Mathematical Proceedings of the Cambridge Philosophical Society

VOL. 163 JULY 2017 PART 1

Math. Proc. Camb. Phil. Soc. (2017), **163**, 1–21 © *Cambridge Philosophical Society* 2016
doi:10.1017/S0305004116000700
First published online 30 August 2016

Measure theory over boolean toposes

BY SIMON HENRY

College de france, ATER chair d'analyse et géométrie, 3 Rue d'Ulm, Paris 75005, France
e-mail: henry@phare.normalsup.org

(*Received* 17 *June* 2015; *revised* 4 *July* 2016)

Abstract

In this paper we develop a notion of measure theory over boolean toposes reminiscent of the theory of von Neumann algebras. This is part of a larger project to study relations between topos theory and noncommutative geometry. The main result is a topos theoretic version of the modular time evolution of von Neumann algebras which take the form of a canonical $\mathbb{R}^{>0}$-principal bundle over any integrable locally separated boolean topos.

1. *Introduction*

This paper is part of a project, starting with the author's thesis [**2**], whose aim is to understand the relations between noncommutative geometry and topos theory, and is more precisely focused on the measure theoretic aspect of this question. We will make extensive use of the internal logic of toposes, one can consult [**7**, VI·5 and VI·6] for an introduction to this concept. By "topos", we will always mean Grothendieck topos and the general framework is classical logic with the axiom of choice. A version of this work avoiding the axiom of choice, and hence applicable in the internal logic of a boolean topos, can be found in [**2**, chapter 2].

If $\mathcal{T}$ is a (Grothendieck) topos, one can define a C^*-category $\mathcal{H}(\mathcal{T})$ of internal $\mathcal{T}$-Hilbert spaces (see section 2·3 for the details), which plays the role of the category of Hilbert bundles over $\mathcal{T}$. This provides an extremely efficient way to relate topos theory to operator algebra theory and, even if it does not produce any strict duality relating the two theories, it will allow to transport certain results and constructions from one side to the other. This paper explains how the notion of "modular time evolution" can be transported this way from the theory of von Neumann algebras to topos theory.

https://doi.org/10.1017/S0305004116000700 Published online by Cambridge University Press

The modular theory of von Neumann algebras is a construction (explained for example in [**11**, chapter VIII]) that associates to any von Neuman algebra a canonical outer action of $\mathbb{R}$ (i.e. a group morphism from $\mathbb{R}$ to the group of outer automorphisms of the algebra). More precisely, M.Tomita associated to any von Neumann algebra endowed with a normal semi-finite weight an action of $\mathbb{R}$ which is characterised by the so called Kubo–Martin–Schwinger (KMS) condition[1] and A.Connes proved that up to conjugation by a continuous cocycle this action actually does not depend on the weight (and hence is well defined when considered as an outer action). This outer action is non-trivial only on the type III part of the algebra. Our goal is to give a purely topos theoretic analogue of this (in Section 6) and to relate it to the classical modular theory through the C^*-category $\mathcal{H}(\mathcal{T})$ (in Section 7).

Of course, the modular time evolution only appears on von Neumann algebras, not on arbitrary C^*-algebras, hence we should restrict ourselves to a certain class of toposes whose attached C^*-categories are in fact W^*-categories (in the sense of [**1**]). It appears that when $\mathcal{T}$ is a boolean topos (i.e. its internal logic satisfies the law of excluded middle) then the C^*-category $\mathcal{H}(\mathcal{T})$ is monotone complete (see 2·4) hence (for example by [**10**, III·3·16]) we just need to be able to construct enough normal states on $\mathcal{H}(\mathcal{T})$ in order to conclude that it is a W^*-category, and we will see in Section 3 that this is possible if and only if $\mathcal{T}$ is "integrable" (in the sense of Definition 3·4).

For these reasons, it seems natural to think of boolean toposes as the topos theoretic analogue of monotone complete C^*-algebras and of integrable boolean toposes as the analogue of von Neumann algebras. This is encouraged by the fact that the "commutative cases" agree: the category of boolean locales is well known[2] to be anti-equivalent to the category of commutative monotone complete C^*-algebras and normal morphisms between them; and, under this correspondence, commutative von Neumann algebras correspond exactly to integrable boolean locales.

This being said we acknowledge the fact that there do exist non-boolean toposes whose category of internal Hilbert spaces is a W^*-category, but it seems to us that these are examples of toposes whose geometry is not reflected by their C^*-categories of internal Hilbert spaces, and that they should be discarded of the present work.

As we are concerned with measure theory, we will only focus on boolean toposes. *All the toposes mentioned in this paper are boolean Grothendieck toposes.*

The main result of this paper (Section 6) is that every boolean integrable locally separated topos carries a completely canonical principal bundle $\chi_{\mathcal{T}}$ for the group $\mathbb{R}^{>0}_{\mathcal{T}}$ called the modular bundle. This modular bundle generates in a natural way a one parameter family of endofunctors of the category $\mathcal{H}(\mathcal{T})$ which constitute our topos theoretic time evolution. The key construction is the notion of "Invariant measure" (as defined in Section 4), a special kind of measures over the objects of $\mathcal{T}$ satisfying certain invariance properties and which appear to be exactly the global sections of $\chi_{\mathcal{T}}$.

[1] This condition is stated in proposition 7·10.

[2] Indeed, by [**5**, lemma III·3·5] the Stone spectrum of complete boolean algebra are the compact extremally disconected spaces, and by [**9**, theorem 2·3·7] those are exactly the Gelfand spectrum of commutative monotone complete C^*-algebras. Moreover by [**5**, lemma III·2·1] bounded functions with values in $\mathbb{R}$ or $\mathbb{C}$ on a boolean locale are the same as functions on its Stone space. Hence any monotone complete algebra A identifies with the algebra of bounded functions on a unique boolean locale, which is the algebra of clopen set of the Gelfand spectrum of A or equivalently the boolean algebra of projection of A. This easily implies the equivalence mentioned.

https://doi.org/10.1017/S0305004116000700 Published online by Cambridge University Press

This gives a classification in types I, II and III of boolean integrable locally separated toposes analogous (but not totally equivalent) to the classification in type of von Neumann algebras: type I corresponds to separated toposes, type II to toposes which are not separated but which have a trivial modular bundle and hence admit an invariant measure, and type III to toposes which have a non-trivial modular bundle and hence have a non-trivial time evolution.

Of course in full generality, one obtains that every boolean integrable locally separated topos decomposes in a disjoint sum of one topos of each of the three types by applying this disjunction internally in its localic reflection.

In the final section of this paper, we consider $\mathcal{T}$ a boolean integrable locally separated topos, X an object of $\mathcal{T}$ such that the slice topos $\mathcal{T}_{/X}$ is separated and $l^2(X)$ the internal Hilbert space of X-indexed square summable sequences. In this situation, we show that the modular time evolution of the von Neumann algebra A of globally bounded endomorphisms of $l^2(X)$ is indeed the time evolution on A described by $\chi_{\mathcal{T}}$, and that an invariant measure on $\mathcal{T}$ induces a trace on A. Hence we relate the purely topos theoretic approach we have developed to the well know concepts of non-commutative geometry.

The following table sums up the dictionary between topos theory (in the left-hand column) and operator algebra (in the right-hand column) that arise in this paper. This is just a vague analogy that we have observed while developing this theory, we do not claim that there is any sort of rigorous correspondence here. In particular we think that this dictionary is meant to be made more precise in the future. For example a more recent work [**3**] highly suggests that it can be made a lot more precise if we take into account, on the right-hand side, the monoidal structure on $\mathcal{H}(\mathcal{T})$ that arises from the internal tensor product of Hilbert spaces.

Boolean topos (locally separated)	Monotone complete C^*-algebra (up to morita equivalence)
Boolean integrable topos (locally separated)	W^*-algebra (up to morita equivalence)
Localic reflection	Center
Measure on an object	Semi-finite normal weight
Measure on an object of mass 1	Normal state
Invariant measure	Normal (semi-finite) trace
The modular bundle	The Δ operator of the Tomita–Takesaki construction
The family of line bundles $(F_t)_{t\in\mathbb{R}}$	The modular time evolution

2. *Preliminaries and notations*

2·1.

The letter $\mathcal{T}$ will denotes a boolean (Grothendieck) topos. Its terminal object is denoted $1_{\mathcal{T}}$, if X is an object of $\mathcal{T}$ we denote by $\mathcal{T}_{/X}$ the slice topos whose objects are couples (Y, f) where f is a morphism from Y to X. It corresponds to the geometric idea of "etale space of X". A family (X_i) of objects of a topos is called a generating family or a family of generators if any object of $\mathcal{T}$ can be covered by epimorphic images of the X_i (in which case on can construct a site of definition of $\mathcal{T}$ on the full sub-category of $\mathcal{T}$ whose objects are the X_i).

https://doi.org/10.1017/S0305004116000700 Published online by Cambridge University Press

2·2.

