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On higher Du Bois singularities and K-regularity

Wanchun Shen

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Abstract

We study the relationship between higher Du Bois singularities and K-regularity, a notion that measures the $\mathbb{A}^1$-invariance of the algebraic K-groups. Building on this relationship, we establish a strengthened form of Vorst's conjecture for local complete intersections in characteristic zero. Our work also provides tools to construct new examples that illustrate various phenomena in the study of K-regularity. The main inputs for our results are vanishing theorems for the Du Bois complexes.

Contents

1 Introduction 1818
2 Preliminaries 1820
3 Vanishing theorems for the Du Bois complexes 1827
4 Higher Du Bois singularities and K-regularity 1828
5 Application to regularity criteria 1834
6 Bass's question and the Du Bois invariants 1837
7 K-regularity for projective varieties 1840
8 Further questions 1846
Acknowledgements 1847
References 1847

1. Introduction

Throughout this paper, we assume F is a field of characteristic zero. By a variety we mean a reduced, separated scheme of finite type over F.

Understanding the $\mathbb{A}^1$-invariance of K-groups is a central problem in algebraic K-theory. A natural way to quantify the failure of $\mathbb{A}^1$-invariance is by studying the cokernels of the maps

$$\varphi_{m,r} : K_m(X) \to K_m(X \times \mathbb{A}^r)$$

for all integers m and $r \geq 0$. A scheme X is said to be *K_m-regular* if the maps $\varphi_{m,r}$ are isomorphisms for all $r \geq 0$. It is a theorem of Quillen [Qui06, § 6, Corollary to Theorem 8] that regular schemes are K_m-regular for all m. In general, the extent to which $K_m(X)$ fails to be $\mathbb{A}^1$-invariant provides a measure of the singularities of X, and this is captured in the notion of K-regularity.

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This paper is devoted to applications of Hodge-theoretic methods to the study of K-regularity. As an illustration of the kind of results obtained by these methods, we highlight a few consequences, the first one being a numerical characterization of K_m-regularity in terms of the *minimal exponent* $\tilde{\alpha}_X$ introduced by Saito [Sai94]. See, for example, [Pop21, § 3.7] for a nice introduction to this numerical invariant that refines the log canonical threshold of X.

Theorem A. *Let X be a complex hypersurface of dimension d with isolated singularities. Then*

$$X \text{ is } K_m\text{-regular} \iff \tilde{\alpha}_X \geq \left\lceil \frac{m+d}{2} \right\rceil.$$

The following example illustrates the use of Theorem A to detect K-regularity.

Example. Consider the hypersurface $X = (x^2 + y^3 + z^7 = 0) \subset \mathbb{A}^3_{\mathbb{C}}$. By [Sai09, 4.1.5], the minimal exponent of X is given by $\tilde{\alpha}_X = 1/2 + 1/3 + 1/7 = 41/42$. Thus, Theorem A says that

$$\begin{aligned} X \text{ is } K_m\text{-regular} &\iff \frac{41}{42} \geq \left\lceil \frac{m+2}{2} \right\rceil \\ &\iff m \leq -2. \end{aligned}$$

Thus, X is K_{-2}-regular, but not K_m-regular for $m \geq -1$. This recovers [Wei01, Example 6.1].

Moreover, we have the following regularity criterion.

Theorem B. *Let X be an affine local complete intersection. If X is K_2-regular, then it is regular.*

Interest in regularity criteria of this kind goes back to the work of Vorst [Vor79], who conjectured that for an affine, finite-type F-scheme of dimension d, regularity and K_{d+1}-regularity are equivalent. This conjecture was confirmed in [CHW08] (as usual, we assume the base field has characteristic zero; over perfect fields of characteristic p, Vorst's conjecture was confirmed in [KST18]). In [HW25], two extensions of Vorst's conjecture were established. First, it was shown that K_2-regular affine varieties are normal. Second, it was shown that for local complete intersections of minimal embedding codimension r (see Definition 5.3), regularity is implied by K_m-regularity for $m > ((d-r)/2) + 1$. Our theorem may be viewed as a further strengthening of Vorst's conjecture along these lines.

The proof of Theorems A and B, along with other applications discussed later in the paper, combines fundamental results of Cortiñas, Haesemeyer, Schlichting, Walker and Weibel [CHS$^+$08, CHW08, CHW$^+$10], now standard in the study of K-regularity, with a new ingredient: the theory of *higher Du Bois singularities*.

Recently, developments in Hodge theory have led to considerable interest in the study of higher analogs of the classical notion of Du Bois singularities for complex algebraic varieties. Following [JKS$^+$22, MOP$^+$23, MP25], a local complete intersection X is said to have *m-Du Bois singularities* if the natural maps

$$\Omega^p_{X/\mathbb{C}} \to \underline{\Omega}^p_{X/\mathbb{C}},$$

from the sheaf of algebraic differential forms to the Du Bois complexes (which are natural objects in the bounded derived category $D^b_{\mathrm{coh}}(X)$ of coherent sheaves on X, arising from Deligne's mixed Hodge theory), are isomorphisms for all $p \leq m$.

For varieties that are not local complete intersections, there is currently no known example satisfying $\Omega^1_{X/\mathbb{C}} \simeq \underline{\Omega}^1_{X/\mathbb{C}}$. For this reason, one works with a less restrictive notion for general

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complex algebraic varieties. Following [SVV24], we say X has *pre-m-Du Bois singularities* if

$$\mathcal{H}^i\underline{\Omega}^p_{X/\mathbb{C}} = 0 \quad \text{for all } i > 0 \text{ and } p \leq m,$$

where we write $\mathcal{H}^i\mathcal{F}$ for the ith cohomology sheaf of an object $\mathcal{F} \in D^b_{\mathrm{coh}}(X)$. Several other conditions are imposed in loc. cit. for the definition of m-Du Bois singularities in general. See § 2.3 for the natural extensions of these notions over more general base fields.

The following theorem relates higher Du Bois singularities to K-regularity. A related result can be found in [HW25], where it is shown that K_m-regularity implies the maps $\Omega^p_X \to \underline{\Omega}^p_X$ are isomorphisms for $p \leq m-1$.

Theorem C. *Let X be a variety of dimension d, and denote $s = \dim X_{\mathrm{sing}}$. Then the following assertions hold.*

(1) *If X has Du Bois singularities, then it is K_{-d+1}-regular.*
(2) *If X is pre-m-Du Bois, then it is K_{-t}-regular for $t \geq \max\{s+1, d-2m-2+s\}$; if X is in addition affine, then the bound can be improved to $t \geq \max\{1, d-2m-2\}$.*
(3) *If X is affine, and either:*
   – *X is a local complete intersection; or*
   – *$\operatorname{depth} \mathcal{H}^0\underline{\Omega}^p_X \geq d-p$ for all $p \geq 0$;*
   and if, furthermore, X is $K_{-d+2m+1+s}$-regular, then it is pre-m-Du Bois.

