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INTEGRATION AND CELL DECOMPOSITION IN P-MINIMAL STRUCTURES

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Abstract. We show that the class of $\mathcal{L}$-constructible functions is closed under integration for any P-minimal expansion of a p-adic field $(K, \mathcal{L})$. This generalizes results previously known for semi-algebraic and subanalytic structures. As part of the proof, we obtain a weak version of cell decomposition and function preparation for P-minimal structures, a result which is independent of the existence of Skolem functions. A direct corollary is that Denef's results on the rationality of Poincaré series hold in any P-minimal expansion of a p-adic field $(K, \mathcal{L})$.

§1. Introduction. One of the main results of this paper is that, for arbitrary P-minimal structures over p-adic fields, the class of constructible functions is closed under integration. This generalizes a result which was previously known only for semi-algebraic and subanalytic structures (and some intermediary cases).

As part of the proof, we obtain the second main result of this paper: a version of cell decomposition and function preparation for P-minimal structures. While our version is somewhat weaker than what was obtained in previous attempts by e.g., Mourgues [20], it does not depend on the existence of definable Skolem functions.

In this introduction we give an informal motivation of our approach, discussing the historical connections between p-adic integration, rationality of Poincaré series, and cell decomposition. We will present exact statements of our main results in the next section.

The study of p-adic integrals was motivated by a number-theoretic question. It was conjectured by Borevich and Shafarevich that for $f(x) \in \mathbb{Z}[x]$, Poincaré series like e.g., $P(T) := \sum_{m \in \mathbb{N}} N_m T^m$, where

$$N_m := \#\{x \in (\mathbb{Z}/p^m\mathbb{Z})^n \mid f(x) \equiv 0 \mod p^m\},$$

are rational functions of T. This was originally proven by Igusa [15–17]. Later, Denef [8] obtained a similar, more general result. He gave two proofs that were based on Macintyre's quantifier elimination for semi-algebraic sets [19], one using resolution of singularities, and one where he introduced cell decomposition techniques. We refer to [10] for a comparison of both approaches.

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A first step towards a proof is the realization that the terms of a Poincaré series can be connected to the measure of certain semi-algebraic sets, and hence to p-adic integrals. For instance, one can check that

$$N_m = p^{nm} \cdot \mu(\{x \in \mathbb{Z}_p^n \mid \operatorname{ord} f(x) \geqslant m\}),$$

where μ is the Haar measure normalized such that $\mu(\mathbb{Z}_p) = 1$, and ord denotes the valuation map $\operatorname{ord} : \mathbb{Q}_p \to \mathbb{Z} \cup \{\infty\}$. To prove rationality, one needs to understand how the measure of this family of definable sets depends on m. (If the dependence is tame enough, identities like $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ can be used to deduce rationality.) Hence the focus shifts to the computation of (families of) p-adic integrals.

In a p-adic integral, the integrand is usually the valuation 'ord f' or the p-adic norm $|f| := p^{-\operatorname{ord} f}$ of a semi-algebraic function f. More generally, one can consider so-called *constructible* functions, i.e., $\mathbb{Q}$-linear combinations of definable functions $\alpha : \mathbb{Q}_p^r \to \mathbb{Z}$ and their induced functions p^α. Note that it is natural here to work in a two-sorted structure $(\mathbb{Q}_p, \mathbb{Z})$, adding the value group $\mathbb{Z} \cup \{\infty\}$ as a separate sort. The word definable should also be interpreted in that context, using a two-sorted language $\mathcal{L}_{\text{ring},2} = (\mathcal{L}_{\text{ring}}, \mathcal{L}_{\text{Pres}}, \text{ord})$, consisting of the ring language $\mathcal{L}_{\text{ring}}$ for the main sort $\mathbb{Q}_p$, the Presburger language $\mathcal{L}_{\text{Pres}} = (+, -, <, \equiv_n)$ for the value group sort $\mathbb{Z} \cup \{\infty\}$, and the valuation map $\operatorname{ord} : \mathbb{Q}_p \to \mathbb{Z} \cup \{\infty\}$ to connect the sorts.

Denef showed in [9] that given a semi-algebraic function f, the integral of $|f|$ equals a constructible function. Cluckers [1] proved a similar result in the subanalytic setting. Further generalizations for these two cases were obtained by Cluckers, Gordon, and Halupczok [6]. We will discuss this in more detail in the next section. In Section 4, we prove that this closure property holds in arbitrary P-minimal structures over p-adic fields. Corollary 4.6 then provides an example of how the rationality of Poincaré series can be deduced from this.

Let us now discuss the connection with cell decomposition techniques. We will assume that K is a p-adically closed field. The general philosophy is to partition a definable set X in somewhat uniform parts, called cells. If X is the domain of a function f, an additional goal may be to *prepare* the function, i.e., choosing the partition in such a way that the function f, when restricted to each of the cells, has some additional nice properties. Cells are generally taken to be sets of the form

$$\left\{ (x,t) \in D \times T \;\middle|\; \begin{array}{l} \text{a condition of a fixed form describing} \\ t \text{ in terms of the other variables } x \end{array} \right\},$$

where D is a definable set and T is one of the sorts. For instance, when working with semi-algebraic sets, this fixed form is a formula stating that t belongs to an interval-like set:

$$\operatorname{ord} a_1(x) \,\square_1\, \operatorname{ord}(t - c(x)) \,\square_2\, \operatorname{ord} a_2(x) \quad \text{and} \quad t - c(x) \in \lambda P_n,$$

where $\square_i$ may denote $<$ or *no condition*, $\lambda \in K$ and the functions $a_i(x)$ are definable functions $D \to K$. We use the symbol P_n for the set of (nonzero) n-th powers. Note that if $\lambda = 0$, what we get is just the graph of the function $c(x)$.

What Denef [8, 12] showed is that given a semi-algebraic function $f : X \subseteq K^{n+1} \to K$, X can be decomposed into cells such that on each cell C, $f_{|C}$ satisfies the following condition:

$$|f(x,t)| = |\lambda(t - c(x))|^{\frac{e}{n}} |h(x)|,$$

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where $h(x)$ and $c(x)$ are definable functions and e, n are integer numbers. The version stated here is a reformulation by Cluckers [5], who also obtained an analogous result for subanalytic functions in [1].

This preparation result is particularly useful for integration purposes. Indeed, if the domain of a function $f : X \to K$ can be partitioned into a finite number of sets $\{(x,t) \in D_i \times T \mid \phi_i(x,t)\}$ on each of which f has the form prescribed above, then one gets that

$$\int_X |f(x,t)||dx||dt| = \sum_i \int_{D_i} |h(x)| \left[\int_{\{t \in T \mid \phi_i(x,t)\}} |\lambda(t - c(x))|^{\frac{e}{n}} |dt| \right] |dx|.$$

Iteration of this theorem allows one to give very accurate descriptions of the value of the integral of f and its dependence on possible parameters. Similar strategies were applied for the subanalytic case, see e.g., [1, 11].