$\mathbb{R}_{\mathcal{T}}$ denotes the object of Dedekind real numbers of the topos $\mathcal{T}$ (see [**7**, VI·8]). When $\mathcal{T}$ is boolean, most of the reasonable definitions of the real numbers agree and give rise to an internal ordered field satisfying the property that every bounded set has a supremum. We denote by $\mathbb{C}_{\mathcal{T}}$ the object of complex numbers of $\mathcal{T}$ (isomorphic to $\mathbb{R}_{\mathcal{T}} \times \mathbb{R}_{\mathcal{T}}$ as an object of $\mathcal{T}$).

2·3.

A (internal) Hilbert space of $\mathcal{T}$ or a $\mathcal{T}$-Hilbert space, is an object E of $\mathcal{T}$ endowed with the structure of a $\mathbb{C}_{\mathcal{T}}$ module with a sesquilinear map $E \times E \to \mathbb{C}_{\mathcal{T}}$ satisfying internally all the axioms of an Hibert space. Completness is taken in the sense of Cauchy filters or equivalently Cauchy approximations (but not Cauchy sequences). A globally bounded operator between two Hilbert spaces E and F of $\mathcal{T}$ is a map $f : E \to F$ which is internally a linear map admitting an adjoint[3] and of norm smaller than N for some external real number N. The smallest possible value of N is denoted $\|f\|_\infty$, while $\|f\|$ will denotes the internal norm of f which, in a boolean topos, is a global section of $\mathbb{R}_{\mathcal{T}}$. The category of Hilbert spaces of $\mathcal{T}$ and globally bounded operator is denoted $\mathcal{H}(\mathcal{T})$ and is a C^*-category (in the sense of [**1**]).

2·4.

Because $\mathcal{T}$ is boolean, $\mathcal{H}(\mathcal{T})$ is not just a C^*-category but a monotone complete C^*-category in the following sense:

PROPOSITION. *Let $\mathcal{T}$ be a boolean topos, then for any $H \in \mathcal{H}(\mathcal{T})$ an Hilbert space of $\mathcal{T}$ the C^*-algebra of bounded self operator of H is monotone complete, i.e. every bounded directed net of positive operators has a supremum.*

Proof. Let us start by the following observation: an external operator $H \to H$ is positive if and only if it is internally positive, indeed, because positivity is characterised by the existence of a square root an externally positive element is also internally a positive operator and as positive square root are unique, an internally positive element has a unique positive square root which is hence also an external square root.

Let $(h_i)_{i \in I}$ be any bounded directed net of positive operators of H. It can also be seen as an internal direct net of positive operators indexed by $p^*(I)$ (where p is the canonical geometric morphism from $\mathcal{T}$ to the topos of sets). Because $\mathcal{T}$ is boolean, this net internally has a supremum, which, because it is uniquely defined, corresponds to a global section of the object of internal operators of H, and hence to an external operator $h : H \to H$. If f is any (external) operator greater than all of the h_i then, because of the remark, it is also greater than all of the h_i internally and hence is greater than h, which proves that h is the supremum of the h_i.

This idea that supremums of directed net of operators can be computed internally will be used constantly in this paper.

2·5.

If X is an object of $\mathcal{T}$ (a boolean topos) we denote by $|X|$ the internal cardinal of X, which is a function from the terminal object of $\mathcal{T}$ to $\mathbb{N} \cup \{\infty\}$ (or equivalently a map from the locale associated to the frame of sub-object of $1_{\mathcal{T}}$ to the discrete locale whose points are $\mathbb{N} \cup \{\infty\}$).

[3] In a boolean topos the existence of an adjoint for a bounded linear map is automatic.

https://doi.org/10.1017/S0305004116000700 Published online by Cambridge University Press

2·6.

An object of a boolean topos will be called *finite* if it is internally finite (either Kuratowski finite or cardinal finite which are equivalent in a boolean topos). A map $f : X \to Y$ between objects of a topos is said to be finite if internally one has $\forall y \in Y$, $f^{-1}(\{y\})$ is finite. It is said to be n-to-1 if internally one has $\forall y \in Y$, $f^{-1}(\{y\})$ has cardinal n, one can eventually take $n = \infty$ in this definition.

2·7.

Let B be a complete boolean algebra with 0 and 1 the bottom and top elemnts. A valuation on B is a non-decreasing function μ from B to the set $\mathbb{R}_+^\infty$ of possibly infinite non-negative real numbers, such that $\mu(U) + \mu(V) = \mu(U \bigcup V) + \mu(U \bigcap V)$, $\mu(0) = 0$ and μ preserve arbitrary directed supremum.

A valuation is said to be finite if $\mu(1) < \infty$, locally finite if 1 is a union of element of finite valuation and well supported if the only element of valuation 0 is 0.

2·8.

If X is a boolean locale, a measure on X is a valuation on the complete boolean algebra $\mathcal{O}(X)$. If $f : X \to \mathbb{C}$ is a continuous map and μ a valuation on X one can integrate f against μ. (see for example [**12**] for a general treatment of the integration over locale in constructive mathematics). This integral satisfies all the properties of the usual notion of integral: it is linear, additive and preserve directed supremum when applied to positive functions. For not necessary positive functions, we need to restrict to functions f for which the integral of $|f|$ is finite and then the integral is linear and satisfy the various form of Lebesgue's dominated convergence theorem.[4]

2·9.

Boolean locales satisfy the Radon–Nikodym theorem: if μ and ν are two locally finite well supported measures on a boolean locale X, then there exists a (unique) continuous map f from X to $\mathbb{R}^{>0}$ such that $\mu = f.\nu$. This last equality means that for all open sublocales $U \subset X$ one has $\mu(U) = \int_U f d\nu$. This result follows directly from the corresponding statement for semi-finite normal weight on commutative von Neumann algebras, and there is also a purely localic proof in [**4**, section 3·4].

3. *Integrable toposes*

3·1.

If X is an object of a topos, then $l^2(X)$ denotes the $\mathcal{T}$-Hilbert space of square sumable sequences of complex numbers indexed by X. Equivalently, $l^2(X)$ is the Hilbert space generated by elements e_x for each $x \in X$ such that $\langle e_x, e_{x'} \rangle = 1$ if $x = x'$ and 0 otherwise.

3·2.

Let $\mathcal{T}$ be a boolean topos, and X be an object of $\mathcal{T}$, and $\mathsf{Sub}(X)$ the complete boolean algebra of subobjects of X. We call *measure* on X a valuation on the complete boolean algebra $\mathsf{Sub}(X)$ as in (2·7).

[4] This last fact can be deduced from the constructivity of [**12**]: it allows to deduce directly lower semi-continuity of integrals with parameters, and using to the domination hypothesis one obtains the lower semi-continuity of $-\int$ and hence the continuity of the integral.

https://doi.org/10.1017/S0305004116000700 Published online by Cambridge University Press

If $f : X \to \mathbb{R}_{\mathcal{T}}^{\geqslant 0}$ is a morphism from X to the object of positive real numbers of $\mathcal{T}$, then f corresponds externally to a continuous function from the locale associated to the frame $\mathsf{Sub}(X)$ to the locale of real numbers. In particular, if μ is a measure on X one can define:

$$\int_X f d\mu \qquad \text{Or} \qquad \int_{x\in X} f(x) d\mu$$

as the integral of this corresponding function with respect to μ (as in (2·8)).

3·3.

This external integration of internal functions allows to relates measures on objects to states of the C^* category of the topos:

LEMMA. *Let $X \in \mathcal{T}$ be an object of $\mathcal{T}$, let $\mathcal{H}$ be an internal Hilbert space of $\mathcal{T}$, μ a measure on X and $v : X \to \mathcal{H}$ a function in $\mathcal{T}$ such that*:

$$\int_{x\in X} \|v(x)\|^2 d\mu = 1$$

then, for $h : \mathcal{H} \to \mathcal{H}$ a globally bounded operator, the formula:

$$\eta(h) := \int_{x\in X} \langle v(x), hv(x)\rangle \, d\mu$$

defines a normal state[5] *η on the monotone complete C^*-algebra $B(\mathcal{H})$ of globally bounded operators on $\mathcal{H}$.*

Proof. η is well defined and of norm smaller than one because:

$$\int_{x\in X} |\langle v(x), hv(x)\rangle| d\mu \leqslant \|h\|_\infty \int_{x\in X} \|v(x)\|^2 = \|h\|_\infty$$

the linearity and the positivity are immediate by linearity and positivity of the scalar product and the integral. The value of of $\eta(1)$ follows from the condition relating v and μ.