This theorem, combined with results from the theory of higher Du Bois singularities, most notably the characterization via the minimal exponent [MOP+23, JKS+22] and the codimension bound on the singular locus (Theorem 2.3), serves as the main technical input in the proofs of Theorems A and B. In addition, parts (1) and (2) produce new classes of K_m-regular varieties, both in the local complete intersection setting and beyond; see Example 4.2.

We conclude with two further applications of Hodge-theoretic methods to K-regularity. First, following [CHW+10, CHW+11], we use results on Du Bois invariants (Proposition 6.3) to obtain new examples in which Bass's question about whether $\mathbb{A}^1$-invariance of the algebraic K-groups implies $\mathbb{A}^2$-invariance has either a positive or a negative answer; see § 6.

Second, we use the degeneration of the Hodge-to-de Rham spectral sequence to characterize K-regularity for projective varieties.

Theorem D. *Let X be a complex projective variety. Then X is K_m-regular if and only if the natural maps*

$$\mathbb{H}^i(X, \mathbb{L}^p_{X/\mathbb{C}}) \xrightarrow{\sim} \mathbb{H}^i(X, \underline{\Omega}^p_{X/\mathbb{C}})$$

are isomorphisms for all $i - p \geq -m+1$. Here, we denote by $\mathbb{H}^i(X, -)$ the hypercohomology functor on X, and by $\mathbb{L}^p_{X/\mathbb{C}}$ the pth derived wedge product of the cotangent complex; see § 2.2.

This leads to examples showing that local K_m-regularity need not imply K_m-regularity; see Example 7.4.

2. Preliminaries

2.1 The cdh differentials and the Du Bois complexes

Let X be a Noetherian scheme over a field F. The *cdh differentials*[1] of X are defined as the sheafification of the Kähler differentials $\Omega^p_{X/F}$ in the cdh topology introduced by Suslin and

[1]Here cdh stands for the completely decomposed h-topology.

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Voevodsky [SV00]. More precisely,

$$\Omega^p_{\mathrm{cdh},X/F} := \mathbf{R}a_* a^* \Omega^p_{X/F},$$

where $a : X_{\mathrm{cdh}} \to X_{\mathrm{zar}}$ is the natural change-of-topology morphism from the cdh site to the Zariski site.

On the other hand, for a complex algebraic variety (i.e. when $F = \mathbb{C}$), Du Bois [Du81] introduced the notion of *Du Bois complexes* (or *filtered de Rham complexes*) following ideas of Deligne. The same construction applies to define the Du Bois complexes of any quasi-excellent Noetherian scheme X over a field F of characteristic zero. Indeed, by Hironaka's theorems, the assumptions on X guarantee the existence of a hyperresolution $\epsilon_\bullet : X_\bullet \to X$. We then define the pth *Du Bois complex* of X over F as

$$\underline{\Omega}^p_{X/F} := \mathbf{R}\epsilon_{\bullet *} \Omega^p_{X_\bullet/F}.$$

We refer readers to [GNP+82, Chapter V] and [PS08, Chapter 7.3] for the construction of hyperresolutions, and for the definition of derived pushforward from a hyperresolution. Being natural objects arising from Deligne's mixed Hodge theory, the Du Bois complexes are mostly studied for complex algebraic varieties. In this case, the Du Bois complexes $\underline{\Omega}^p_{X/\mathbb{C}}$ in our notation are usually denoted simply by $\underline{\Omega}^p_X$ in the literature.

The relationship between the Du Bois complexes and the cdh differentials is observed in [CHW+11, Lemma 2.1].

Lemma 2.1. *Let X be a finite-type F-scheme, and let $k \subset F$ be a subfield. Then for $p \geq 0$, there is an isomorphism*

$$\underline{\Omega}^p_{X/k} \simeq \Omega^p_{\mathrm{cdh},X/k}.$$

Proof. When $k = F$, this is exactly [CHW+11, Lemma 2.1]. For $k \subset F$, the result follows by the same proof, using [CHW08, Corollary 2.5] and [SV00, Lemma 12.1]. □

For an F-scheme X, we denote by $\underline{\Omega}^{\leq p}_{X/F}$ the (derived) cdh sheafification of the following complex of quasi-coherent sheaves:

$$\mathcal{O}_X \to \Omega^1_{X/F} \to \cdots \to \Omega^p_{X/F}$$

supported in cohomological degrees $0, 1, \ldots, p$. Explicitly, $\Omega^{\leq p}_{\mathrm{cdh},X/F} := \mathbf{R}a_* a^* \Omega^{\leq p}_{X/F}$, where $a : X_{\mathrm{cdh}} \to X_{\mathrm{zar}}$ is the natural change-of-topology morphism from the cdh site to the Zariski site.

When X is of finite type over F, this represents an object in the derived category of differential complexes on X. When $F = \mathbb{C}$, it follows from Lemma 2.1 that for each $p \geq 0$,

$$\underline{\Omega}^{\leq p}_{X/\mathbb{C}} = \underline{\Omega}^\bullet_{X/\mathbb{C}} / F^{p+1}$$

agrees with the quotient of the filtered de Rham complex by the Hodge filtration. It is immediate from the definition that we have the following exact triangle for each $p \geq 0$:

$$\underline{\Omega}^p_{X/F}[-p] \to \underline{\Omega}^{\leq p}_{X/F} \to \underline{\Omega}^{\leq p-1}_{X/F} \xrightarrow{+1}. \tag{1}$$

In the proof of [CHS+08, Theorem 6.1] and [CHW08, Proposition 2.6], the following useful properties of the objects $\underline{\Omega}^p_{X/F}$ and $\underline{\Omega}^{\leq p}_{X/F}$ are established.

Proposition 2.2. *Let X be a separated, finite-type F-scheme, and let $k \subset F$ be a subfield. Then for each $p \geq 0$, the following assertions hold:*

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(1) $\dim \operatorname{supp} \mathcal{H}^q \underline{\Omega}^p_{X/k} \leq \dim X - q - 1$ *if* $q > 0$*; in particular,* $\mathcal{H}^i \underline{\Omega}^p_{X/k} = 0$ *for all* p *and all* $i \geq d$*;*

(2) $\mathbb{H}^q(X, \underline{\Omega}^p_{X/k}) = 0$ *for all* $q > \dim X$*;*

(3) $\mathbb{H}^q(X, \underline{\Omega}^{\leq p}_{X/k}) = 0$ *for all* $q > \dim X + p$.

