset$. Denote by z_n a point $z_n \in X^{(\eta_n)} \cap X_2$ such that $T(z_n) \notin B_\varepsilon(a_t)$ for some t. Since there are only finitely many different points a_j, we may assume that t is the same for all n. We may also assume that $z_n \to z$ and clearly by the definition $z \in X_2$. Observe that also $z \in X^{(\xi)}$. That is, $z \in X_2 \cap X^{(\xi)}$, which is a contradiction with the assumption that $X_2 \cap X^{(\eta)} = \emptyset$ for each ordinal $\xi \le \eta \le \xi + \gamma$. □

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Now we are ready to prove Lemma 5.14.

Lemma 5.14. Let (X, T) be a countable i.t.d.s. consisting of periodic points and with $d(X) = \alpha$. Assume that $A = X^{(\alpha)} = \{a_1, a_2, \ldots, a_m\}$ consists of fixed points of T. Let $\varepsilon > 0$ be such that $B_\varepsilon(A) = \overline{B_\varepsilon(A)}$, $B_\varepsilon(a_j) = \overline{B_\varepsilon(a_j)}$ for each $a_j \in A$ and $T(B_\varepsilon(a_i)) \cap B_\varepsilon(a_j) = \emptyset$ when $i \neq j$. Denote:

(i) $X_1 = \{x \in X : x \notin B_\varepsilon(A)\}$;

(ii) $X_2 = \bigcup_{j=1}^m X_{2,j}$, where $X_{2,j} = \{x \in X : x \in B_\varepsilon(a_j) \text{ and } \mathrm{orb}(x, T) \not\subset B_\varepsilon(a_j)\}$;

(iii) $X_3 = \bigcup_{j=1}^m X_{3,j}$, where $X_{3,j} = \{x \in X : x \in B_\varepsilon(a_j) \text{ and } \mathrm{orb}(x, T) \subset B_\varepsilon(a_j)\}$.

Then there exists an integer $n > 0$ such that if we denote $\widetilde{T} = T^n$ and:

(I) $\widetilde{X}_1 = \{x \in X : x \notin B_\varepsilon(A)\}$;

(II) $\widetilde{X}_2 = \bigcup_{j=1}^m \widetilde{X}_{2,j}$, where $\widetilde{X}_{2,j} = \{x \in X : x \in B_\varepsilon(a_j) \text{ and } \mathrm{orb}(x, \widetilde{T}) \not\subset B_\varepsilon(a_j)\}$;

(III) $\widetilde{X}_3 = \bigcup_{j=1}^m \widetilde{X}_{3,j}$, where $\widetilde{X}_{3,j} = \{x \in X : x \in B_\varepsilon(a_j) \text{ and } \mathrm{orb}(x, \widetilde{T}) \subset B_\varepsilon(a_j)\}$,

then $\widetilde{X}_2 = \emptyset$.

Proof. If $\alpha < \omega$, where ω is the first limit ordinal number with cardinality $\aleph_0$, then the result follows by Lemma B.1 with $\xi = \gamma = 0$. If $\alpha = \beta_1 + \alpha_1$ for some limit ordinal β_1 and $\alpha_1 < \omega$, then, by Lemma B.1 and next by Lemma B.2, we can find $n_1 > 0$ and an ordinal $\hat{\beta}_1 < \beta_1$ such that if we denote $\widetilde{T} = T^{n_1}$ and:

(I) $\widetilde{X}_1 = \{x \in X : x \notin B_\varepsilon(A)\}$;

(II) $\widetilde{X}_2 = \bigcup_{j=1}^m \widetilde{X}_{2,j}$, where $\widetilde{X}_{2,j} = \{x \in X : x \in B_\varepsilon(a_j) \text{ and } \mathrm{orb}(x, \widetilde{T}) \not\subset B_\varepsilon(a_j)\}$;

(III) $\widetilde{X}_3 = \bigcup_{j=1}^m \widetilde{X}_{3,j}$, where $\widetilde{X}_{3,j} = \{x \in X : x \in B_\varepsilon(a_j) \text{ and } \mathrm{orb}(x, \widetilde{T}) \subset B_\varepsilon(a_j)\}$,

then $\widetilde{X}_2 \cap X^{(\eta)} = \emptyset$ for each ordinal $\hat{\beta}_1 \leq \eta \leq \alpha$. But $\hat{\beta}_1 = \beta_2 + \alpha_2$ or $\hat{\beta}_1 = \alpha_2$ for some limit ordinal β_2 and ordinal $\alpha_2 < \omega$. In the later case, using Lemma B.1, we are done, and, in the first case, again by Lemmas B.1 and B.2, we can find an ordinal $\hat{\beta}_2 < \beta_2$ and an $n_2 > 0$ such that if we replace $\widetilde{T} = T^{n_1 n_2}$ and:

(I) $\widetilde{X}_1 = \{x \in X : x \notin B_\varepsilon(A)\}$;

(II) $\widetilde{X}_2 = \bigcup_{j=1}^m \widetilde{X}_{2,j}$, where $\widetilde{X}_{2,j} = \{x \in X : x \in B_\varepsilon(a_j) \text{ and } \mathrm{orb}(x, \widetilde{T}) \not\subset B_\varepsilon(a_j)\}$;

(III) $\widetilde{X}_3 = \bigcup_{j=1}^m \widetilde{X}_{3,j}$, where $\widetilde{X}_{3,j} = \{x \in X : x \in B_\varepsilon(a_j) \text{ and } \mathrm{orb}(x, \widetilde{T}) \subset B_\varepsilon(a_j)\}$,

then $\widetilde{X}_2 \cap X^{(\eta)} = \emptyset$ for each ordinal $\hat{\beta}_2 \leq \eta \leq \alpha$.

Since the derived degree $d(X)$ is countable, we have $0 \leq \alpha < \omega_1$, where ω_1 is the first uncountable limit ordinal with cardinality $\aleph_1 = \aleph_0^{\aleph_0}$. Then, by [**32**, Theorem 5, p. 258], α can be exactly represented in the form of $\alpha = \omega^{\theta_1}\zeta_1 + \omega^{\theta_2}\zeta_2 + \cdots + \omega^{\theta_n}\zeta_n$, where $n \in \mathbb{N}$, $\theta_n < \cdots < \theta_2 < \theta_1 < \omega$ and $0 \leq \zeta_i < \omega$ for $i = 1, \ldots, n$. Hence, applying inductively the above reduction a finite number of times, the result follows. □

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