The normality of η follow from the fact that the supremums are computed internally and that the integral defined in [**12**] commute to directed supremums.

3·4.

Definition. A boolean topos $\mathcal{T}$ is said to be *integrable* if for all non-zero object $X \in \mathcal{T}$ there exists a non zero finite measure on X.

3·5.

THEOREM. *Let $\mathcal{T}$ be boolean topos, then the C^*-category of $\mathcal{T}$-Hilbert spaces is a W^*-category if and only if $\mathcal{T}$ is integrable.*

Proof. Assume first that $\mathcal{T}$ is integrable. As the category of $\mathcal{T}$-Hilbert spaces has bi-product, it is enough to show that the algebra of endomorphism of any internal Hilbert space is a W^*-algebra (because of [**1**, proposition 2·6]), and as it is already know that these are monotone

[5] i.e. a positive linear form such that $\eta(1) = 1$ and which send supremums of bounded increasing net of positive operators to supremums in $\mathbb{R}$.

https://doi.org/10.1017/S0305004116000700 Published online by Cambridge University Press

complete, we only have to prove that they admit enough normal state (see [**10**, theorem III·3·16]).

Let $\mathcal{H}$ be a Hilbert space over $\mathcal{T}$ and $h : \mathcal{H} \to \mathcal{H}$ be a non zero positive self adjoint operator, let $X \subset \mathcal{H}$ the set of[6] x such that $\langle x, h(x)\rangle > 0$ and $\|x\| \leqslant 1$. If $X = 0$ then $h = 0$ hence, X is a non zero subset of $\mathcal{H}$ which hence admit a positive finite measure. After rescaling, Lemma 3·3 gives a state η such that $\eta(h) > 0$ which concludes the proof of the first implication.

Conversely, assume that the category of Hilbert spaces is a W^*-category, let $X \in \mathcal{T}$ a non-zero object of $\mathcal{T}$, then there is a state η on the algebra of endomorphisms of $l^2(X)$. For any subobject $S \subset X$ one can define P_S to be the endomorphism of $l^2(X)$ defined by $P_S(e_x) = e_x$ if $x \in S$ and 0 otherwise. Then $\mu(S) = \eta(P_S)$ defines a non zero finite measure on X.

3·6.

COROLLARY. *Let $\mathcal{T}$ be an integrable boolean topos, then there exists a von Neumann algebra A (uniquely determined up to Morita equivalence of von Neumann algebras) such that the category of $\mathcal{T}$-Hilbert spaces is equivalent to the category of self-dual Hermitian A-modules (in the sense of* [**1**, example 1·4]*).*

Proof. As the category of $\mathcal{T}$-Hilbert spaces has arbitrary orthogonal sums and splitting of projections (they are computed internally) it is enough, by [**1**, proposition 7·6], to check that it has a generator in the sense of [**1**, proposition 7·3].

We fix G a set of generators of $\mathcal{T}$. Let E be the family of all isomorphisms class of triples (H, g, f) where H is a $\mathcal{T}$-Hilbert space, g is an element of G and f is a map $g \to H$ such that internally the image of this map spans a dense subspace of H. To an element $(\mathcal{H}, g, f)$ of E one can associate a continuous function on $g \times g$ defined by $(x, y) \to \langle f(x), f(y)\rangle$, and if (H, g, f) and (H', g, f') define the same function on $g \times g$ then they are isomorphic. In particular the isomorphism class of elements of E form a set.

Now, for any $f : A \to B$ an operator between two $\mathcal{T}$-Hilbert spaces, there exists $g \in G$ and a map $\lambda : g \to A$ such that $f \circ \lambda \neq 0$, the adherence of the span of λ gives an element H of E and a map i from H to A such that $f \circ i \neq 0$. This proves that elements of E form a family of generators of $\mathcal{H}(\mathcal{T})$, i.e. their orthogonal sum is a generator of this W^*-category. This concludes the proof.

Even if this von Neumann algebra is naturally attached to the topos $\mathcal{T}$ and uniquely determined by it up to Morita equivalence, it is in most case "too big" for example, if $\mathcal{T}$ is the topos of G-sets for some discrete group G, then $\mathcal{H}(\mathcal{T})$ is the category of unitary representations of G and the associated von Neumann algebra is the enveloping von Neumann algebra of the maximal C^*-algebra of the group, which is an enormous algebra.

It seems that a more reasonable algebra to consider in practice is the algebra of operators on a space $l^2(X)$ for X a separating bound of $\mathcal{T}$. In the case of G-sets, this gives the Von Neumann algebra of the group, eventually up to Morita equivalence depending on the choice of the object X. The results of the last section suggest this algebra can be controlled by the geometry of $\mathcal{T}$, whereas in general an algebra of operators on an arbitrary Hilbert space over a topos $\mathcal{T}$ can have nothing in common with the geometry of $\mathcal{T}$. For example, any

[6] We are using the internal logic here, what we mean is "let X be the sub-object of $\mathcal{H}$ defined by the following formula.

https://doi.org/10.1017/S0305004116000700 Published online by Cambridge University Press

von Neumann algebra arises as the algebra of globally bounded endomorphisms of some representation of a discrete group.

4. *Invariant measures*

4·1.

Definition. Let $\mathcal{T}$ be a boolean topos. An invariant measure[7] on $\mathcal{T}$ is a function which to every object X of $\mathcal{T}$ associates a (possibly infinite) real number $\mu(X) \in [0, \infty]$ such that:

(IM1) for each $X \in \mathcal{T}$, the restriction of μ to sub-objects of X defines a locally finite valuation on $\mathsf{Sub}(X)$;

(IM2) there exists a generating family of $\mathcal{T}$ of objects X such that the valuation induced by μ on $\mathsf{Sub}(X)$ is well supported;

(IM3) if $f : Y \to X$ is a n-to-1 map in $\mathcal{T}$ and if $\mu(Y) < \infty$ then:

$$\mu(X) = \frac{\mu(Y)}{n}.$$

Note that the third condition implies that $\mu(X) = \mu(Y)$ when X and Y are isomorphic. In the case $n = \infty$, the axiom $(IM3)$ has to be interpreted as $\mu(X) = 0$ if $\mu(Y) < \infty$.

4·2.

This definition, which might seems ad-hoc, essentially comes from the study of explicit examples of topos where the associated Von Neumann algebra have an explicite time evolution and found his justification in the results of Section 6. Some detail on its origin can be found in [**2**, chapter I and II].

4·3.

The following proposition largely extend the strength of axiom (IM3).

PROPOSITION. *Let $\mathcal{T}$ be a boolean topos endowed with an invariant measure μ. Let $f : Y \to X$ be a finite map and h be a complex valued function on Y (i.e. a function from Y to the object of complex numbers of $\mathcal{T}$).*

Assume that h is positive or that:

$$\int_{y \in Y} |h(y)| d\mu < \infty.$$

Then one has:

$$\int_{y \in Y} h(y) d\mu = \int_{x \in X} \left(\sum_{y \in f^{-1}(x)} h(y) \right) d\mu.$$

Proof. We will proceed in a series of steps:

(i) assume first that h is the constant equal to one function, that $\mu(Y) < \infty$ and that f is a n-to-1 map for some (external) integer n. Then the results is exactly (IM3) applied to $f : Y \to X$;

[7] It should probably be more correct to call this a "well supported locally finite invariant measure".

https://doi.org/10.1017/S0305004116000700 Published online by Cambridge University Press

(ii) still assume that $h = 1$ and $\mu(Y) < \infty$, but f is an arbitrary finite map. Let:

$$X_n = \{x \in X | x \text{ has exactly } n \text{ antecedent by } f\},$$

$$Y_n = f^{-1}(X_n)$$

and let f_n be the restriction of f as a map from Y_n to X_n. One has $X = \coprod X_n$ and $Y = \coprod Y_n$. For all $n > 0$ one can apply the first point to the function f_n to get that $\mu(Y_n) = n\mu(X_n)$. This formula also holds for $n = 0$ as Y_n is empty by definition. Using the decomposition of X and Y into the X_n and Y_n one obtain that the left-hand side of the formula is $\sum \mu(Y_n)$ and the right-hand side is $\sum n\mu(X_n)$ which proves they are equals;

(iii) assume now that Y is arbitrary and that h is the characteristic function of $S \subset Y$ with $\mu(S) < \infty$. Then the function $\sum_{f(y)=x} h(y)$ is supported on $f(S)$ hence the result follows from the previous point applied to $f : S \to f(S)$;

(iv) by linearity, this automatically imply the result for any linear combination of such characteristic functions. And as every positive function is a directed supremum of such linear combinations and that both the integral and the sum commute to directed supremum of positive functions it imply that the result holds for any positive function h;

(v) if h is an arbitrary complex function such that $\int |h| < \infty$ then we can decompose it into $a - b + ic - id$, where a, b, c, d are four positive functions, and the result follows by linearity of the integral and of the sum.