When Haskell and Macpherson developed the notion of P-minimality [13], it was natural to ask how much of the above ideas could be generalized to that setting. One of the most notable results so far in this context is Mourgues' cell decomposition theorem [20]. She showed that in a P-minimal structure which admits definable Skolem functions, any definable set $A \subseteq K^{r+1}$ can be partitioned in cells of the form

$$\{(x,t) \subseteq S \times K \mid \operatorname{ord} a(x) \,\square_1\, \operatorname{ord}(t - c(x)) \,\square_2\, \operatorname{ord} b(x);\ t - c(x) \in \lambda P_n\},$$

where $a, b, c : S \to K$ are definable functions, $\square_i$ are either $<$ or *no condition*, P_n denotes the set of n-th powers and $\lambda \in K$. Moreover, she showed that the existence of definable Skolem functions is a necessary condition for the existence of a decomposition using cells of this form. Note that it is currently not known whether all P-minimal structures admit definable Skolem functions. Work by the second author on reducts of p-adically closed fields [18] seems to indicate that this may not be the case, and hence some caution is warranted when making this assumption.

One way to deal with this uncertainty is to replace P-minimality by a (possibly) more restrictive notion, explicitly adding the existence of Skolem function as a condition. An example of this approach is the recent attempt of Darnière (see [7] for his preprint), who suggests a notion of so-called P-optimal structures. Even with these stronger assumptions, it is still an open problem whether some version of the Denef–Cluckers preparation theorem holds for arbitrary P-minimal structures.

In this paper, we take an alternative approach: the results we present here do not rely on the existence of definable Skolem functions. Instead, we decided to shift the emphasis back to possible applications. While our versions of cell decomposition/preparation are certainly weaker than versions known for individual structures, they are still strong enough to prove the theorems that initially motivated their development. Theorem 4.1 is an illustration of this.

The rest of the paper is organized as follows. In Section 2, we explain our results in more detail. The proof of the cell decomposition and preparation theorems will be given in Section 3. In Section 4 we show that constructible functions form a class that is stable under integration. This result will be used to derive the rationality of Poincaré series.

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Our arguments use the fact that on every model M of Presburger arithmetic, there exists a definable total order $\lhd$ of M, such that every definable subset of M has a $\lhd$-minimal element. We present this result in an appendix, as part of a more general framework. As a corollary, we obtain that Presburger arithmetic has elimination of imaginaries (a result already proven in [2]).

§2. Overview of main results. In this section we state the main results of this paper. Proofs will be deferred to later sections.

We first fix some notations. Let K be a p-adically closed field (that is, elementarily equivalent to a p-adic field). We use the notation q_K for the number of elements of the residue field k_K, $\mathcal{O}_K$ for the valuation ring of K, and π_K for a uniformizing element. Write $\mathrm{ac}_m : K \to (\mathcal{O}_K/\pi_K^m\mathcal{O}_K)^\times \cup \{0\}$ for the m-th angular component map, which can be defined as

$$\mathrm{ac}_m(x) := \begin{cases} \tilde{x} \mod \pi_K^m & \text{if } 0 \neq x = \pi_K^{\mathrm{ord}\, x}\tilde{x}, \\ 0 & \text{if } x = 0. \end{cases}$$

In every expansion of a p-adically closed field K, such angular component maps exist and can be defined in a unique way, as was shown in Lemma 1.3 of [3]. For notational purposes, we will fix a definable set $S \subseteq K^{m_0} \times \Gamma_K^{m_0}$ which we call a *parameter set*. Given a set $X \subseteq S \times K$ and $s \in S$, we write

$$X_s := \{x \in K \mid (s, x) \in X\}$$

to denote the fiber over s. Analogously, for a definable function $f : X \to \Gamma_K$, we use the notation $f_s(\cdot)$ for the function $f(s, \cdot) : X_s \to \Gamma_K$. Given two sets A and B, we write $\Pi_A : A \times B \to A$ for the projection onto A, and $\Pi_B : A \times B \to B$ for the projection onto B. For a positive integer $n \geq 1$, $A^{\leq n}$ denotes $\bigcup_{i=1}^n A^i$.

We will work with a two-sorted version of P-minimality, where we consider both the field sort and the value group sort $\Gamma_K \cup \{\infty\}$ to be of equal importance. Let $(K, \Gamma_K; \mathcal{L}_2)$ be a two-sorted structure, with language $\mathcal{L}_2 = (\mathcal{L}, \mathcal{L}_{Pres}, \mathrm{ord})$. Here $\mathcal{L}$, the language for the K-sort, is assumed to be an expansion of the ring language $\mathcal{L}_{\mathrm{ring}}$. For the value group sort $\Gamma_K \cup \{+\infty\}$, we use the language of Presburger arithmetic $\mathcal{L}_{Pres} = (+, -, <, \equiv_n)$. The sorts are connected through the valuation map $\mathrm{ord} : K \to \Gamma_K \cup \{+\infty\}$. If the language $\mathcal{L}_2$ is clear from the context, we will just write (K, Γ_K). By a definable set we mean definable with parameters.

DEFINITION 2.1. A two-sorted structure $(K, \Gamma_K; \mathcal{L}_2)$ with $\mathcal{L}_2 = (\mathcal{L}, \mathcal{L}_{\mathrm{Pres}}, \mathrm{ord})$ and $\mathcal{L}_{\mathrm{ring}} \subseteq \mathcal{L}$ is said to be *P-minimal* if the underlying structure $(K, \mathcal{L})$ is P-minimal, that is, for every $(K', \mathcal{L})$ elementarily equivalent to $(K, \mathcal{L})$, the $\mathcal{L}$-definable subsets of K' are $\mathcal{L}_{\mathrm{ring}}$-definable.

This definition is motivated by the following observation, which was based on Wagner's results on definable functions in one variable on certain ordered abelian groups [21].

THEOREM 2.2 (Cluckers [2], Lemma 2 and Theorem 6). *Let $(K, \mathcal{L})$ be a P-minimal field.*

For any $\mathcal{L}$-definable set $X \subseteq (K^\times)^m$, the set

$$\mathrm{ord}(X) := \{(\mathrm{ord}\, x_1, \ldots, \mathrm{ord}\, x_m) \in \Gamma_K^m \mid (x_1, \ldots, x_m) \in X\}$$

is $\mathcal{L}_{Pres}$-definable.

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Let $S \subseteq \Gamma_K^m$ be a Presburger-definable set. Then the set

$$\operatorname{ord}^{-1}(S) := \{(x_1, \dots, x_m) \in (K^\times)^m \mid \operatorname{ord} x \in S\}$$

is $\mathcal{L}_{ring}$-definable.

This theorem implies that, given a (mono-sorted) P-minimal structure $(K, \mathcal{L})$, the valuation map $\operatorname{ord} : K \to \Gamma_K \cup \{\infty\}$ induces a two-sorted structure (K, Γ_K) where every definable subset of Γ_K^m is $\mathcal{L}_{\text{Pres}}$-definable, and hence it is natural to take $\mathcal{L}_{\text{Pres}}$ as the language for the value group sort.

Note that a two-sorted P-minimal structure (K, Γ_K) cannot have definable Skolem functions since any definable section of ord contradicts the assumption of P-minimality.