Proof. Assertion (1) is stated in the proof of [CHS+08, Theorem 6.1] when $p = 0$ (in which case $\underline{\Omega}^0_{X/k} = \underline{\Omega}^0_{X/F}$). The proof for $p \geq 1$ is identical when $k = F$, and the case for general $k \subset F$ follows from Lemma 2.11(1). Assertion (2) follows from (1) and the spectral sequence

$$E_1^{ij} = H^i(X, \mathcal{H}^j \underline{\Omega}^p_{X/F}) \Longrightarrow \mathbb{H}^{i+j}(X, \underline{\Omega}^p_{X/F}).$$

Finally, (3) follows from (2) by induction using the exact triangle (1). □

2.2 Derived de Rham complex and the cotangent complex

Following Bhatt [Bha12, Construction 4.1], for an F-algebra A, we define the *(Hodge-completed) derived de Rham complex of* A *over* F as

$$\widehat{\mathrm{dR}}_{A/F} := \lim_{p \in \mathbb{N}^{op}} L\Omega^{\leq p}_{A/F},$$

where $L\Omega^{\leq p}_{A/F} := \mathrm{Tot}(\Omega^{\leq p}_{P/F} \otimes_P A)$ is the single complex associated to the double complex $\Omega^{\leq p}_P \otimes_P A$, in which $P \to A$ is a polynomial F-algebra resolution of A. It can be shown that the objects $L\Omega^{\leq p}_{A/F}$ and $\widehat{\mathrm{dR}}_{A/F}$ are independent of the choice of the polynomial resolution $P \to A$ (up to homotopy), and that the construction globalizes to give definitions of $L\Omega^{\leq p}_{X/F}$ and $\widehat{\mathrm{dR}}_{X/F}$ for any F-scheme X.

By construction, there is a natural *derived filtration* Fil_H on $\widehat{\mathrm{dR}}_{X/F}$, with respect to which the graded pieces are given by

$$\mathrm{gr}^p_H \widehat{\mathrm{dR}}_{X/F} = \mathbb{L}^p_{X/F}[-p]$$

where $\mathbb{L}^p_{X/F}$ is the pth derived wedge product of the cotangent complex, introduced by Illusie [Ill71].

As explained in [Bha12, § 5], for a finite-type F-scheme X, there is a map

$$\varphi : \widehat{\mathrm{dR}}_{X/F} \to \underline{\Omega}^\bullet_{X/F}$$

preserving the derived filtration on $\widehat{\mathrm{dR}}_{X/F}$ and the Hodge filtration on $\underline{\Omega}^\bullet_{X/F}$. Taking associated graded pieces with respect to the two filtrations, we obtain a natural map

$$\mathbb{L}^p_{X/F} \to \underline{\Omega}^p_{X/F}$$

for each p. It is a natural question when these maps are isomorphisms. For local complete intersections over $F = \mathbb{C}$, this question is studied in the theory of higher Du Bois singularities. When X is not a local complete intersection, this map is rarely an isomorphism unless $p = 0$; see Remark 2.5.

2.3 Background in higher Du Bois singularities

In the literature, higher Du Bois singularities are defined and studied for complex algebraic varieties. For the purpose of relating to K-regularity, we follow the notation in the previous subsections, and work with these singularities over more general base fields.

Let X be a local complete intersection over F, and $k \subset F$ a subfield. Following [MOP+23, JKS+22] (with the obvious replacement of $\mathbb{C}$ by F), we say X has *m-Du Bois singularities over k*

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if the natural maps $\Omega^p_{X/k} \to \underline{\Omega}^p_{X/k}$ are isomorphisms for all $p \leq m$, where $\Omega^p_{X/k}$ is the (underived) pth wedge product of the Kähler differential $\Omega^1_{X/k}$ relative to k. One important property of m-Du Bois singularities is the following codimension bound on the singular locus, obtained by combining [MP22, Theorem F] and [MP22, Corollary 9.26]. See e.g. [FL24, Theorem 3.8].

Theorem 2.3 (Mustaţă–Popa). *Let X be a local complete intersection. If X has m-Du Bois singularities, then*

$$\operatorname{codim} X_{\mathrm{sing}} \geq 2m+1.$$

As a consequence, for local complete intersections one can replace $\Omega^p_{X/k}$ in the definition of m-Du Bois singularities by the more natural object $\mathbb{L}^p_{X/k}$, the pth derived wedge product of the cotangent complex of X over k.

Lemma 2.4. *Let X be a local complete intersection over F, and $k \subset F$ a subfield. Then the following assertions are equivalent:*

(1) *X has m-Du Bois singularities over k;*
(2) *X has m-Du Bois singularities over F;*
(3) *X has m-Du Bois singularities over F', for any field extension $F \subset F'$;*
(4) *the natural maps $\mathbb{L}^p_{X/k} \to \underline{\Omega}^p_{X/k}$ are isomorphisms for all $p \leq m$;*
(5) *the natural maps $\mathbb{L}^p_{X/F} \to \underline{\Omega}^p_{X/F}$ are isomorphisms for all $p \leq m$;*
(6) *the natural maps $\mathbb{L}^p_{X\times_F F'/F'} \to \underline{\Omega}^p_{X\times_F F'/F'}$ are isomorphisms for all $p \leq m$, and any field extension $F \subset F'$.*

Proof. The equivalences (1) $\iff$ (2) and (4) $\iff$ (5) follow from Lemma 2.11(1), and the equivalences (2) $\iff$ (3) and (5) $\iff$ (6) follows from Lemma 2.11(2).

We will show (2) $\iff$ (5): using Lemma 2.11, we reduce to the case $k = \mathbb{C}$.[2] For ease of notation, we then write $\mathbb{L}^p_{X/\mathbb{C}}$ and $\underline{\Omega}^p_{X/\mathbb{C}}$ simply as $\mathbb{L}^p_X$ and $\underline{\Omega}^p_X$.

If the natural maps $\mathbb{L}^p_X \to \underline{\Omega}^p_X$ are isomorphisms for all $p \leq m$, taking cohomology in degree greater than or equal to 0 shows that X has m-Du Bois singularities.