4·4.

COROLLARY. *Let $\mathcal{T}$ be a topos endowed with an invariant measure and let $f : Y \twoheadrightarrow X$ be an arbitrary epimorphism, with $\mu(Y) < \infty$ then*:

$$\mu(X) = \int_{y \in Y} \frac{1}{|f^{-1}(f(y))|} d\mu.$$

Proof. If f is a finite map then the result follow directly from 4·3 applied to $h(y) := 1/|f^{-1}(f(y))|$. In the general case, we decompose X and Y into two components: one where f is finite and the other where f is ∞-to-1. on the first component the previous result apply, on the second component $\mu(X) = 0$ because of $(IM3)$ and the integral is also 0 because $h = 0$.

4·5.

PROPOSITION. *Let $\mathcal{C}$ be a class of objects of $\mathcal{T}$ such that if $C \in \mathcal{C}$ and $f : X \to C$ is any map in $\mathcal{T}$ then $X \in \mathcal{C}$. Also assume that every object of $\mathcal{T}$ can be covered by objects in $\mathcal{C}$.*

Let μ be a function which associate to every object in $\mathcal{C}$ a non-negative, possibly infinite, real number, and which satisfy all the axiom of the definition of an invariant measure when restricted to objects of $\mathcal{C}$ and maps between them. Then μ extend to a unique invariant measure on $\mathcal{T}$.

We insist on the fact that any subobject of an object in $\mathcal{C}$ is also in $\mathcal{C}$, hence axiom (IM1) do mean something when we restrict to $\mathcal{C}$.

https://doi.org/10.1017/S0305004116000700 Published online by Cambridge University Press

Proof. First observe that the results of Proposition 4·3 and Corollary 4·4 still hold for our function μ when $X, Y \in \mathcal{C}$. Let $\mathcal{C}_f$ the subclass of $C \in \mathcal{C}$ such that $\mu(C) < \infty$.

Let X be an object of $\mathcal{T}$ and assume that there is two epimorphisms $f : C \twoheadrightarrow X$ and $f' : C' \twoheadrightarrow X$ with $C, C' \in \mathcal{C}_f$. Then one has:

$$\int_{c\in C} \frac{1}{|f^{-1}f(c)|} d\mu = \int_{c\in C'} \frac{1}{|f'^{-1}f'(c)|} d\mu.$$

Indeed, to see that, let $P = C \times_X C'$ and g and g' the two maps from P to C and C', let h be the function on P defined by $h(c, c') = (1/|f'^{-1}f'(c')|)(1/|f^{-1}f(c)|)$. Now, (internally) for any $c \in C$ one has:

$$\sum_{\substack{x\in P \\ g(x)=c}} h(x) = \sum_{\substack{c'\in C' \\ f'(c')=f(c)}} \frac{1}{|f'^{-1}f'(c')|} \frac{1}{|f^{-1}f(c)|} = \frac{1}{|f^{-1}f(c)|}$$

and the same computation exchanging C and C' together with Proposition 4·3, yields that:

$$\int_{c\in C} \frac{1}{|f^{-1}f(c)|} d\mu = \int_P h = \int_{c'\in C'} \frac{1}{|f'^{-1}f'(c')|} d\mu.$$

In particular, for any object X of $\mathcal{T}$ which is an epimorphic image of $C \in \mathcal{C}_f$ there is a uniquely defined extension of μ to X as:

$$\mu(X) = \int_{c\in C} \frac{1}{|f^{-1}f(c)|} d\mu.$$

If $S \subset X$ then S is the surjective image of $f^{-1}(S)$ hence one also have a definition of $\mu(S)$ and this defines a valuation on X. If X and X' are two subobjects of Y begin epimorphic images of objects in $\mathcal{C}_f$, then the measure defined on them agree on their intersection by the uniqueness property, hence this measure extend as a measure on their union, and as every object is covered by surjective image of object of $\mathcal{C}_f$ by hypothesis this defines a unique measure on every object of $\mathcal{T}$. We already proved that the extension satisfies (IM1), and it clearly satisfy (IM2) because $\mathcal{C}$ already contains a generating family of objects on which the measure is well supported.

In order to conclude, we just need to prove that this extension satisfy (IM3). Let $f : Y \to X$ be a n-to-1 map. Let $C \in \mathcal{C}_f$ and $v : C \to X$. Let $C' := C \times_X Y$. The map $\pi_1 : C' \to C$ is again n-to-1 hence:

$$\mu(v(C)) = \int_{c\in C} \frac{1}{|v^{-1}v(c)|} d\mu = \frac{1}{n} \int_{c'\in C'} \frac{1}{|v^{-1}v(\pi_1(c'))|} d\mu.$$

On the other hand:

$$\mu(\pi_2(C')) = \int_{c'\in C'} \frac{1}{|\pi_2^{-1}(\pi_2(c')|} d\mu.$$

But if $c' = (c, y)$ then $\pi_2^{-1}(\pi_2(c'))$ is the set of (a, y) with $a \in v^{-1}v(c)$ hence, $|\pi_2^{-1}(\pi_2(c'))| = |v^{-1}v(\pi_1(c'))|$ so that:

$$\mu(v(C)) = \frac{1}{n}\mu(\pi_2(C')) = \frac{1}{n}\mu(f^{-1}(v(c))).$$

As the measure on X is defined out of the measure on such $v(C)$ this shows that $\mu(X) = \mu(Y)/n$ and concludes the proof.

https://doi.org/10.1017/S0305004116000700 Published online by Cambridge University Press

4·6.

Let $\mathcal{T}$ be a topos, if X is an object of $\mathcal{T}$ we denote by $M(X)$ the set[8] of invariant measures on the slice topos $\mathcal{T}_{/X}$. If $f : Y \to X$ is a map in $\mathcal{T}$, and $\mu \in M(X)$ then we define $f^*\mu \in M(Y)$ by the formula $f^*(\mu)(V) = \mu(f_!V)$. Where $f_!$ is the "composition with f" functor from $\mathcal{T}_{/Y}$ to $\mathcal{T}_{/X}$. This defines a contravariant functor from $\mathcal{T}$ to Set.

THEOREM. *Let $\mathcal{T}$ be a boolean topos, then the contravariant functor M of invariant measures is representable by an object denoted χ or $\chi_{\mathcal{T}}$ that we will call the modular bundle.*

Proof. As $\mathcal{T}$ is a Grothendieck topos we just need to prove that M is a sheaf for the canonical topology of $\mathcal{T}$. More precisely: let $f : X \twoheadrightarrow Y$ be an epimorphism, $P = X \times_Y X$ and $\pi_1, \pi_2 : P \rightrightarrows X$ the two projections, we need to prove that for any $\mu \in M(X)$ such that $\pi_1^*\mu = \pi_2^*\mu$ there exists a unique $\nu \in M(Y)$ such that $\mu = f^*\nu$.

By working with $\mathcal{T}_{/Y}$ instead of $\mathcal{T}$ one can freely assume that Y is the terminal object of $\mathcal{T}$ and $P = X \times X$.

Let $\mathcal{C}$ be the class of object V of $\mathcal{T}$ such that there exists a map from V to X. Let $V \in \mathcal{C}$ and let $g, h : V \rightrightarrows X$ two maps from V to X, then (V, g, h) is an object of $\mathcal{T}_{/P}$ and one has by hypothesis $\pi_1^*\mu(V, g, h) = \pi_2^*\mu(V, g, h)$. On the other hand, $\pi_1^*\mu(V, g, h) = \mu(V, g)$ and $\pi_2^*\mu(V, g, h) = \mu(V, h)$. Hence $\mu(V, g)$ does not depend on $g : V \to X$ and we define, for $V \in \mathcal{C}$, $\nu(V) = \mu(V, g)$ for any $g : V \to X$. It is easy to check that this satisfies the axiom of an invariant measure restricted to $\mathcal{C}$: for each axiom, one can chose a coherent family of functions to X and apply the corresponding axiom for the measure μ in $\mathcal{T}_{/X}$. Hence by Proposition 4·5 this extends to a unique measure ν on $\mathcal{T}$ which by definition satisfies $f^*\nu = \mu$. Conversely, any invariant measure ν such that $f^*\nu = \mu$ has to satisfy that for any object V with an arrow $g : V \to X$, one has $\mu(V, g) = f^*\nu(V, g) = \nu(V)$, hence is equal to the ν we just constructed.

4·7.