We will now explain our notion of cells and cell decomposition. We distinguish the following two kinds of cells:

DEFINITION 2.3 (Cells). Let (K, Γ_K) be an $\mathcal{L}_2$-structure.

- A subset $C \subseteq S \times K$ is a *K-cell* if it is of the form

$$C = \left\{ (s,t) \in D \times K \;\middle|\; \begin{array}{l} \alpha(s) \square_1 \operatorname{ord}(t - c(s)) \square_2 \beta(s), \\ \operatorname{ord}(t - c(s)) \equiv k \mod n, \\ \operatorname{ac}_m (t - c(s)) = \xi \end{array} \right\},$$

  where D, the *base* of the cell, is a definable subset of S, c is a definable function $c : D \to K$, α, β are $\mathcal{L}_2$-definable functions $D \to \Gamma_K$, $k, n, m \in \mathbb{N}$, $\xi \in \operatorname{ac}_m(K)$ and the symbols $\square_i$ may denote $<$ or *no condition*.

  If $\operatorname{ac}_m (t - c(s)) = 0$ (and hence $t = c(s)$), one should ignore the first two conditions.
- A subset $B \subseteq S \times \Gamma_K$ is a *Γ-cell* if it is of the form

$$B = \left\{ (s,\gamma) \in D \times \Gamma_K \;\middle|\; \begin{array}{l} \alpha(s) \square_1 \gamma \square_2 \beta(s), \\ \gamma \equiv k \mod n \end{array} \right\},$$

  where D, the *base* of the cell, is a definable subset of S, α, β are definable functions $D \to \Gamma_k$, $k, n \in \mathbb{N}$ and the squares $\square_i$ may denote $<$ or *no condition*.

To each cell, one can associate a tuple $(\square, k, n, m, \xi)$, respectively $(\square, k, n)$, where $\square = (\square_1, \square_2) \in \{\emptyset, <\}^2$, $\xi \in \operatorname{ac}_m(K)$, and (k, n, m) is a triple of nonnegative integers such that $k < n$. These tuples will be referred to as the *type* of the respective cells. We denote by P_K (resp. P_Γ) the set of all possible types of K-cells (resp. Γ-cells).

We obtain cell decomposition results for definable sets both of the form $X \subseteq S \times K$ and $X \subseteq S \times \Gamma_K$. Let us first consider the case where the last variable belongs to the Γ_K-sort. When X is a set $X \subseteq S \times \Gamma_K$, we obtain a partition into a finite union of Γ-cells. Moreover, for functions $f : X \subseteq S \times \Gamma_K \to \Gamma_K$, we describe explicitly how, on each cell in the decomposition of X, the value of f depends on the last variable.

PROPOSITION 2.4 (Function preparation). *Let $f : X \subseteq S \times \Gamma_K \to \Gamma_K$ be definable in a P-minimal structure (K, Γ_K). There exists a finite partition of X in Γ-cells C, such that on each cell C with type $(\square, k, n)$, the function f has the form*

$$f(x, \gamma) = a \left(\frac{\gamma - k}{n} \right) + \delta(x),$$

where $a \in \mathbb{Z}, n, k \in \mathbb{N}$ and δ is a definable function $S \to \Gamma_K$.

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If X is a subset of $S \times K$, the statement of our cell decomposition result is more subtle. The main difference between classical cell decomposition and K-cell decomposition arises at the level of centers. In the classical definition, the centers appear as the images of definable functions from the parameter set S. Instead, a K-cell decomposition provides a partition of X into sets X_i, which are essentially a finite union of cells, together with a definable Σ_i containing all their possible tuples of centers. This type of decomposition is sufficiently strong for the computation of integrals: the Haar measure is translation invariant, and hence the centers are not of great importance here.

THEOREM-DEFINITION 2.5 (K-cell decomposition). *Let (K, Γ_K) be a P-minimal structure, and $X \subseteq S \times K$ be a definable set. There exists a finite partition of X in sets $X_i \subseteq S_i \times K$. On each part X_i, there is an integer r and r associated K-cells C_j of the form*

$$C_j := \left\{ (s,t) \in S_i \times K \;\middle|\; \begin{array}{l} \alpha_j(s) \,\square_{1,j}\, \operatorname{ord} t \,\square_{2,j}\, \beta_j(s) \wedge \\ \operatorname{ord} t \equiv k_j \mod n \wedge \\ ac_m(t) = \xi_j \end{array} \right\},$$

where $\alpha_j, \beta_j : S_i \to \Gamma_K$ are definable functions and $(\square_j, k_j, n, m, \xi_j) \in P_K$. To each X_i, we associate a definable set $\Sigma_i \subseteq S_i \times K^r$, which has the following property. To any function

$$\sigma : S_i \to K^r : s \mapsto (\sigma_1(s), \ldots, \sigma_r(s)),$$

whose graph is contained in Σ_i, we can associate a (bijective) translation $T_\sigma : \sqcup_j C_j \to X_i$, defined by

$$T_\sigma(s,t) = (s, t - \sigma_j(s)) \qquad \text{for all } (s,t) \in C_j.$$

The tuple $\{(X_i)_i, (\Sigma_i)_i, (C_{\delta_{ij}})_{i,j}\}$, will be called a K-cell decomposition of X.

Notice that the cells C_j are not necessarily disjoint (in fact, some of them may even coincide.) What we obtain is a family of bijective translations T_σ between the disjoint union $\sqcup_j C_j$ and one of the parts X_i. Also note that while the sets Σ_i are definable, we cannot assure that any of the individual curves σ contained in it will be definable. (If a definable σ exists and X only consists of K-variables, a cell decomposition with the functions $\sigma(x)$ as centers will be very similar to what Mourgues obtained.)

In the second part of the paper (Section 4), we discuss applications of the preparation and cell decomposition theorems. We will restrict our attention to the case where K is a p-adic field. The results in other sections are valid for arbitrary p-adically closed fields.

Inspired by his rationality results, Denef decided to introduce the class of constructible functions:

DEFINITION 2.6. Let X be an $\mathcal{L}_2$-definable set. Write $\mathbb{A}_{q_K}$ for the ring

$$\mathbb{A}_{q_K} := \mathbb{Z}\left[q_K, q_K^{-1}, \left(\frac{1}{1 - q_K^{-i}} \right)_{i \in \mathbb{N}, i>0} \right].$$

We say that a function $f : X \to \mathbb{Q}$ is $\mathcal{L}_2$-*constructible* if it is contained in the $\mathbb{A}_{q_K}$-algebra generated by functions of the forms

$$\alpha : X \to \mathbb{Z} \qquad \text{and} \qquad X \to \mathbb{Z} : x \mapsto q_K^{\beta(x)},$$

where α and β are $\mathcal{L}_2$-definable and $\mathbb{Z}$-valued.