For the converse, suppose X is a local complete intersection subvariety of a smooth variety Y, and that X has m-Du Bois singularities. Denote by $\mathcal{I} = \mathcal{I}_{X/Y}$ the ideal sheaf defining X in Y. Then the cotangent complex $\mathbb{L}^p_X$ can be represented by

$$\mathcal{I}^p/\mathcal{I}^{p+1} \to \mathcal{I}^{p-1}/\mathcal{I}^p \otimes \Omega^1_Y|_X \to \cdots \to \mathcal{I}/\mathcal{I}^2 \otimes \Omega^{p-1}_Y|_X \to \Omega^p_Y|_X,$$

supported in cohomological degrees $-p, -p+1, \ldots, -1, 0$.

This is the Eagon–Northcott complex associated to (the dual of) the map $\mathcal{I}/\mathcal{I}^2 \to \Omega^1_Y|_X$. As explained in [MP22] before the proof of Theorem F,[3] by the depth sensitivity of the Eagon–Northcott complex, it is a resolution of Ω^p_X when $\operatorname{codim} X_{\mathrm{sing}} \geq p$. This is satisfied when X has m-Du Bois singularities for $m \geq 1$ (see Theorem 2.3). Since the Lemma is clear when $m = 0$ (in which case $\mathbb{L}^0_X = \mathcal{O}_X = \Omega^0_X$), this completes the proof. □

[2] More precisely, given a variety X/k with m-Du Bois singularities over k, we can write $X \simeq X(k_0) \times_{k_0} k$ for some extension field k_0 of $\mathbb{Q}$ of finite transcendence degree. Then the field k_0 embeds into $\mathbb{C}$. By the equivalences established above, X has m-Du Bois singularities if and only if the complex algebraic variety $X(k_0) \times_{k_0} \mathbb{C}$ has m-Du Bois singularities. If we can show this is equivalent to $\mathbb{L}^p_{X(k_0)\times_{k_0}\mathbb{C}/\mathbb{C}} \to \underline{\Omega}^p_{X(k_0)\times_{k_0}\mathbb{C}/\mathbb{C}}$ being isomorphisms for $p \leq m$, applying the equivalence (5) $\iff$ (6) then gives the result.

[3] In loc. cit., this complex shows up as the Grothendieck dual of $\mathrm{Gr}^E_{-p}\,\mathrm{DR}(\mathcal{H}^r_X\mathcal{O}_Y)$, the graded piece of the Hodge module $\mathcal{H}^r_X\mathcal{O}_Y$ with respect to the Ext filtration.

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In view of the equivalence (1) $\iff$ (2), we will sometimes simply say X has *m-Du Bois singularities* when the base field causes no confusion.

When X is not necessarily a local complete intersection, as explained in [SVV24], the conditions $\Omega^1_{X/F} \xrightarrow{\sim} \underline{\Omega}^1_{X/F}$ and $\mathbb{L}^1_{X/F} \xrightarrow{\sim} \underline{\Omega}^1_{X/F}$ are too restrictive due to the bad behaviors of the Kähler differentials.

Remark 2.5. For a variety X over F, the sheaf $\mathcal{H}^0\underline{\Omega}^1_{X/F}$ (even $\mathcal{H}^0\underline{\Omega}^p_{X/F}$ for all $p \geq 1$) is:

- torsion-free by [HJ14, Remark 3.8];
- reflexive when X has rational singularities, by [KS21, Corollary 1.2] and the identification of $\mathcal{H}^0\underline{\Omega}^1_{X/F}$ with the h-differential in [HJ14].

On the other hand, when X is not a local complete intersection, the Kähler differentials typically have torsion, and are not reflexive even when X has mild singularities; see [GR11]. For such varieties, one does not have $\Omega^1_{X/F} \simeq \mathcal{H}^0\underline{\Omega}^1_{X/F}$.

As a consequence, when X is not a local complete intersection, the maps $\Omega^1_{X/F} \to \underline{\Omega}^1_{X/F}$ and $\mathbb{L}^1_{X/F} \to \underline{\Omega}^1_{X/F}$ need not be isomorphisms even when X has mild singularities.

In fact, currently there is no known example of a variety satisfying $\Omega^1_X \simeq \mathcal{H}^0\underline{\Omega}^1_X$ that is not a local complete intersection. Therefore, for arbitrary varieties, we work with the following definition. The equivalences follow from Lemma 2.11.

Definition 2.6. Let X be a variety over F, and $k \subset F$ a subfield. We say that X has *pre-m-Du Bois singularities* if one of the following equivalent conditions holds:

(1) $\mathcal{H}^i\underline{\Omega}^p_{X/F} = 0$ for all $p \leq m$ and $i > 0$;

(2) $\mathcal{H}^i\underline{\Omega}^p_{X/k} = 0$ for all $p \leq m$ and $i > 0$;

(3) $\mathcal{H}^i\underline{\Omega}^p_{X\times_F F'/F'} = 0$ for all $p \leq m$ and $i > 0$, for some (any) field extension F' of F.

Remark 2.7. Criteria for various classes of varieties (that are typically not local complete intersections) to have pre-m-Du Bois singularities have been worked out. See [GNP+82, V Theorem 4.6] for toric varieties, [CHW+13, PS] for cones over projective varieties, and [ORS24] for secant varieties.

For local complete intersections, one can characterize m-Du Bois singularities using the minimal exponent [JKS+22, MOP+23, CDM+24, CDM24], which is an invariant that refines the log canonical threshold.

On the other hand, necessary and sufficient criteria for an arbitrary variety to have pre-m-Du Bois singularities are not known at this moment.

Following [SVV24], we say X has *m-Du Bois singularities* if it is seminormal, pre-m-Du Bois, the sheaves $\mathcal{H}^0\underline{\Omega}^p_{X/F}$ are reflexive for all $p \leq m$, and $\operatorname{codim} X_{\text{sing}} \geq 2m+1$. As explained in [SVV24, § 5], this new definition agrees with the well-studied notion in the case of local complete intersections, and includes more examples in general. Varieties with m-Du Bois singularities in this sense provide examples that are K_{-d+m+1}-regular; see Example 4.2.

2.4 Background in K-regularity

Let X be a scheme. We say X is *K_m-regular* if the natural maps

$$K_m(X) \to K_m(X \times \mathbb{A}^r)$$

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between algebraic K-groups are isomorphisms for all $r \geq 0$. It is a fact that if X is regular, then it is K_m-regular for all m. Also, K_m-regularity implies K_{m-1}-regularity for all m; see [Vor79] and [DW80, Corollary 4.4].