Of course, this object χ comes with several structures (due to structures on M). The most important is the multiplicative action of $\mathbb{R}_{\mathcal{T}}^{\geq 0}$ on χ which corresponds to the fact that if μ is an invariant measure on $\mathcal{T}_{/X}$ and f a function from X to $\mathbb{R}_{\mathcal{T}}^{\geq 0}$ then one can define, exactly as in the case of classical measure theory, $f.\mu$ by $f.\mu(Y, p) = \int_Y f(p(y))d\mu$. One can also define an addition on χ, but there is no "zero" element because it is ruled out by (IM2).

5. *Separated and locally separated boolean toposes*

In this section we introduce the notion of separated and locally separated toposes due to I.Moerdjik and C.J.Vermeulen in [**8**], and we prove a new characterisation of separation for boolean toposes (Theorem 5·2).

5·1.

The following definitions are due to Moerdijk and Vermeulen (see [**8**]).

[8] It is indeed a set because one can see from Corollary 4·4 that an invariant measure is fully determined by its values on the sub-objects some generating family of the topos.

https://doi.org/10.1017/S0305004116000700 Published online by Cambridge University Press

Definition.

(i) A topos is said to be compact if its localic reflection is compact.
(ii) A geometric morphism $f : \mathcal{E} \to \mathcal{T}$ is said to be proper if, internally in $\mathcal{T}$, the $\mathcal{T}$-topos $\mathcal{E}$ is compact.
(iii) A geometric morphism $f : \mathcal{E} \to \mathcal{T}$ is said to be separated if its diagonal map $\Delta : \mathcal{E} \to \mathcal{E} \times_{\mathcal{T}} \mathcal{E}$ is a proper map.
(iv) A topos is said to be separated or Hausdorff if its canonical geometric morphism to the base topos is separated.

Being separated is a strong property, for example the topos of G-sets for a discrete group G is separated if and only if G is finite.

5·2.

The main result of this section is the following characterisation of separation in the case of boolean toposes:

THEOREM. *Let $\mathcal{T}$ be a boolean topos. Then $\mathcal{T}$ is separated if and only if $\mathcal{T}$ has a generating family of internally finite objects.*

The proof of this theorem will be dived in various lemmas, one implication is proved in Lemma 5·3 and the other by Corollary 5·5.

5·3.

One implication is given by the following lemma:

LEMMA. *Let $\mathcal{T}$ be a topos in which the cardinal finite objects are generating, then $\mathcal{T}$ is separated.*

By cardinal finite, we mean objects X which satisfy internally: $\exists n \in \mathbb{N}, X \simeq \{1, \ldots, n\}$.

Proof. This is proved in [**8**, during the proof of proposition IV·3·4], as the main argument for the proof of the implication (iii) $\Rightarrow$ (i). In the terminology of this references, cardinal finite objects are called locally finite objects and the class of locally finite objects of $\mathcal{T}$ is denoted $LF(\mathcal{T})$.

5·4.

PROPOSITION. *Let $\mathcal{T}$ be a hyperconnected[9] separated topos. Then $\mathcal{T}$ is atomic and all its atoms are (internally) finite.*

Proof. Let $\mathcal{T}$ be a hyperconnected topos. Let $\mathcal{B}$ be a non zero boolean locale endowed with a geometric morphism to $\mathcal{T}$. In the logic of $\mathcal{B}$, the pullback of the topos $\mathcal{T}$ is hyperconnected separated and has a point. Applied internally in $\mathcal{B}$, [**8**, theorem II·3·1] shows that the pullback of $\mathcal{T}$ is equivalent to the topos of G-sets for G a compact localic group in $\mathcal{B}$. In particular the pullback of $\mathcal{T}$ is atomic in the logic of $\mathcal{B}$ and as the map from $\mathcal{B}$ to the base topos is an open

[9] This means that its localic reflection is the point, i.e. that $1_{\mathcal{T}}$ has no non-trivial subobjects, see [**6**, A4·6].

https://doi.org/10.1017/S0305004116000700 Published online by Cambridge University Press

surjection (because the base topos is boolean) it implies that $\mathcal{T}$ is atomic in the base topos (by [**6**, C5·1·7]).

Let now a be an atom of a separated topos $\mathcal{T}$, then $\mathcal{T}_{/a} \to *$ is a proper map (because as a is an atom it is hyperconnected), and $\mathcal{T} \to *$ is separated by hypothesis. Hence, by [**8**, proposition II·2·1(iv)] this proves that the map $\mathcal{T}_{/a} \to \mathcal{T}$ is proper, i.e. that a is internally finite.

5·5.

We are now ready to prove the other implication of Theorem 5·2.

COROLLARY. *Let $\mathcal{T}$ be a boolean separated topos, then $\mathcal{T}$ admit a generating family of internally finite objects.*

Proof. Let $\mathcal{L}$ be the localic reflection of $\mathcal{T}$. The canonical geometric morphism from $\mathcal{T}$ to $\mathcal{L}$ is hyperconnected by definition and separated by [**8**, II·2·3 or II·2·5], hence we can apply Proposition 5·4 internally in $\mathcal{L}$ (which is boolean) and this proves that the map from $\mathcal{T}$ to $\mathcal{L}$ is atomic and internally with finite atoms.

We need to externalise this result: Let X be any object of $\mathcal{T}$. Internally in $\mathcal{L}$, the object X still corresponds to an object of $\mathcal{T}$ (now seen as a topos internally in $\mathcal{L}$). Because of the first observation, it is true internally in $\mathcal{L}$ that X is a union of finite subobjects (even finite atoms). Externally this mean that X can be covered by subobjects S defined over an open sublocale $U \subset \mathcal{L}$ with the map $S \to U$ finite (such a S corresponds to a section over U of the object of finite subobject of X). But as U is complemented, S is finite in $\mathcal{T}$: internally in $\mathcal{T}$ if U then S is finite but if $\neg U$ then S is empty hence also finite. This proves that any object of X can be covered by finite objects and this concludes the proof.

5·6.

We conclude this section with the notion of locally separated topos.

Definition. An object X of $\mathcal{T}$ is said to be separating if the slice topos $\mathcal{T}_{/X}$ is separated. A topos is said to be locally separated if it has a generating family of separating objects.

A slice of a separated topos (by a decidable object) is again separated, hence in a boolean topos as soon as one has an arrow $X \to Y$ with Y separating, X is again separating. As moreover a coproduct of separating object is separating, a boolean topos (or more generally a locally decidable topos) is locally separated if an only if the terminal object can be covered by a separating object.

5·7.

In addition, the class of separating objects also enjoy another stability property:

LEMMA. *Let $f : X \twoheadrightarrow Y$ be a surjection in $\mathcal{T}$. Assume that f is a finite map and that X is separating, then Y is separating.*

Proof. The fact that f is a finite surjection means that the induced map $\mathcal{T}_{/X} \to \mathcal{T}_{/Y}$ is a proper surjection (indeed, internally in $\mathcal{T}_{/Y}$ this means that the object $X \to Y$ is inhabited

https://doi.org/10.1017/S0305004116000700 Published online by Cambridge University Press

and finite). The result then follows immediately from [**8**, proposition II·2·1.(iii)], asserting that if $f \circ g$ is separated with g a proper surjection then f is separated.

6. *Main result and examples*

6·1.

LEMMA. *Let $\mathcal{T}$ be a boolean topos endowed with an invariant measure which is well supported on the terminal object, then $\mathcal{T}$ is separated.*

Proof. By (IM1), $\mathcal{T}$ is generated by objects with $\mu(X) < \infty$. For such an object X, let P be the subobject of $1_{\mathcal{T}}$ corresponding to the proposition "X is infinite". The map from $X \times P$ to P is ∞-to-1, hence by (IM3) one has $\mu(P) = 0$. As μ is supposed to be well supported on $1_{\mathcal{T}}$ this imply that $P = 0$ hence X is internally finite, and hence $\mathcal{T}$ is generated by finite objects and hence separated by Theorem 5·2.

Note that a local application of this result automatically shows that if X is an object of a topos $\mathcal{T}$ endowed with an invariant measure which is well supported on X then X is a separating object. In particular $\mathcal{T}$ is automatically locally separated. We will improve this with Theorem 6·3.

6·2.

PROPOSITION. *Let $\mathcal{T}$ be a boolean separated topos, and μ a well supported locally finite valuation on $\mathsf{Sub}(1_{\mathcal{T}})$, then there is an invariant measure $\tilde{\mu}$ on $\mathcal{T}$ defined by*:

$$\tilde{\mu}(X) = \int_{1_{\mathcal{T}}} |X| d\mu.$$

Moreover μ is the restriction of $\tilde{\mu}$ to subobjects of $1_{\mathcal{T}}$ and every invariant measure on $\mathcal{T}$ is of the form $\tilde{\mu}$ for some locally finite well supported valuation on $\mathsf{Sub}(1_{\mathcal{T}})$.