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When $\mathcal{L}$ is $\mathcal{L}_{\text{ring}}$, the subanalytic language $\mathcal{L}_{an}$ on K (see [11] for a definition), or some intermediary languages as in [4], the class of $\mathcal{L}_2$-constructible functions is known to be stable under integration (see [10, 1, 6] for the most convenient dealing with integrability conditions). We show that (see Theorem 4.1), whenever $(K,\mathbb{Z})$ is a P-minimal structure, the class of $\mathcal{L}_2$-constructible functions is stable under integration:

THEOREM. *Let K be a p-adic field, $(K,\mathbb{Z})$ a P-minimal structure, and $f : X \subseteq S \times K^m \to \mathbb{A}_{q_K}$ an $\mathcal{L}_2$-constructible function such that $f(s,\cdot)$ is measurable and integrable on Y_s for all $s \in S$. There exists a constructible function $g : S \to \mathbb{A}_{q_K}$, such that*

$$g(s) = \int_{X_s} f(s,x)|dx|,$$

for all $s \in S$.

We also extend the rationality results known so far only for the semi-algebraic [8] and subanalytic setting [11] (and thus also for any sublanguage), obtaining the following:

THEOREM. *Suppose that $(K,\mathbb{Z})$ is P-minimal. Let X be a definable subset of $\mathcal{O}_K^n \times \mathbb{N}$, and let a_n be the Haar measure of $X_n := \{x \in \mathcal{O}_K^n \mid (x,n) \in X\}$ for each $n \geq 0$. Then the series $\sum_{i\geq 0} a_i T^i$ is rational.*

Here we normalize the Haar measure on K^n so that $\mathcal{O}_K^n$ has measure 1. For a more precise statement we refer to Corollary 4.6.

§3. Cell decomposition and function preparation. In this section we will give a proof of the cell decomposition and preparation theorems. For the comfort of the reader, we will restate (an abbreviated version of) the theorems. To ease notation, we will assume that for any definable set $X \subseteq S \times K$, the projection onto the parameter set S is surjective, replacing S by $\Pi_S(X)$ if necessary. The following notation will also be used in the proofs of this section.

DEFINITION 3.1. A *K-cell condition* is a formula $C_\delta(x,y,\alpha,\beta;s)$ of the form

$$C_\delta(x,y,\alpha,\beta;s) := \begin{pmatrix} \alpha(s) \,\square_1\, \operatorname{ord}(x-y) \,\square_2\, \beta(s) \wedge \\ \operatorname{ord}(x-y) \equiv k \mod n \wedge \\ \operatorname{ac}_m(x-y) = \xi \end{pmatrix},$$

where $(\square, k, n, m, \xi) = \delta \in P_K$, and α, β are definable functions $S \to \Gamma_K$. When no s appears, α, β are just elements of Γ_K.

THEOREM (Theorem-Definition 2.5). *Let $X \subseteq S \times K$ be a set definable in a P-minimal structure (K,Γ_K). There exists a K-cell decomposition $\{(X_i)_i, (\Sigma_i)_i, (C_{\delta_{ij}})_{i,j}\}$ of X.*

PROOF. Fix a parameter $s \in S$. By the cell decomposition theorem for semi-algebraic sets, see e.g., [6, Theorem 3.3.2], there exists a finite partition of X_s into K-cells

$$C_s = \left\{ t \in K \;\middle|\; \begin{array}{l} \alpha_s \,\square_1\, \operatorname{ord}(t-c_s) \,\square_2\, \beta_s, \\ \operatorname{ord}(t-c_s) \equiv k_s \mod n_s, \\ \operatorname{ac}_{m_s}(t-c_s) = \xi_{m_s,s} \end{array} \right\}, \tag{1}$$

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where $\alpha_s, \beta_s \in \Gamma_k, c_s \in K$ and $(\square, k_s, n_s, m_s, \xi_{m_s,s}) \in P_K$. Note that the cell decomposition of X_s may contain multiple cells of the same type.

CLAIM 3.2. *There is a natural number $N \geq 1$ such that for every $s \in S$, the set X_s can be partitioned as a union of at most N K-cells, and for each of these cells we can assure that $n_s, m_s < N$.*

The claim will follow by a standard compactness argument. Recall that P_K consists of elements $\delta = (\square_\delta, k_\delta, n_\delta, m_\delta, \xi_\delta)$, encoding the type of a K-cell. For each positive integer N, put

$$P_{K,N} := \{\delta \in P_K \mid n_\delta < N, m_\delta < N\},$$

and write

$$E_{K,N} := \bigsqcup_{i=1}^{N} P_{K,N},$$

for the disjoint union of N copies of $P_{K,N}$. Note that $E_{K,N}$ is a finite set. For every $J \subseteq E_{K,N}$, fix an enumeration $\{\delta_1, \ldots, \delta_{|J|}\}$ of J. Given $y = (y_1, \ldots, y_{|J|}) \in K^{|J|}$, and $\alpha = (\alpha_1, \alpha_2) \in \Gamma_K^{2|J|}$ we will write

$$C_J(y, \alpha) := \begin{cases} \bigcup_{i=1}^{|J|} C_{\delta_i}(K, y_i, \alpha_{1i}, \alpha_{2i}) & \text{if the sets } C_{\delta_i}(K, y_i, \alpha_{1i}, \alpha_{2i}) \\ & \text{are disjoint,} \\ \emptyset & \text{otherwise,} \end{cases}$$

making use of K-cell conditions C_{δ_i} as defined in Definition 3.1. Consider the set of formulas

$$\Sigma(x) := \left\{ \bigwedge_{J \subseteq E_{K,N}} \neg(\exists y \in K^{|J|})(\exists \alpha \in \Gamma_K^{2|J|})[X_x = C_J(y, \alpha)] \;\middle|\; N \in \mathbb{N}^* \right\}.$$

Since each X_s can be partitioned in semi-algebraic cells as in (1), $\Sigma(x)$ is inconsistent. Hence, by compactness there exists a finite subset $\Sigma_0(x)$ which is inconsistent. Since $\Sigma_0(x)$ is a finite subset of $\Sigma(x)$, one can find a positive integer N_0 such that

$$\left[\bigwedge_{J \subseteq E_{K,N_0}} \neg(\exists y \in K^{|J|})(\exists \alpha \in \Gamma_K^{2|J|})[X_x = C_J(y, \alpha)] \right] \models \Sigma_0(x).$$

This implies that there must exist $N > 0$ such that for every $s \in S$

$$(K, \Gamma_K) \models \left[\bigvee_{J \subseteq E_{K,N}} (\exists y \in K^{|J|})(\exists \alpha \in \Gamma_K^{2|J|})[X_s = C_J(y, \alpha)] \right],$$

which completes the claim.