In a sequence of foundational papers [CHS+08, CHW+10, CHW08] in this area, it was discovered that K_m-regularity of X is controlled by the Kähler differentials, the cdh differentials and the Hochschild homology of X. Following these papers, we denote by

$$NK_m(X) = \operatorname{coker}(K_m(X) \to K_m(X \times \mathbb{A}^1))$$

the NK-groups of Bass. For $r \geq 2$, the groups $N^r K(X)$ are inductively defined by

$$N^r K_m(X) := N(N^{r-1}K_m)(X) := \operatorname{coker}(N^{r-1}K_m(X) \to N^{r-1}K_m(X \times \mathbb{A}^1))$$

By [CHW+10, Theorem 0.1], we have

$$\begin{aligned} N^r K_m(X) = 0 &\iff NK_m(X) = NK_{m-1}(X) = \cdots = NK_{m-r+1}(X) = 0 \\ &\iff K_m(X) \xrightarrow{\sim} K_m(X \times \mathbb{A}^r) \text{ is an isomorphism.} \end{aligned}$$

This reduces the study of K-regularity to that of the NK-groups. Another theorem in [CHW+10] expresses NK, or more precisely, $NK^{(i)}$, the ith eigenspace of the λ-operation on NK, in terms of the Kähler differentials, the cdh differentials and the Hochschild homology of X. We rephrase the statement with the notation of Du Bois complexes, using Lemma 2.1.

Theorem 2.8 [CHW+10, Theorem 0.4]. *Let $X = \operatorname{Spec} R$ be an affine scheme over $\mathbb{Q}$. Then*

$$NK_n^{(i)}(X) \cong TK_n^{(i)}(X) \otimes_{\mathbb{Q}} t\mathbb{Q}[t]$$

where t is the coordinate on $\mathbb{A}^1$ in the definition of NK, and

$$TK_n^{(i)}(X) \simeq \begin{cases} \mathcal{H}^{i-n-1}\underline{\Omega}^{i-1}_{X/\mathbb{Q}} & n \leq i-2, \\ \operatorname{coker}(\Omega^i_{X/\mathbb{Q}} \to \mathcal{H}^0\underline{\Omega}^i_{X/\mathbb{Q}}) & n = i-1, \\ \ker(\Omega^i_{X/\mathbb{Q}} \to \mathcal{H}^0\underline{\Omega}^i_{X/\mathbb{Q}}) & n = i, \\ HH_{n-1}^{(i-1)}(X) & n \geq i+1. \end{cases}$$

Remark 2.9. Since $HH_n^{(i)}(X) \simeq \mathcal{H}_{n-i}\mathbb{L}^i \simeq \mathcal{H}^{i-n}\mathbb{L}^i$, this theorem can be rephrased in the following way. Let C^i be the cone of the natural map $\mathbb{L}^i_{X/\mathbb{Q}} \to \underline{\Omega}^i_{X/\mathbb{Q}}$. Then the $TK_n^{(i)}(X)$ are determined by the cohomologies of C^i:

$$TK_{i-j}^{(i+1)}(X) \simeq \mathcal{H}^j C^i.$$

As a consequence, X is K_m-regular if and only if

$$\mathcal{H}^j C^i = 0 \quad \text{for all } j - i \geq -m.$$

Thus, in view of Lemma 2.4, the notion of K-regularity is naturally related to higher singularities, at least for local complete intersections.

To study the K-regularity for varieties that are not necessarily affine, it is convenient to work with a local version of Bass's NK-groups, denoted by $\mathcal{NK}_m$, and defined as the sheafification of

$$U \mapsto NK_m(\Gamma(U))$$

on the étale site of X in [vdK86]. As shown in loc. cit., we have

$$H^i_{\text{ét}}(X, \mathcal{NK}_m) = H^i(X, \mathcal{NK}_m),$$

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where H^i denotes the cohomology taken in the Zariski topology. When X is affine,

$$H^i_{\text{ét}}(X, \mathcal{NK}_m) = \begin{cases} NK_m(X) & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

We have a local-to-global spectral sequence relating local and global NK-groups

$$E_2^{pq} = H^p(X, \mathcal{NK}_q(X)) \Longrightarrow NK_{q-p}(X).$$

In view of these results, Theorem 2.8 can be stated in this local setting as follows.

Theorem 2.10 [CHW+10]. *Let X be any scheme over $\mathbb{Q}$. Then*

$$\mathcal{NK}_m(X) \simeq \bigoplus_{j-i=-m} \mathcal{H}^j C^i \otimes t\mathbb{Q}[t],$$

where $C^i = \text{Cone}(\mathbb{L}^i_{X/\mathbb{Q}} \to \underline{\Omega}^i_{X/\mathbb{Q}})$. In particular, X is locally K_m-regular, that is, $\mathcal{NK}_q(X) = 0$ for all $q \le m$, if and only if

$$\mathcal{H}^j C^i = 0 \quad \text{for } j - i \ge -m.$$

2.5 Base change results

We saw that the theory of higher Du Bois singularities concerns the study of the Du Bois complex $\underline{\Omega}^p_{X/F}$ for a variety over F, whereas the K-regularity of X is related to $\mathbb{L}^p_{X/\mathbb{Q}}$ and $\underline{\Omega}^p_{X/\mathbb{Q}}$. Thus, to relate the two theories we will need base change results for the Du Bois complex and the cotangent complex over various fields, which we collect below.

Lemma 2.11. *Let X be a variety over F.*

(1) ([KS92, 4.3a], [CHW08, Lemmas 4.1, 4.2].) *If $k \subset F$ is a subfield, then we have spectral sequences*

$$E_1^{ij} = \Omega^i_{F/k} \otimes_F \mathcal{H}^{i+j}\mathbb{L}^{p-i}_{X/F} \Longrightarrow \mathcal{H}^{i+j}\mathbb{L}^p_{X/k},$$
$$E_1^{ij} = \Omega^i_{F/k} \otimes_F \mathcal{H}^{i+j}\underline{\Omega}^{p-i}_{X/F} \Longrightarrow \mathcal{H}^{i+j}\underline{\Omega}^p_{X/k},$$
$$E_1^{ij} = \Omega^i_{F/k} \otimes \mathbb{H}^j(X, L\Omega^{\le p-i}_{X/F}) \Longrightarrow \mathbb{H}^{i+j}(X, L\Omega^{\le p}_{X/k}),$$
$$E_1^{ij} = \Omega^i_{F/k} \otimes \mathbb{H}^j(X, \underline{\Omega}^{\le p-i}_{X/F}) \Longrightarrow \mathbb{H}^{i+j}(X, \underline{\Omega}^{\le p}_{X/k}).$$

(2) *If $F \subset F'$ is a field extension, then we have isomorphisms*

$$\mathbb{L}^p_{X\times_F F'/F'} \simeq \mathbb{L}^p_{X/F} \otimes_F F',$$
$$\underline{\Omega}^p_{X\times_F F'/F'} \simeq \underline{\Omega}^p_{X/F} \otimes_F F'.$$

Proof.