Proof. We will first show that $\tilde{\mu}$ satisfies the three points (IM1)–(IM3) of Definition 4·1.

(IM1) One easily sees that $\tilde{\mu}$ is a valuation on $\mathsf{Sub}(X)$ for any $X \in |\mathcal{T}|$ essentially because the cardinal is internally a valuation on X and that the integral of an internal valuation is again a valuation because the notion of integral we are using is linear and preserve directed supremums of positive functions.
It remains to prove that this valuation is locally finite. If X is a finite object of $\mathcal{T}$, then $|X|$ is everywhere finite hence we can cover $1_{\mathcal{T}}$ by subobjects U such that $\int_U |X| < \infty$, and for such U, the object $X_U = X \times U$ satisfy $\tilde{\mu}(X_U) = \int_U |X| < \infty$ and as the X_U form a covering of X and that every object of $\mathcal{T}$ can be covered by finite objects this concludes the proof.

(IM2) Let X such that $\tilde{\mu}(X) = 0$, then as μ is well supported this imply that $|X| = 0$ hence X is the zero object. This proves that $\tilde{\mu}$ is well supported on every object.

(IM3) if $f : X \to Y$ is a n-to-1 map then internally $|X| = n|Y|$. Hence, if $\int |X| < \infty$, as μ is well supported, one has $|X| < \infty$ everywhere, and hence one indeed get $\int |Y| = \int |X|/n$.

https://doi.org/10.1017/S0305004116000700 Published online by Cambridge University Press

Now, if $U \in \mathsf{Sub}(1_{\mathcal{T}})$ then $|U|$ is just the characteristic function of U hence $\tilde{\mu}(U) = \mu(U)$.

Conversely, if μ is an invariant measure on $\mathcal{T}$ and if X is any object such that $\mu(X) < \infty$ then, as observed in the proof of Lemma 6·1, X is finite and, by Proposition 4·3 applied to $f : X \to 1_{\mathcal{T}}$ (the terminal object) and $h = 1$, one has:

$$\mu(X) = \int_X 1 d\mu = \int_{1_{\mathcal{T}}} \left(\sum_{x \in X} 1 \right) d\mu = \int_{1_{\mathcal{T}}} |X| d\mu.$$

Hence μ is of the announced form. We just have to check that the measure induced by μ on $\mathsf{Sub}(1_{\mathcal{T}})$ is well supported, but if $V \subset 1_{\mathcal{T}}$ is of zero measure any object defined over V has measure zero because of the previous formula, hence axiom $(IM2)$ imply that V is empty.

6·3.

THEOREM. *Let $\mathcal{T}$ be a boolean topos then χ is well supported (i.e. internally inhabited) if and only if $\mathcal{T}$ is integrable and locally separated. Moreover, if $\mathcal{T}$ is integrable and locally separated then χ is a principal bundle for the action of $\mathbb{R}_{\mathcal{T}}^{>0}$.*

Proof. First let $\mathcal{T}$ be a boolean topos such that χ is well supported. We will prove that $\mathcal{T}$ is integrable and locally separated.

Let $X \to \chi$ be any map, then by definition $\mathcal{T}_{/X}$ is endowed with an invariant measure. In particular $\mathcal{T}_{/X}$ admit a generating family of objects Y on which the measure is well supported, and hence by lemma 6·1 a generating family of separating objects of $\mathcal{T}_{/X}$. Such objects are also separating in $\mathcal{T}$, and as χ is assumed to be well supported they form a generating family of $\mathcal{T}$. This proves that $\mathcal{T}$ is locally separated. Moreover, as we have constructed a generating family of objects of $\mathcal{T}$ which admit well supported measures this also proves that $\mathcal{T}$ is integrable.

Conversely assume that $\mathcal{T}$ is integrable and locally separated, we will prove that χ is well supported and is a principal $\mathbb{R}_{\mathcal{T}}^{>0}$-bundle.

Let X be any separating object of $\mathcal{T}$. As $\mathcal{T}$ is integrable one can cover X by subobject $S \subset X$ such that S is endowed with a well supported measure. Such an object S is separating and endowed with a well supported measure, hence there is an invariant measure on $\mathcal{T}_{/S}$ by Proposition 6·2, and hence a map from S to χ. As such objects from a generating family of $\mathcal{T}$, they in particular form a covering of $1_{\mathcal{T}}$ and hence χ is inhabited.

Moreover, still by Proposition 6·2 any two functions from such an object S to χ correspond to two measures on S, and hence (as $\mathsf{Sub}(S)$ is boolean) the Radon-Nikodym theorem for complete boolean algebras imply that they differ only by multiplication by a function from S to $\mathbb{R}_{\mathcal{T}}^{>0}$. As objects of this form are generating this exactly proves that χ is a $\mathbb{R}_{\mathcal{T}}^{>0}$-principal bundle.

6·4.

Let us now give some examples of boolean integrable locally separated toposes and see what is this modular bundle.

(a) First if $\mathcal{T}$ is a topos of sheaves over a boolean locale $\mathcal{L}$ then $\mathcal{T}$ is always separated and χ is the sheaf of locally finite well supported measure on $\mathcal{L}$ because of Proposition 6·2. It is

https://doi.org/10.1017/S0305004116000700 Published online by Cambridge University Press

inhabited if and only if $\mathcal{L}$ is integrable and in this case it always has a section (assuming the axiom of choice, every inhabited sheaves over a boolean locale has a section). Moreover, it is a principal bundle exactly because of the ordinary Radon–Nikodym theorem.

This corresponds to the localic point of view on ordinary measure theory: if (X, μ) is a σ-finite measured space then one can consider the von Neumann algebra $L^\infty(X, \mu)$ which corresponds through the duality mentioned in the introduction to a boolean locale $\mathcal{L}$ (the complete boolean algebra of projections of $L^\infty(X, \mu)$) and μ gives rise to a locally finite well supported measure μ on $\mathcal{L}$, and hence an invariant measure on the corresponding topos by proposition 6·2. The locale $\mathcal{L}$ can also be described directly as the algebra of measurable sets modulo μ-null sets. Other locally finite valuation on $\mathcal{L}$ corresponds to the measure on X that are absolutely continuous with respect to μ (as they have to be zero on μ-null sets).

(b) If $\mathcal{T}$ is the topos of G-sets for G a discrete group, then χ is the trivial bundle:

indeed, for any G-set X, X can be recovered as the set of morphism from G to X (the G-action being obtained by the right multiplication G-action on G) But $\mathcal{T}_{/G}$ is isomorphic to the topos of sets, hence G is separating and the set of morphisms from G to χ can be identified with the set of invariant measure on the topos of sets, i.e. $\mathbb{R}^{>0}$, and one easily check that the action of G is trivial. Going through the proof that χ is a sheaf one can see that the invariant measure corresponding to a real number $\alpha \in \mathbb{R}^{>0}$ is given by:

$$\mu(X) = \sum_{x \in (X/G)} \frac{\alpha}{|\mathrm{Stab}(x)|},$$

where X/G is the set of orbit of X under the action of G, and $|\mathrm{Stab}(x)|$ is the cardinal of the stabiliser of any element in the orbit x.

Finally, $\mathcal{T}$ is separated if and only if G is finite, in which case the previous formula can be rewritten more trivially as $|X|/|G|$.

The fact that the modular bundle is trivial in this case corresponds to the fact that discrete group von Neumann algebras are (when they are factor) type II, and hence have a trivial modular automorphism group. This also highlight the difference between the separated case, where the invariant measure is just the cardinal and the "type II" case where it involves some sort of renormalized cardinal.

(c) More generally, If $\mathcal{T}$ is the topos of G-equivariant sheaves over a boolean locale $\mathcal{L}$ endowed with an action of a discrete group G, then there is an object X such that $\mathcal{T}_{/X}$ is isomorphic to the topos of sheaves over $\mathcal{L}$. In particular morphisms from X to χ are the same as measures on $\mathcal{L}$, and a short calculation involving this object X allows to deduce from this that χ is the sheaves of measure on $\mathcal{L}$ endowed with the natural action of G on it.

In particular, an invariant measure on $\mathcal{T}$ is exactly a well supported locally finite measure on $\mathcal{L}$ invariant by the action of G (this is where the name "invariant measure" comes from). Note that in this case the formula giving the invariant measure on $\mathcal{T}$ in terms of the G-invariant measure on $\mathcal{L}$ might be complicated: the value of an invariant measure on a G-equivariant sheaf X corresponds to the integration over a fundamental domain of X for the action of G of the inverse of the cardinal of the stabiliser. This can be generalized to any boolean etendu (boolean etale localic groupoids).