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Now choose an integer N satisfying the requirements of Claim 3.2. Let W_N denote the power set of $E_{K,N}$. Since W_N is finite, one can put a total ordering $\preccurlyeq$ on it. We will also put an alternative ordering $\lhd$ on the value group Γ_K, which is defined by:

$$x \lhd y \Leftrightarrow (0 \leqslant x < y) \vee (0 < x \leqslant -y) \vee (0 < -x < y) \vee (0 < -x < -y). \quad (2)$$

This produces a total ordering on Γ_K which can be extended to Γ_K^k lexicographically. We will also denote this extension by $\lhd$. The important property of the order $\lhd$ is that every definable set of Γ_K has a $\lhd$-smallest element (for a proof of this, see the appendix, in particular A.2). Now consider the map

$$\tau : S \to W_N \times (\Gamma_K)^{\leq 2|W_N|} : s \mapsto (\tau_1(s), \tau_2(s)),$$

where τ_1, τ_2 are defined as follows:

- put $\tau_1(s) = J$, if J is the $\preccurlyeq$-smallest element of W_N such that

$$(K, \Gamma_K) \models \left[(\exists y \in K^{|J|})(\exists \alpha \in \Gamma_K^{2|J|})[X_s = C_J(y, \alpha)]\right].$$

  The claim ensures the existence of at least one such J in W_N.
- let $\tau_2(s)$ be the $\lhd$-smallest tuple $\alpha \in \Gamma_K^{2|\tau_1(s)|}$ such that

$$(K, \Gamma_K) \models (\exists y \in K^{|J|})[X_s = C_{\tau_1(s)}(y, \alpha)].$$

It is clear that the function τ will be definable, using some fixed representation for the finite index set W_N. For each $J \in \tau_1(S)$, let S_J be the set $\{s \in S \mid \tau_1(s) = J\}$. These sets induce a partition of X into sets $X_J := \{(s, x) \in X : s \in S_J\}$. We show that this partition satisfies all conditions stated in the theorem. Fix $\{\delta_1, \ldots, \delta_{|J|}\} = J \in \tau_1(S)$. The integer r associated to X_J is precisely $r := |J|$. Let $\Sigma_J := \bigcup_{s \in S_J} \{s\} \times \Sigma_{J,s}$ be the set consisting of fibers

$$\Sigma_{J,s} := \{y \in K^r \mid \left[X_s = C_J(y, \tau_2(s))\right]\}.$$

Note that these sets are nonempty by definition of τ_1. Given any function $\sigma : S_J \to K^r : s \mapsto (\sigma_1(s), \ldots, \sigma_r(s))$ whose graph is contained in Σ_J, one then has that $X_s = C_J(\sigma(s), \tau_2(s))$ for all $s \in S_J$. For $1 \leq j \leq r$ and $i \in \{1, 2\}$, define $\alpha_{ij} : S_J \to \Gamma_K$ to be the ij^{th}-component in the tuple $\tau_2(s)$. We obtain that

$$X_J = \bigcup_{j=1}^{r} \{(s, x) \in S_J \times K \mid C_{\delta_j}(x, \sigma_j(s), \alpha_{1j}, \alpha_{2j}, s)\}.$$

Taking the r K-cells associated to X_J given by

$$C_j := \{(s, x) \in S_J \times K \mid C_{\delta_j}(x, 0, \alpha_{1j}, \alpha_{2j}, s)\}, 1 \leq j \leq r,$$

it is clear that the translation map T_σ stated in the theorem gives the required bijection. ⊣

We now present the preparation theorem for definable subsets of the form $X \subseteq S \times \Gamma_K$. The proof follows a similar scheme as the previous one.

Proposition (Proposition 2.4). *Let $(K, \Gamma_k, \mathcal{L}_2)$ be P-minimal. Let $X \subseteq S \times \Gamma_K$ and $f : X \to \Gamma_K$ a definable function. There exists a finite decomposition of X into*

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Γ-cells C such that on each such cell C, there exists a constant $a_C \in \mathbb{Z}$ and a definable function $\delta : D \to \Gamma_K$, such that for all $(x, \gamma) \in C$,

$$f(x,\gamma) = a_C\left(\frac{\gamma - n_0}{n}\right) + \delta(x).$$

Proof. Since $(K, \mathcal{L})$ is P-minimal, it follows from Theorem 2.2 that each of the fibers X_s is Presburger definable. Cluckers [2] obtained a cell decomposition theorem for Presburger structures. Applying this to the sets X_s, yields that each X_s can be partitoned into a finite union of cells of the form

$$C_s := \{\gamma \in K \mid \alpha_s \,\square_1\, \gamma \,\square_2\, \beta_s \quad \text{and} \quad \gamma \equiv n_0 \mod n_s\},$$

where $\alpha_s, \beta_s \in \Gamma_K$ and $n_s \in \mathbb{N}$ are constants depending on s. Also note that for any $s \in S$, the graph of the function f_s will be a Presburger set, by the assumption of P-minimality. Indeed, the related set

$$G_s := \{(x, y) \in (K^\times)^2 \mid f_s(\operatorname{ord} x) = \operatorname{ord} y\}$$

is definable in a P-minimal structure, and hence by Theorem 2.2, the set

$$\mathit{Graph}(f_s) = \{(\operatorname{ord} x, \operatorname{ord} y) \in \Gamma^2 \mid (x, y) \in G_s\},$$

is Presburger definable. This means that each f_s is a Presburger definable function, and hence must be piecewise linear (with coefficients in $\mathbb{Q}$). In particular, the above partition can be taken such that on each C_s, there exist constants $a_s \in \mathbb{Z}, \delta_s \in \gamma_K$ such that for all $\gamma \in C_s$, we have that

$$f_s(\gamma) = a_s\left(\frac{\gamma - n_0}{n_s}\right) + \delta_s.$$

Claim 3.3. *There is a natural number $N \geq 1$ such that for every $s \in S$, the set X_s can be partioned as a union of at most N Γ-cells, and for each of these cells we can assure that $n_s, |a_s| < N$.*

The claim follows by compactness and P-minimality using an analogous argument to the one presented in Claim 3.2.

For an integer N satisfying the requirements of the claim, let

$$P_{\Gamma,N} := \{(\square, k, n) \in P_\Gamma \mid n < N\} \quad \text{and} \quad E_{\Gamma,N} := \bigsqcup_{i=1}^{N} P_{\Gamma,N}.$$

Recall that P_Γ consists of elements $\delta = (\square_\delta, k_\delta, n_\delta)$, encoding the type of a Γ-cell. We will use the notation

$$C_\delta(\alpha, \beta) := \{\gamma \in K \mid \alpha \,\square_{\delta,1}\, \gamma \,\square_{\delta,2}\, \beta \wedge \gamma \equiv k_\delta \mod n_\delta\}.$$

For every $J \subseteq E_{\Gamma,N}$, fix an enumeration $\{\delta_1, \ldots, \delta_{|J|}\}$ of J. Given $\alpha = (\alpha_1, \alpha_2) \in \Gamma_K^{2|J|}$, we put

$$C_J(\alpha) := \begin{cases} \bigcup_{i=1}^{|J|} C_{\delta_i}(\alpha_{1i}, \alpha_{2i}) & \text{if the sets } C_{\delta_i}(\alpha_{1i}, \alpha_{2i}) \\ & \text{are disjoint,} \\ \emptyset & \text{otherwise.} \end{cases}$$

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Let $a \in \mathbb{Z}^{|J|}$ be such that $|a_i| < N$ for all $1 \leq i \leq |J|$. Let x be a tuple of variables of the same length (and sorts) as elements in S, and γ a Γ_K-variable of length 1. The tuple $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ consists of (tuples of) Γ_K-variables: α_1, α_2, and α_3 all have length $|J|$. We define the formula $\phi_{J,a}(x, \alpha)$ as

$$\phi_{J,a}(x,\alpha) := \left(\begin{array}{l} X_x = C_J(\alpha_1, \alpha_2) \wedge \\ \bigwedge\limits_{1 \leq i \leq |J|} (\forall \gamma) \left[\gamma \in C_{\delta_i}(\alpha_{1i}, \alpha_{2i}) \rightarrow \left(f_x(\gamma) = a_i \left(\dfrac{\gamma - k_{\delta_i}}{n_{\delta_i}} \right) + \alpha_{3i} \right) \right] \end{array} \right).$$