(1) The first spectral sequence follows from [KS92, 4.3a], using that $HH^{(i)}_n(X) \simeq \mathcal{H}_{n-i}\mathbb{L}^i \simeq \mathcal{H}^{i-n}\mathbb{L}^i$. The second one is exactly [CHW08, Lemma 4.2], by the comparison between cdh differentials and the Du Bois complexes (see Lemma 2.1). The last two spectral sequences are proved along the same lines as in [KS92, 4.3a] and [CHW08, Lemma 4.2], using instead the filtration $\mathcal{G}$ on $\Omega^{\le p}_{X/\mathbb{Q}}$ given by

$$\mathcal{G}^i/\mathcal{G}^{i+1} \simeq \Omega^i_{\mathbb{C}/\mathbb{Q}} \otimes_{\mathbb{C}} \Omega^{\le p-i}_{X/\mathbb{C}}[-i]$$

when X is smooth.

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(2) For Kähler differentials, we have the isomorphism

$$\Omega^p_{X\times_k k'/k'} \simeq \Omega^p_{X/k} \otimes_k k'.$$

This gives the two desired isomorphisms, taking the left derived functor for the first, and cdh sheafification for the second. □

3. Vanishing theorems for the Du Bois complexes

In this section, we collect vanishing theorems for the Du Bois complexes, with a focus on the effect of the change of base fields on the statements. These will be applied later to study K-regularity.

The first general result is due to Steenbrink when $F = \mathbb{C}$. The general case follows from Lemma 2.11.

Theorem 3.1 [Ste85]. *Let X be a variety over F. Then*

$$\mathcal{H}^q \underline{\Omega}^p_{X/F} = 0$$

for $p + q > \dim X$.

In view of Lemma 2.11(1), Steenbrink's vanishing theorem need not hold for Du Bois complexes over a subfield $k \subset F$. Instead, we have the following weaker statement.

Proposition 3.2. *Let X be a variety over F, and $k \subset F$ a subfield. If X has pre-m-Du Bois singularities, then*

$$\mathcal{H}^q \underline{\Omega}^p_{X/k} = 0$$

for all $q \geq \dim X - m - 1$ and $q > 0$.

Proof. It suffices to prove the result when $p \geq m + 1$. By [PSV, Proposition C] and Steenbrink vanishing (and reduction to the case $F = \mathbb{C}$ as usual), we have that $\mathcal{H}^q \underline{\Omega}^p_{X/F} = 0$ for $q \geq \dim X - p \geq \dim X - m - 1$. It then follows from the spectral sequence in Lemma 2.11(1) that $\mathcal{H}^q \underline{\Omega}^p_{X/k} = 0$ for $q \geq \dim X - m - 1$. □

Complementary to Steenbrink's theorem, we have vanishing results for $\mathcal{H}^q \underline{\Omega}^p_{X/k}$ when $p + q$ is small.

Theorem 3.3 [MP25]. *Let X be a local complete intersection over F, and $k \subset F$ a subfield. Denote by X_{sing} the singular locus of X. Then for all $p \geq 0$, we have*

$$\mathcal{H}^q \underline{\Omega}^p_{X/k} = 0$$

for $1 \leq q < \operatorname{codim} X_{\mathrm{sing}} - p - 1$.

Proof. When $F = \mathbb{C}$, this is [MP25, Corollary 13.9]. The result for F follows from Lemma 2.11(2), and the statement for a subfield $k \subset F$ follows by analyzing the spectral sequence in Lemma 2.11(1). □

Beyond local complete intersections, we have the following result.

Theorem 3.4 [PSV, Kov25]. *Let X be a variety over F, and $k \subset F$ be a subfield. Then the following assertions hold:*

(1) *$\mathcal{H}^q \underline{\Omega}^0_{X/F} = \mathcal{H}^q \underline{\Omega}^0_{X/k} = 0$ for $1 \leq q < \operatorname{depth} \mathcal{O}_X - \dim X_{\mathrm{sing}} - 1$;*

(2) *if X is pre-$(p-1)$-Du Bois with $\dim X_{\mathrm{sing}} = s$, then*

$$\mathcal{H}^q \underline{\Omega}^p_{X/F} = 0$$

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for $1 \le q < \operatorname{depth} \mathcal{H}^0 \underline{\Omega}^p_{X/F} - s - 1$*; in particular, if* $\operatorname{depth} \mathcal{H}^0 \underline{\Omega}^p_{X/F} \ge \dim X - p$ *for all* $p \ge 0$*, then*

$$\mathcal{H}^q \underline{\Omega}^p_{X/k} = 0$$

for $1 \le q < \dim X - p - s - 1$*, and any subfield* $k \subset F$*.*

Proof. Assertion (1) follows from [PSV, Corollary B], using Lemma 2.11(2) and the fact $\underline{\Omega}^0_{X/F} = \underline{\Omega}^0_{X/k}$. Assertion (2) follows from [PSV, Theorem A], [Kov25, Theorem 1.1] and Lemma 2.11. □

4. Higher Du Bois singularities and K-regularity

We start by establishing a general relationship between higher Du Bois singularities and K-regularity. Later, we will apply vanishing theorems for the Du Bois complexes to improve these results under an additional hypothesis.

Proposition 4.1 ($\Longrightarrow$ Theorem C(1)). *Let X be a variety of dimension d.*

(1) *If X has m-Du Bois singularities, then it is K_{-d+m+1}-regular.*
(2) *Suppose X either is affine or has isolated singularities. If X is K_m-regular, then it is pre-$(m+1)$-Du Bois.*

Proof. (1) Since X has m-Du Bois singularities in the sense of [SVV24], the following conditions are satisfied:

- X is seminormal;
- X has pre-m-Du Bois singularities;
- $\operatorname{codim} X_{\mathrm{sing}} \ge m+2$ if $m \ge 1$.

To show X is K_{-d+m+1}-regular, by the local-to-global spectral sequence

$$E_2^{pq} = H^p(X, \mathcal{NK}_q(X)) \Longrightarrow NK_{q-p}(X),$$

it suffices to show

$$H^p(X, \mathcal{NK}_q(X)) = 0 \quad \text{for all } q - p \le -d + m + 1.$$

By Theorem 2.10, this is equivalent to showing

$$H^p(X, \mathcal{H}^s C^t) = 0 \quad \text{for } s - t = -q \text{ and } q - p \le -d + m + 1,$$

where C^t is the cone of the natural map $\mathbb{L}^t_{X/\mathbb{Q}} \to \underline{\Omega}^t_{X/\mathbb{Q}}$.