In this case an invariant measure might or might not exists, hence one can get a non trivial modular bundle.

https://doi.org/10.1017/S0305004116000700 Published online by Cambridge University Press

(d) Consider now the case where G is a etale complete localic group[10] and $\mathcal{T}$ is the topos of continuous G-sets. On can then see that $\mathcal{T}$ is always integrable (because it is atomic), it is separated if and only if G is compact and locally separated if and only if G is locally compact.

A short computation yields that χ is simply $\mathbb{R}^{>0}$ endowed with the action of G trough its modular character, or more naturally χ is the bundle of right Haar measure with G acting by translation on the left. In particular, $\mathcal{T}$ has an invariant measure if and only if G is unimodular (i.e. if the left invariant Haar measure are also right invariant) and in this case the invariant measure is given by the same formula as for discrete group by replacing $|\mathrm{Stab}(x)|$ by $\mu(\mathrm{Stab}(x))$ for a bi-invariant[11] Haar measure μ. For non unimodular group there is no invariant measure, but instead on gets a non trivial modular bundle, corresponding to the modular character (this is essentially where the name "modular" bundle comes from, it is also already used for the "modular" operator Δ of the Tomita–Takesaki construction).

(e) In the general case where $\mathcal{T}$ is the topos of equivariant sheaves over an etale complete groupoid with a boolean space of objects, then one can think of χ as the sheave of right invariant Haar systems endowed with the left action of the groupoid, and hence an invariant measure on $\mathcal{T}$ can be thought of as a bi-invariant Haar system. But this cannot be made into a precise statement without developing first a theory of Haar systems well adapted to the localic framework.

7. *Time evolution of Hilbert bundles and relation to the modular theory.*

7·1.

From the modular principal bundle χ one can define the time evolution of Hilbert bundle in the following way:

χ is $\mathbb{R}^{>0}_{\mathcal{T}}$ bundle, and for each $t \in \mathbb{R}$, as $x \mapsto x^{it}$ defines a continuous morphism from $\mathbb{R}^{>0}$ to the circle group, one can define for each[12] t a principal bundle χ^{it} for the circle group. A principal bundle for the circle group is the same as an internal one dimensional Hilbert space, hence for each t one has a internal one dimension Hilbert space F_t in $\mathcal{T}$. One can then easily see that because $\chi^{it} \otimes \chi^{it'} = \chi^{i(t+t')}$ one has $F_t \otimes F_{t'} = F_{t+t'}$. Hence $\sigma_t : \mathcal{H} \mapsto \mathcal{H} \otimes F_t$ defines a one parameter family of endofunctor of the W^*-category of $\mathcal{T}$-Hilbert space.

7·2.

One can give a more explicit formula for these constructions: For $t \in \mathbb{R}$, F_t is the sheaf defined as:

$$F_t = \{f : \chi \to \mathbb{C} | \forall \alpha \in \chi, \forall r \in \mathbb{R}^{>0}, f(r.\alpha) = r^{-it} f(\alpha)\}.$$

The addition, multiplication and scalar product are defined pointwise, and the internal choice of a $v \in \chi$ induce an internal isomorphism[13] between F_t and $\mathbb{C}$ preserving all these structures.

[10] See [**6**, just after the proof of C5·3·15 and example C.5·3·17]. A reader unfamiliar with this terminology can replace "etale complete localic group" by pro-discrete topological group.

[11] One can observe that as x is an element of X/G and not of X this only make sense if μ is bi-invariant.

[12] For $t = 0$ the morphism is constant equal to 1 and the bundle is trivial.

[13] This does not mean that F_t is trivial, but only that it is "locally trivial of rank one", unless of course χ has a global section.

https://doi.org/10.1017/S0305004116000700 Published online by Cambridge University Press

The isomorphism between $F_t \otimes F'_t$ and $F_{t+t'}$ is given by sending $u \otimes v$ to the pointwise product of u and v.

7·3.

The time evolution is then defined by: $\sigma_t(\mathcal{H}) = F_t \otimes \mathcal{H}$ for $\mathcal{H}$ a $\mathcal{T}$-Hilbert space. Equivalently, one can define:

$$\sigma_t\mathcal{H} = \{f : \chi \to \mathcal{H} | \forall \alpha \in \chi, \forall r \in \mathbb{R}^{>0}, f(r.\alpha) = r^{-it} f(\alpha)\}.$$

The σ_t are endofunctors of the category of $\mathcal{T}$-Hilbert spaces.

7·4.

The final part of this paper is devoted to relate this "time evolution" to the modular time evolution of von Neumann algebra, and to explain how global sections of χ give rise to traces. It is actually not true that this family of functors σ_t is the modular time evolution of the full W^*-categories of $\mathcal{T}$-Hilbert spaces. This category is too big and there are Hilbert spaces in it which have nothing to do with the geometry of $\mathcal{T}$ (this should be related to the remark made after the proof of corollary 3·6). In order to get a result in this spirit we need to focus on a smaller, more reasonable, full subcategory, essentially corresponding to the notion of "square-integrable" representations of a group. More precisely, we will focus our attention the subcategory which is generated (in the sense of the generators of [**1**, Proposition 7·3]) by $\mathcal{T}$-Hilbert spaces of the form $l^2(X)$ for X a separating object, and hence it is enough to study the Hilbert spaces of the form $l^2(X)$ for X a separating object.

7·5.

For now on, X denotes a *separating* object of a boolean integrable locally separated topos $\mathcal{T}$. We also choose μ a locally finite well supported measure on X, hence an invariant measure on $\mathcal{T}_{/X}$, which corresponds to a morphism λ from X to χ.

We denote by $l^2(X)$ the Hilbert space internally defined as the set of square summable X-indexed sequences, with (internally) its generator $(e_x) \in l^2(X)$ for each $x \in X$. We also denote by $A = \mathcal{B}(l^2(X))$ the (external) von Neumann algebra of globally bounded operators on $l^2(X)$.

7·6.

μ can be used to construct a normal semi-finite weight on A, also denoted μ defined by:

$$\forall f \in A^+, \mu(f) = \int_{x \in X} \langle x, fx \rangle \, d\mu.$$

μ is semi-finite because if V is a subset of finite measure of X and, if $P_V \in A$ denotes the orthogonal projection on $l^2(V) \subset l^2(X)$, then $P_V f P_V$ has measure smaller than the measure of V times the norm of f and (assuming f is positive) when V vary among all finite measure subsets of X, this constitute a directed net of operators whose supremum is f.

7·7.

λ can be used to construct an isomorphism $\phi_t : l^2(X) \simeq \sigma_t l^2(X)$.

Indeed, one can define (internally):

$$\phi_t(e_x) := \left(\alpha \longmapsto \left(\frac{\alpha}{\lambda(x)}\right)^{-it} e_x\right) \in \sigma_t l^2(X),$$

https://doi.org/10.1017/S0305004116000700 Published online by Cambridge University Press

where $\alpha/\lambda(x)$ denotes the unique element r of $\mathbb{R}^{>0}$ such that $\alpha = r.\lambda(x)$. Defined this way $\phi_t(e_x)$ indeed satisfies the relation $\phi_t(e_x)(r.\alpha) = r^{-it}\phi_t(e_x)$, showing that it is an element of $\sigma_t(l^2(X))$. Moreover the $\phi_t(e_x)$ are of norm one and pairwise orthogonal, hence ϕ_t indeed defines an isometric map $l^2(X) \to \sigma_t(l^2(X))$.

It satisfies in particular, $\phi_0 = Id$ and $\sigma_t(\phi_{t'}) \circ \phi_t = \phi_{t+t'}$ hence $\sigma_t \phi_{-t}$ constitutes an inverse for ϕ_t showing that it is an isomorphism.

7·8.

Moreover, as $l^2(X)$ is "fixed" by the time evolution (as attested by the isomorphism ϕ_t) one obtains an action of $\mathbb{R}$ directly on A, via:

$$\forall a \in A, \theta_t(a) = \phi_t^{-1}\sigma_t(a)\phi_t.$$

One easily checks that it is an action of $\mathbb{R}$, either directly or from the following proposition.

7·9.

This time evolution on A can be more explicitly described on the matrix elements by:

PROPOSITION. *For a an element of A and x, y internal elements of X*:

$$\langle e_y, \theta_t(a)e_x \rangle = \left(\frac{\lambda(y)}{\lambda(x)}\right)^{-it} \langle e_y, ae_x \rangle.$$

With $\lambda(y)/\lambda(x)$ denoting the unique element $r(x, y)$ of $\mathbb{R}^{>0}$ such that $r(x, y)\lambda(x) = \lambda(y)$.