Roughly, this formula states that the set X_x can be decomposed into finitely many disjoint Γ-cells, on each of which the function f_s satisfies the required preparation condition. Define the set W_N by

$$W_N := \{(J, a) : J \subseteq E_{\Gamma,N}, a \in \mathbb{Z}^{|J|}, |a_i| < N \text{ for all } 1 \leq i \leq |J|\}.$$

Since W_N is finite, one can put a total ordering $\preccurlyeq$ on it. As before we work with an alternative total ordering $\lhd$ on the value group Γ_K defined as in equation (2). We proceed as in the K-cell decomposition theorem and define a map

$$\sigma : S \rightarrow W_N \times (\Gamma_K)^{\leq 3|W_N|} : s \mapsto (\sigma_1(s), \sigma_2(s)),$$

where σ_1, σ_2 are defined as follows:

- put $\sigma_1(s) = (J, a)$, if (J, a) is the $\preccurlyeq$-smallest element of W_N such that
$$(K, \Gamma_K) \models (\exists \alpha \in \Gamma_K^{3|J|}) \phi_{J,a}(s, \alpha).$$
Claim 3.3 ensures the existence of at least one such (J, a) in W_N.
- let $\sigma_2(s)$ be the $\lhd$-smallest tuple $\alpha \in \Gamma_K^{3|\sigma_1(s)|}$ such that
$$(K, \Gamma_K) \models \phi_{\sigma_1(s)}(s, \alpha).$$

It is easy to see that the function σ will be definable, using some fixed representation of the finite index set W_N. We recover the Γ-cell decomposition for X and the linear functions satisfying the preparation condition in the following way. For each $\lambda = (J, a) \in \sigma_1(S)$, we define sets S_λ and X_λ, as

$$S_\lambda := \{s \in S \mid \sigma_1(s) = \lambda\} \quad \text{and} \quad X_\lambda := \{(s, x) \in X : s \in S_\lambda\}.$$

This gives us a finite partition of X as $X = \cup_\lambda X_\lambda$. We will now partition the sets X_λ as a finite union of Γ-cells, on each of which f will have the required form.

For $1 \leq j \leq |J|$ and $i \in \{1, 2, 3\}$, define $\alpha_{\lambda ij} : S_\lambda \rightarrow \Gamma_K$ to be the ij^{th}-coordinate of $\sigma_2(s)$. Note that these functions are indeed definable, since σ_2 is definable. The above construction now implies that

$$X_\lambda = \bigcup_{1 \leqslant j \leqslant |J|} C_{\lambda,j},$$

where $C_{\lambda,j} := \left\{(s, \gamma) \in S_\lambda \times K \mid \alpha_{\lambda 1 j} \sqsubset_{\delta_j,1} \gamma \sqsubset_{\delta_j,2} \alpha_{\lambda 2 j} \wedge \gamma \equiv k_{\delta_j} \mod n_{\delta_j}\right\}$. The formula $\phi_\lambda(s, \alpha)$ then ensures that for all $(s, \gamma) \in C_{\lambda,j}$, it holds that

$$f_{|C_{\lambda,j}}(s, \gamma) = a_j \left(\frac{t - k_{\delta_j}}{n_{\delta_j}} \right) + \alpha_{\lambda 3 j},$$

which completes the proof. ⊣

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REMARK. Proposition 2.4 can be used to *translate* theorems for parametrized Presburger definable sets $X \subseteq S \times \Gamma_K^m$ to two-sorted P-minimal structures, in the following sense. The same theorems will hold in any two-sorted P-minimal structure, where the parameter set S can now be any $\mathcal{L}_2$-definable set containing variables in both K and Γ_K, and the involved Presburger-definable functions should be replaced by functions which are piecewise linear in the Γ_K-variables, in the sense of Proposition 2.4. The corollary stated below is an example of this.

Given a P-minimal structure $(K, \Gamma_K, \mathcal{L}_2)$, we call a definable function $f : X \subseteq S \times \Gamma_K^m \to S \times \Gamma_K^l$ *linear over* S if there is a definable function $g : S \to \Gamma_K^l$ and a linear definable function $a : \Gamma_K^m \to \Gamma_K^l$ such that $f(s,t) = (s, g(s) + a(t))$ for all $(s,t) \in X$. Note that the parameter set S may contain both Γ_K- and K-variables. We write H for the set $H := \{x \in \Gamma_K \mid x \geqslant 0\}$.

COROLLARY 3.4 (Parametric rectilinearization). *Let $(K, \Gamma_K, \mathcal{L}_2)$ be a P-minimal structure and $X \subseteq S \times \Gamma_K^m$ be a definable set. There exists a finite partition of X into definable sets such that the following holds.*

For each part A, there is a set $B \subseteq S \times \Gamma_K^m$ and a definable bijection $\rho : A \to B$ which is linear over S such that, for each $s \in S$, the set B_s is a set of the form $\Lambda_s \times H^l$ for a bounded subset $\Lambda_s \subseteq H^{m-l}$, depending on s (in a definable way), and for an integer $l \geq 0$ only depending on A.

PROOF. The proof is almost word for word the proof the same as the proof of the Parametric rectilinearization Theorem for Presburger definable sets (Theorem 3 in [2]). One just needs to replace every application of the Presburger function preparation theorem (Theorem 1 in [2]) by Proposition 2.4. (Indeed, both theorems are formally very similar, the main difference being that Theorem 2.4 allows the parameter set S to be a general P-minimal set, whereas Theorem 1 from [2] was stated for the Presburger setting, and hence only allows S to range over Γ_k-variables.) ⊣

§4. Integration and rationality. In this section, K denotes a p-adic field, so the value group Γ_K will just be $\mathbb{Z}$. Two types of integrals will appear. When integrating over (subsets of) K^m, the Haar measure μ is used. When integrating over $\mathbb{Z}^n$, we use the counting measure. The notation $\int_X |dx|$ will be used in both contexts, adapting the measure $|dx|$ to the sort of the variables involved.

The results below are stated for an $\mathcal{L}_2$-constructible function $f : X \subseteq S \times Y \to \mathbb{A}_{q_K}$, where both S and Y are definable sets and S is considered a parameter set. Note that both S and Y may contain variables in both the K-sort and the $\mathbb{Z}$-sort, unless explicitly stated otherwise. Recall that the definition of constructible functions was given in Definition 2.6. For such a function f, we define its *locus of integrability* as the set

$$\mathrm{Int}(f, S) := \{s \in S \mid f(s, \cdot) \text{is measurable and integrable on } Y_s\}.$$

The main result of this section is the following theorem.