We verify this through a case-by-case analysis.

- If $s > 0$ and $t \le m$, we have $\mathcal{H}^s C^t = 0$ because X is pre-m-Du Bois.
- If $s > 0$ and $t \ge m+1$, then $p \ge q + d - m - 1 = t - s + d - m - 1 \ge d - s$. By Proposition 2.2,

$$\dim \operatorname{supp} \mathcal{H}^s C^t = \dim \operatorname{supp} \mathcal{H}^s \underline{\Omega}^t_{X/\mathbb{Q}} \le d - s - 1.$$

Thus, we have $H^p(\mathcal{H}^s C^t) = 0$.

- If $s \le 0$, then $q \ge s + q = t \ge 0$, and $p \ge q + d - m - 1 \ge d - m - 1$. By assumption, the sheaves $\mathcal{H}^s C^t$ are supported on X_{sing}, which has dimension $\le d - m - 2$ when $m \ge 1$, so we have $H^p(\mathcal{H}^s C^t) = 0$.

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For $m=0$, some further work is needed.
- If $t=0$, then $\mathcal{H}^s C^t = 0$.
- If $t \geq 1$, then $p \geq q+d-1 = d-1-s+t \geq d$, so $H^p(\mathcal{H}^s C^t) = 0$, because $\mathcal{H}^s C^t$ is supported on X_{sing}, which has dimension $\leq d-1$.

This completes the proof that X is K_{-d+m+1}-regular.

(2) Suppose X is affine and K_m-regular. By Theorem 2.8, we have

$$\mathcal{H}^s C^t = 0 \quad \text{for } s-t \geq -m.$$

In particular, taking $s>0$, this implies

$$\mathcal{H}^s \underline{\Omega}^p_{X/\mathbb{Q}} = 0$$

for all $p \leq m+1$ and $s>0$, that is, that X is pre-$(m+1)$-Du Bois.

Now, suppose X has isolated singularities. Then the spectral sequence

$$E_2^{pq} = H^p(X, \mathcal{NK}_q(X)) \implies NK_{q-p}(X)$$

degenerates, giving that the local NK-groups $\mathcal{NK}_q(X) = 0$ for $q \leq m$. It then follows from the affine case that X is pre-$(m+1)$-Du Bois. □

Proposition 4.1(1) allows us to construct many examples of K_{-d+m+1}-regular varieties.

Example 4.2. We indicate some ways to construct varieties with m-Du Bois singularities.

(1) A codimension r local complete intersection X has m-Du Bois singularities if and only if its minimal exponent $\tilde{\alpha}_X \geq m+r$; see [Sai94, JKS+22, MOP+23, CDM24]. Playing around with the minimal exponents, one can construct varieties with m-Du Bois singularities.

(2) For toric varieties and cones over smooth projective varieties, one can find criteria for them to have m-Du Bois singularities in [SVV24, §§ 6 and 7].

(3) Criteria for secant varieties to have m-Du Bois singularities are worked out in [ORS24].

By Proposition 4.1(1), these provide examples of K_{-d+m+1}-regular varieties.

Next, applying vanishing theorems in § 3, we obtain the following strengthening of Proposition 4.1 in the local setting.

Proposition 4.3 ($\Longrightarrow$ Theorem C(2)). *Let X be a d-dimensional variety over F with pre-m-Du Bois singularities. Then the following assertions hold.*

(1) *The variety X is locally K_{-t}-regular if $t \geq \max\{1, d-2m-2\}$.*

(2) *If X is seminormal, then it is locally K_0-regular if $m \geq (d-2)/2$.*

(3) *If X is seminormal, and the natural map $\Omega^1_{X/F} \to \mathcal{H}^0\underline{\Omega}^1_{X/F}$ is surjective, then it is locally K_1-regular if $m \geq (d-1)/2$.*

(4) *Let $s = \dim X_{\mathrm{sing}}$. Then X is K_{-t}-regular if $t \geq \max\{s+1, d-2m-2+s\}$; if X is in addition seminormal, then the bound can be improved to $t \geq \max\{s, d-2m-2+s\}$.*

(5) *If X is either affine, or has isolated singularities, then we can remove the adjective 'locally' in (1), (2) and (3).*

Proof.

(1) By Theorem 2.10, we need to show

$$\mathcal{H}^q C^p = 0 \quad \text{for } q-p \geq \max\{1, d-2m-2\},$$

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where $C^p = \mathrm{Cone}(\mathbb{L}^p_{X/\mathbb{Q}} \to \underline{\Omega}^p_{X/\mathbb{Q}})$. Since $q \geq q-p \geq 1$, this is equivalent to

$$\mathcal{H}^q \underline{\Omega}^p_{X/\mathbb{Q}} = 0 \quad \text{for } q-p \geq d-2m-2.$$

Since X has pre-m-Du Bois singularities, this holds when $p \leq m$. When $p \geq m+1$, we have $q \geq d-2m-2+p \geq d-m-1$, in which case the vanishing follows from Proposition 3.2.

(2) By (1), it suffices to show $\mathcal{H}^0 C^0 \simeq \mathrm{coker}(\mathcal{O}_X \to \mathcal{H}^0 \underline{\Omega}^0_{X/\mathbb{Q}}) = 0$. This is true by [Sai00, Corollary 0.3], since X is assumed to be seminormal.

(3) By (1) and (2), since $\mathcal{H}^{-1} C^0 = 0$, it suffices to show $\mathcal{H}^0 C^1 \simeq \mathrm{coker}(\Omega^1_{X/\mathbb{Q}} \to \mathcal{H}^0 \underline{\Omega}^1_{X/\mathbb{Q}}) = 0$. This follows by assumption and Lemma 2.11(1).

(4) Consider the local-to-global spectral sequence

$$E_2^{pq} = H^p(X, \mathcal{NK}_q) \implies NK_{q-p}(X).$$

Since $\mathcal{NK}_q$ is supported on X_{sing} and $\dim X_{\mathrm{sing}} = s$, we have

$$H^p(X, \mathcal{NK}_q) = 0 \quad \text{for } p > s.$$

On the other hand, if $p-q \geq d-2m-2+s$, then for every $0 \leq p \leq s$ we have

$$q = (q-p)+p \leq -(d-2m-2+s)+s = -d+2m+2.$$

Note also that $q = p-t \leq p-s-1 < 0$. Hence $\mathcal{NK}_q = 0$ by part (1). It follows that $NK_{-t}(X) = 0$ for $t \geq 1$ and $t \geq d-2m-2+s$. That is, X is K_{-t}-regular. When X is in addition seminormal, the assertion follows by invoking (2) in place of (1).