Proof. One has by definition of θ and as the ϕ_t are isometric:

$$\langle e_y, \theta_t(a)e_x \rangle = \langle \phi_t(e_y), \sigma_t(a)\phi_t(e_x) \rangle.$$

But:

$$\phi_t(e_x) = \alpha \longmapsto \left(\frac{\alpha}{\lambda(x)}\right)^{-it} e_x.$$

And similarly for y, hence:

$$\begin{aligned} \langle \phi_t(e_y), \sigma_t(a)\phi_t(e_x) \rangle &= \left\langle \left(\frac{\alpha}{\lambda(y)}\right)^{-it} e_y, \left(\frac{\alpha}{\lambda(x)}\right)^{-it} ae_x \right\rangle \\ &= \left(\frac{\lambda(y)}{\lambda(x)}\right)^{-it} \langle e_y, ae_x \rangle. \end{aligned}$$

7·10.

Finally all these structures satisfy the modular, or Kubo–Martin–Schwinger (KMS) condition, which characterises the relation between a weight and its associated modular automorphism group in the Tomita-Takesaki construction, more precisely:

PROPOSITION. *For each $u \in A_+$ such that $\mu(u)$ is finite one has $\mu(\theta_t(u)) = \mu(u)$.*

https://doi.org/10.1017/S0305004116000700 Published online by Cambridge University Press

Let $u, v \in A$ such that $\mu(u^*u), \mu(uu^*), \mu(v^*v)$ and $\mu(vv^*)$ are all finite. Then there exists a continuous complex valued function $F_{u,v}$ defined on $\{z \in \mathbb{C} | Im(z) \in [-1, 0]\}$ and holomorphic on its interior such that for all real numbers t:

$$F_{u,v}(t) = \mu(\theta_t(u)v) \qquad F_{u,v}(t - i) = \mu(v\theta_t(u)).$$

This proves that θ_t is indeed the modular group of automorphisms of the algebra A, associated to the semi finite normal weight μ. See [**11**, chapter VIII]. Moreover, if $\mathcal{T}$ is endowed with an invariant measure, then χ has a global section and λ can be chosen to be constant equal to this global section. In this case, the formula given in Proposition7·9 shows that θ_t is the identity for all t and hence this last result shows that μ is a normal semi-finite trace on A constructed out of the invariant measure.

Proof. From the formula given in Proposition 7·9 one can see that θ_t left unchanged the diagonal coefficients of u, and as μ is defined as the integral of the diagonal coefficients one immediately has that $\mu(\theta_t(u)) = \mu(u)$.

Let, for any $a \in A$, the function a_x^y of matrix coefficients be defined internally by $a_x^y = \langle e_y, ae_x \rangle$ (it is a function on $X \times X$). A general formal computation gives that for $a, b \in A$:

$$\begin{aligned}
\mu(ab) &= \int_{x \in X} \langle e_x, abe_x \rangle \, d\mu \\
&= \int_{x \in X} \left\langle e_x, \sum_{y \in X} ab_x^y e_y \right\rangle d\mu \\
&= \int_{x \in X} \sum_{y \in X} b_x^y a_y^x d\mu \\
&= \int_{(x,y) \in X^2} b_x^y a_y^x d\pi_1^* \mu.
\end{aligned}$$

The last equality, corresponds to Proposition 4·3, and holds only if $b_x^y a_y^x$ is a positive function, or if the integral is finite when we replace $b_x^y a_y^x$ by $|b_x^y a_y^x|$.

In particular, it holds when a, b are $(u^*, u), (u, u^*), (v^*, v)$ or (v, v^*), and this together with the finiteness hypothesis on u and v shows that all the four integrals of $|a_x^y|^2$ and $|b_x^y|^2$ with respect to both $d\pi_1^*\mu$ and $d\pi_2^*\mu$ on $X \times X$ are finite. Also note that under the correspondence between invariant measures on $X \times X$ and functions from $X \times X$ to χ, the measure $\pi_1^*\mu$ corresponds to the function $(x, y) \mapsto \lambda(x)$ and $\pi_2^*\mu$ to the function $(x, y) \mapsto \lambda(y)$. Hence one has:

$$\pi_2^*\mu = \left(\frac{\lambda(y)}{\lambda(x)}\right) \pi_1^*\mu. \tag{7·1}$$

For any complex number z such that $Im(z) \in [-1, 0]$ one has:

$$\begin{aligned}
\left| \left(\frac{\lambda(y)}{\lambda(x)}\right)^{iz} u_y^x v_x^y \right| &= \left(\frac{\lambda(y)}{\lambda(x)}\right)^{-Im(z)} |u_y^x||v_x^y| \\
&\leqslant |u_y^x|^2 + |v_x^y|^2 + \left(\frac{\lambda(y)}{\lambda(x)}\right) |u_y^x|^2 + \left(\frac{\lambda(y)}{\lambda(x)}\right) |v_x^y|^2. \quad (7·2)
\end{aligned}$$

https://doi.org/10.1017/S0305004116000700 Published online by Cambridge University Press

and the four terms on the right have a finite integral on $X \times X$ with respect to $\pi_1^*\mu$ (because of (7·1) for the last two), hence one can define the following function $F_{u,v}$ which is finite and continuous for $Im(z) \in [-1, 0]$ and the previous formal computation can be applied to it.

$$F_{u,v}(z) = \int_{(x,y)\in X\times X} \left(\frac{\lambda(y)}{\lambda(x)}\right)^{iz} u_y^x v_x^y d\pi_1^*\mu.$$

Putting together the formal computation done for $\mu(ab)$ and the expression given in (7·8) for the matrix coefficients of $\theta_t(a)$, one has for t real $F_{u,v}(t) = \mu(\theta_t(u)v)$, and using equation (7·1) one gets that:

$$\begin{aligned} F_{u,v}(t-i) &= \int_{(x,y)\in X\times X} \left(\tfrac{\lambda(y)}{\lambda(x)}\right)^{it} u_y^x v_x^y \tfrac{\lambda(y)}{\lambda(x)} d\pi_1^*\mu \\ &= \int_{(x,y)\in X\times X} \left(\tfrac{\lambda(y)}{\lambda(x)}\right)^{it} u_y^x v_x^y d\pi_2^*\mu \\ &= \int_{(y,x)\in X\times X} \left(\tfrac{\lambda(y)}{\lambda(x)}\right)^{it} u_y^x v_x^y d\pi_1^*\mu \\ &= \mu(v\theta_t(u)). \end{aligned}$$

It remains to be proven that $F_{u,v}$ is holomorphic. Let V_n be the subobject of $X \times X$ on which the function $\lambda(y)/\lambda(x)$ is between $(1/n)$ and n. one has $\bigcup V_n = X \times X$ and consider:

$$F_{u,v}^n = \int_{(x,y)\in V_n} \left(\frac{\lambda(y)}{\lambda(x)}\right)^{iz} u_y^x v_x^y d\pi_1^*\mu.$$

The functions $F_{u,v}^n$ are holomorphic, and the inequality (7·2) shows that $F_{u,v}^n$ converges to $F_{u,v}$ uniformly in z on all its domain of definition, showing that $F_{u,v}$ is holomorphic on the interior of its domain.

REFERENCES

[**1**] P. GHEZ, R. LIMA and J. E ROBERTS. W^*-categories. *Pacific J. Math.* **120**(1) (1985), 79–109.
[**2**] S. HENRY. Des topos à la géométrie non commutative par l'étude des espaces de Hilbert internes. PhD thesis. Université Paris 7 (2014).
[**3**] S. HENRY. Toward a non-commutative Gelfand duality: Boolean locally separated toposes and monoidal monotone complete C^*-categories. arXiv preprint arXiv:1501.07045 (2015).
[**4**] M. JACKSON. A sheaf theoretic approach to measure theory. PhD thesis. University of Pittsburgh (2006).
[**5**] P. T. JOHNSTONE. *Stone spaces* (Cambridge University Press, 1986).
[**6**] P. T. JOHNSTONE. *Sketches of an Elephant: a Topos Theory Compendium* (Clarendon Press, 2002).
[**7**] S. MACLANE and I. MOERDIJK. *Sheaves in Geometry and Logic: a First Introduction to Topos Theory* (Springer, 1992).
[**8**] I. MOERDIJK and J. J. C. VERMEULEN. *Proper Maps of Toposes*, vol. 705 (AMS Bookstore, 2000).
[**9**] K. SAITÔ and J. D. M. WRIGHT. Monotone complete C^*-algebras and generic dynamics. *Proc. London Math. Soc.* (2013), page pds 084.
[**10**] M. TAKESAKI. *Theory of Operator Algebras I*, vol. 1 (Springer, 2003).
[**11**] M. TAKESAKI. *Theory of Operator Algebras II*, vol. 2 (Springer, 2003).
[**12**] S. VICKERS. A localic theory of lower and upper integrals. *Mathematical Logic Quarterly*. **54**(1) (2008), 109–123.

https://doi.org/10.1017/S0305004116000700 Published online by Cambridge University Press