THEOREM 4.1. *Let K be a p-adic field and $(K, \mathbb{Z}, \mathcal{L}_2)$ be a P-minimal structure. Let S be a definable set, and $f : X \subseteq S \times Y \to \mathbb{A}_{q_K}$ an $\mathcal{L}_2$-constructible function such that $Int(f, S) = S$. There exists an $\mathcal{L}_2$-constructible function $g : S \to \mathbb{A}_{q_K}$, such that*

$$g(s) = \int_{X_s} f(s, y)|dy|,$$

for all $s \in S$.

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For semi-algebraic and subanalytic sets, Cluckers, Gordon, and Halupczok [6] recently obtained a stronger version of this result. Their version is more general in the sense that they do not require the assumption $\mathrm{Int}(f, S) = S$, instead relying on an interpolation lemma, replacing f by a function $\tilde{f}$ that coincides with f on its locus of integrability, and for which $\mathrm{Int}(\tilde{f}, S) = S$. If a similar interpolation lemma can be proven to hold in general P-minimal structures, the assumption that $\mathrm{Int}(f, S) = S$, can be removed from our result as well.

PROPOSITION 4.2. *Theorem* 4.1 *holds when* $X \subseteq \mathbb{Z}^r$.

PROOF. Note that in this case, X is $\mathcal{L}_{\mathrm{Pres}}$-definable, by Theorem 2.2. Proofs can be found in [6], Theorem 2.1.6. ⊣

PROPOSITION 4.3. *Theorem* 4.1 *holds when* $\mathcal{L}_2 = \mathcal{L}_{ring,2}$ *or* $\mathcal{L}_{an,2}$.

PROOF. See [6], Theorem 3.1.1. ⊣

As a first step towards a general proof of Theorem 4.1, we show that it already holds when $Y \subseteq \mathbb{Z}^r$:

PROPOSITION 4.4. *Theorem* 4.1 *holds when* $Y \subseteq \mathbb{Z}^r$.

PROOF. This is essentially a consequence of Proposition 2.4 (see also the remark on page 1135). In [6], this proposition was proven under the assumption that $\mathcal{L}_2 = \mathcal{L}_{\mathrm{ring},2}$ or $\mathcal{L}_{\mathrm{an},2}$. Part (1) corresponds to Theorem 3.4.5 and part (2) to Theorem 3.1.1 in [6]. If one replaces their Parametric rectilinearization Theorem (Proposition 3.4.4 in [6]) by Corollary 3.4, the same proof also works for two-sorted P-minimal structures. ⊣

We will reduce the general case to Proposition 4.4 using the following observation on the measure of definable sets.

PROPOSITION 4.5. *Let* $(K, \mathbb{Z}, \mathcal{L}_2)$ *be a* P*-minimal structure. Let* $X \subseteq S \times T$ *be a definable set, where* T *is* K *or* $\mathbb{Z}$. *There exists a constructible function* $g : S \to \mathbb{A}_{q_K}$, *such that* g *uniformly measures the fibers* X_s, *that is,*

$$g(s) = \int_{X_s} |dt|,$$

whenever X_s *has finite measure. Moreover, the set*

$$\tilde{S} := \{s \in S \mid X_s \text{ has finite measure}\},$$

is definable.

PROOF. When $T = \mathbb{Z}$, this is a consequence of Proposition 4.4, where f is the constant function $f(x) = 1$ for all $x \in X$. The fact that the set $\tilde{S}$ is definable follows from the cell-decomposition part of Proposition 2.4. Indeed, a Γ-cell $B \subseteq \mathbb{Z}$ (as in definition 2.3) has finite measure if and only if both $\square_1$ and $\square_2$ are $<$ on such cell. This is a definable condition.

Let us now consider the case where $T = K$. By (the translation version of) the K-cell decomposition theorem (i.e., Theorem 2.5), we can partition X in parts $X_i \subseteq S_i \times K$. On each of these parts, for any choice of a function σ with image contained in Σ, we have that

$$\int_{(X_i)_s} |dt| = \int_{T_\sigma(\sqcup(C_j)_s)} |dt| = \sum_j \left[\int_{(C_j)_s} |dt| \right],$$

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and hence it suffices to compute the integral $\int_{(C_j)_s} |dt|$. Assume that C_j is the zero-centered K -cell

$$C_j := \left\{ (s,t) \in S_i \times K \;\middle|\; \begin{array}{l} \alpha_j(s) \square_{j,1} \operatorname{ord} t \square_{j,2} \beta_j(s), \\ \operatorname{ord} t \equiv k_j \mod N, \\ \operatorname{ac}_M(x) = \xi_j \end{array} \right\}.$$

Computing the measures of these cells, we get that

$$\begin{aligned} \int_{(C_j)_s} |dt| &= \sum_{\tau \in T_j} \mu\left(\xi_j \pi_K^{k_j+\tau N}(1+\pi^M \mathcal{O}_K)\right), \\ &= |\xi| q_K^{-(k_j+M)} \sum_{\tau \in T_j} (q_K^{-N})^\tau, \end{aligned}$$

where π_K is a uniformizing element for K and T_j is the set

$$T_j := \{\tau \in \mathbb{Z} \mid \alpha_j(s) \square_{j,1} k_j + \tau N \square_{j,2} \beta_j(s)\}.$$

It is easy to see that $(C_j)_s$ (and hence X) can only have finite measure if $\square_{j,1}$ denotes $<$ for $j = 1, \ldots, l$. Since this is a property of the cell, this is a definable condition.

We get the following results for this sum. If we put $\tilde{\alpha} := \lfloor \frac{\alpha - k}{N} \rfloor + 1$, and $\tilde{\beta} := \lceil \frac{\beta - k}{N} \rceil - 1$ (clearly these are still definable functions), then we get that

$$\sum_{\tau \in T_j} (q_K^{-N})^\tau = \begin{cases} \frac{q_K^{-N\tilde{\alpha}_j}}{1-q_K^{-N}} & \text{if } \square_{j,2} = \emptyset, \\ \frac{1}{1-q_K^{-N}}(q_K^{-N\tilde{\alpha}_j} - q_K^{-N\tilde{\beta}_j}) & \text{if } \square_{j,2} = <. \end{cases} \tag{3}$$

In both cases we obtain an $\mathcal{L}_2$-constructible function. Hence, we can conclude that $\mu(X_s)$ is given by a constructible function as well. ⊣

We can now complete the proof of the main theorem:

Proof of Theorem 4.1. Since $\operatorname{Int}(f, S) = S$, by Fubini's theorem, the general result can be obtained by iteration and we may assume that either $Y = \mathbb{Z}$, or $Y = K$. The first case is already included in Proposition 4.4, so we only need to consider the case $Y = K$.

A general constructible function $f : X \subseteq S \times K \to \mathbb{A}_{q_K}$ has the form

$$f(s,x) = \sum_{i=1}^{r} a_i q_K^{f_{i0}(s,x)} \prod_{j=1}^{r'} f_{ij}(s,x),$$

where the f_{ij} are definable functions $X \to \mathbb{Z}$, and $a_i \in \mathbb{A}_{q_K}$. Now put $\gamma = (\gamma_{ij})_{i,j}$ and consider the set

$$G := \{(s, \gamma, x) \in S \times \mathbb{Z}^{(r'+1)r} \times K \mid \gamma_{ij} = f_{ij}(s,x)\},$$

which is a permutated version of the combined graphs of the functions generating f. To ease notations, we will sometimes consider G as a subset of $D \times K$, where $D = \Pi_{S \times \mathbb{Z}^{(r'+1)r}}(G)$. Let μ denote the usual Haar measure. The integral of f_s can be written as a sum ranging over $\operatorname{Im}(f_s)$:

$$\int_{X_s} f(s,x)|dx| = \sum_{\delta \in \operatorname{Im}(f_s)} \delta \cdot \mu\{x \in X_s \mid f_s(x) = \delta\},$$

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and this sum can be expressed in terms of the variables γ, to obtain a sum

$$\sum_{\gamma \in D_s} \left[\left(\sum_{i=1}^{r} a_i q_K^{\gamma_{i0}} \prod_{j=1}^{r'} \gamma_{ij} \right) \cdot \mu \left(\{ x \in X_s \mid \bigwedge_{ij} f_{ij}(s,x) = \gamma_{ij} \} \right) \right].$$

This reduces the integral to a sum

$$\int_{X_s} f(s,x)|dx| = \sum_{\gamma \in D_s} \left(\sum_{i=1}^{r} a_i q_K^{\gamma_{i0}} \prod_{j=1}^{r'} \gamma_{ij} \right) \cdot \mu(G_{s,\gamma}). \tag{4}$$

Applying Proposition 4.5, we know that $\mu(G_{s,\gamma})$ is given by a constructible function, whenever $G_{s,\gamma}$ has finite measure. Since this is a definable condition, we may as well assume that the measure of $G_{s,\gamma}$ is finite for all $s \in S$ and $\gamma \in \mathrm{Im}(f_s)$. Hence, we can conclude that

$$\int_{X_s} f(s,x)|dx| = \sum_{\gamma \in D_s} h(s,\gamma), \tag{5}$$

for some constructible function $h : D \to \mathbb{A}_{q_K}$. Noticing that

$$\sum_{\gamma \in D_s} h(s,\gamma) = \int_{D_s} h(s,\gamma)|d\gamma|,$$

the result follows by Proposition 4.4 applied to the constructible function h. ⊣

As a consequence of Theorem 4.1, we obtain the following rationality result.

COROLLARY 4.6. *Suppose that $(K,\mathbb{Z})$ is P-minimal. Let X be a definable subset of $\mathbb{N} \times K^m$, for which all fibers X_n have finite measure. Then the series $\sum_{n\geq 0} \mu(X_n)T^n$ is a rational function. More precisely,*

$$\sum_{n\geq 0} \mu(X_n)T^n = \frac{Q(T)}{\prod_{i=1}^{r}(1 - q_K^{-m_i} T_i^N)},$$

for certain integers $m_i, r \in \mathbb{N}$, $N_i > 0$ and $Q(T) \in \mathbb{A}_{q_K}[T]$.

PROOF. Applying Theorem 4.1 to the set $X \subseteq \mathbb{N} \times D$, one can find a constructible function $g : \mathbb{N} \to \mathbb{A}_{q_K}$, such that

$$g(n) = \int_{X_n} |dx|.$$

This function must have the form

$$g(n) = \sum_{i=1}^{r} a_i q_K^{\alpha_i(n)} \prod_j \beta_{ij}(n),$$

where $a_i \in \mathbb{A}_{q_K}$, and the functions α_i and β_{ij} are Presburger-definable functions $\mathbb{N} \to \mathbb{Z}$, and hence it is actually $\mathcal{L}_{\mathrm{ring},2}$-constructible. Our claim now follows from Denef's rationality results in the semi-algebraic case, for which we refer to eg., [8,9]. ⊣

We finish by presenting as a conjecture a version of interpolation for P-minimal constructible functions:

CONJECTURE 4.7 (Interpolation). *Let K be a p-adic field and $(K,\mathbb{Z},\mathcal{L}_2)$ be a P-minimal structure. For every $\mathcal{L}_2$-constructible function $f : X \subseteq S \times Y \to \mathbb{A}_{q_K}$ there*

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exists an $\mathcal{L}_2$-constructible function $g : X \subseteq S \times Y \to \mathbb{A}_{q_K}$ such that $Int(g, S) = S$ and $f(s, y) = g(s, y)$ whenever $s \in Int(f, S)$.

Assuming this conjecture, Theorem 4.1 implies the full generalization of the stability result in [6].

COROLLARY 4.8. *Let K be a p-adic field and $(K, \mathbb{Z}, \mathcal{L}_2)$ be a P-minimal structure and suppose that the interpolation conjecture is true. Let S be a definable set, and $f : X \subseteq S \times Y \to \mathbb{A}_{q_K}$ an $\mathcal{L}_2$-constructible function. There exists an $\mathcal{L}_2$-constructible function $g : S \to \mathbb{A}_{q_K}$, such that*

$$g(s) = \int_{X_s} f(s, y)|dy|,$$

whenever $s \in Int(f, S)$.

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§A. Appendix. Definably well-ordered structures.

DEFINITION A.1. A structure $(M, \mathcal{L})$ is said to be *definably well-ordered* if there exists a definable linear order $\lhd$ on M, such that every definable subset of M has a $\lhd$-minimal element.

Note that this is a very strong property. Theories having definably well-ordered models have definable choice, so in particular they have definable Skolem functions and they eliminate imaginaries (for definitions we refer to [23] and [14]).

For a structure, being definably well-ordered is a stronger property than having definable choice: the real field has definable choice, yet by a result of Ramakrishnan in [22] every definable order embeds in $(\mathbb{R}^n, <_{\text{lex}})$. Therefore, no definable linear order has minimal elements for all definable subsets of the real line. Using this, one can show that no reduct of the real field is definably well-orderable.

Even though definably well-ordered structures have strong model-theoretic properties, they are not always model-theoretically tame. For instance, the theory of arithmetic is definably well-orderable and yet model-theoretically wild. The main example of a tame well-orderable theory is Presburger Arithmetic.

PROPOSITION A.2. *Let $\mathcal{L}$ be a language containing $\{\leq, -\}$. Then every model of $Th(\mathbb{Z}, \mathcal{L})$ is definably well-orderable.*

PROOF. Notice that if $(M, \mathcal{L})$ is a definably well-ordered structure and the definable linear order witnessing this property is 0-definable, then every $N \equiv M$ is also definably well-ordered. Now consider the following

$$x \lhd y \Leftrightarrow (0 \leqslant x < y) \vee (0 < x \leqslant -y) \vee (0 < -x < y) \vee (0 < -x < -y).$$

This formula defines the following well-order on $\mathbb{Z}$ (which completes the proof):

$$0 \lhd 1 \lhd -1 \lhd 2 \lhd -2 \lhd \cdots . \qquad \dashv$$

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The linear ordering defined in the above proposition does not necessarily define a well-order on every $\mathbb{Z}$-group G. However, it does define a linear order such that for any definable subset $A \subseteq G$, A has a $\lhd$-minimal element. As a corollary we get a result from [2]:

COROLLARY A.3. *Presburger arithmetic has elimination of imaginaries.*

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