(5) If X is either affine, or has isolated singularities, the local-to-global spectral sequence

$$E_2^{pq} = H^p(X, \mathcal{NK}_q(X)) \implies NK_{q-p}(X)$$

degenerates, so local $K_{-d+2m+2}$-regularity implies $K_{-d+2m+2}$-regularity. □

As an application of this proposition, we give some examples that are K_0-regular.

Example 4.4. We list some varieties with pre-k-Du Bois singularities for all k. These have 'mild' singularities from the perspective of Hodge theory. By Proposition 4.3, such an X is locally:

(1) K_n-regular for all $n < 0$;

(2) K_0-regular if X is seminormal.

Examples include toric varieties [GNP+82], finite or more generally geometric quotients [Du81], and cones over varieties satisfying Bott vanishing ([CHW+13], [SVV24, Corollary 7.6]). Since all these varieties are normal, they are locally K_0-regular.

Next, we investigate the converse to the implication in Proposition 4.3.

Proposition 4.5 ($\Longrightarrow$ Theorem C(3)). *Let X be a d-dimensional affine variety with $\dim X_{\mathrm{sing}} = s$. Suppose either of the following assertions holds:*

(1) *X is a local complete intersection;*

(2) *$\mathrm{depth}\, \mathcal{H}^0 \underline{\Omega}^p_X \geq d-p$ for all $p \geq 0$.*

If X is $K_{-d+2m+1+s}$-regular, then it is pre-m-Du Bois.

Proof. Since X is $K_{-d+2m+1+s}$-regular, we know by Theorem 2.8 that

$$\mathcal{H}^i \underline{\Omega}^j_{X/\mathbb{Q}} = 0 \quad \text{for } i \geq d-2m-1-s+j \text{ and } i > 0.$$

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Applying Theorem 3.3 in case (1), and Theorem 3.4 in case (2) (with $k = \mathbb{Q}$ and F the field of definition of X), we have

$$\mathcal{H}^i \underline{\Omega}^j_{X/\mathbb{Q}} = 0 \quad \text{for } 1 \le i \le d - s - 2 - j.$$

For $j \le m$, the two inequalities of i cover all indices $i \ge 1$. It follows that X is pre-m-Du Bois. □

Combining the two propositions above, we obtain the following corollary.

Corollary 4.6. *Let X be a d-dimensional variety with isolated singularities. Suppose one of the following assertions holds:*

(1) *X is a local complete intersection;*
(2) *$\operatorname{depth} \mathcal{H}^0 \underline{\Omega}^p_X \ge d - p$ for all $p \ge 0$.*

Then

$$\begin{aligned} X \text{ has pre-}m\text{-Du Bois singularities} &\iff X \text{ is } K_{-d+2m+1}\text{-regular} \\ &\iff X \text{ is } K_{-d+2m+2}\text{-regular} \end{aligned}$$

when $-d + 2m + 2 < 0$.

Proof. By Propositions 4.3 and 4.5 (applied with $s = 0$), we have the implications

$$K_{-d+2m+1}\text{-regular} \implies \text{pre-}m\text{-Du Bois} \implies K_{-d+2m+2}\text{-regular}.$$

Since $K_{-d+2m+2}$-regular implies $K_{-d+2m+1}$-regular, all three conditions are in fact equivalent. □

When $m = 0$, this gives the following result.

Corollary 4.7. *Let X be an affine variety of dimension $d \ge 2$. Suppose X is normal, Cohen–Macaulay with isolated singularities. Then the following assertions are equivalent:*

(1) *X has Du Bois singularities;*
(2) *X has pre-0-Du Bois singularities;*
(3) *X is K_{-d+1}-regular;*
(4) *X is K_{-d+2}-regular.*

Proof. Since X is Cohen–Macaulay, condition (2) in Corollary 4.6 is satisfied with $m = 0$. We thus have (2) $\iff$ (3) $\iff$ (4).

Since X is normal, hence seminormal, the equivalence (1) $\iff$ (2) follows from the well-known fact that $\mathcal{H}^0 \underline{\Omega}^0_{X/\mathbb{Q}} \simeq \mathcal{H}^0 \underline{\Omega}^0_{X/\mathbb{C}} \simeq \mathcal{O}_{X^{sn}}$, the structure sheaf of the seminormalization of X. □

Remark 4.8. The assumptions in Corollary 4.7 are satisfied for normal surfaces, so we have the following equivalences:

(1) X is K_{-1}-regular;
(2) X is K_0-regular;
(3) X has Du Bois singularities;
(4) X has pre-0-Du Bois singularities;
(5) X has pre-1-Du Bois singularities.

The equivalence of (1)–(3) with (4) is remarked in [PSV, Example 3.5]. See also Lemma 6.2.

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In [Wei01, Theorem 5.9], characterizations for K_{-1}-regularity that are more geometric in nature are obtained. For example, given a resolution of singularities $\tilde{X} \to X$ with exceptional divisor E, it is shown in loc. cit. that X is K_{-1}-regular if and only if

$$H^1(E, \mathcal{I}/\mathcal{I}^n) = 0 \quad \text{for all } n \geq 2,$$

where $\mathcal{I}$ is the ideal of $\mathcal{O}_{\tilde{X}}$ defining E.

One can directly see that this condition characterizes Du Bois singularities for normal surfaces, as follows: since X is seminormal, we have $\mathcal{H}^0\underline{\Omega}^0_X = \mathcal{O}_X$; moreover, by dimension considerations, $\mathcal{H}^2\underline{\Omega}^0_X = 0$. It follows that X has Du Bois singularities if and only if $\mathcal{H}^1\underline{\Omega}^0_X = 0$.

By Steenbrink's triangle for the Du Bois complexes,

$$\underline{\Omega}^0_X \to \mathcal{R}f_*\mathcal{O}_{\tilde{X}}(-E) \to \mathcal{O}_x \xrightarrow{+1},$$

where x is the isolated singularity of X. We thus see that

$$\mathcal{H}^1\underline{\Omega}^0_X = R^1 f_*\mathcal{O}_{\tilde{X}}(-E),$$

which by the formal function theorem is

$$R^1 f_*\mathcal{O}_{\tilde{X}}(-E) = \varprojlim_n H^1(E_n, \mathcal{O}_{\tilde{X}}(-E)|_{E_n}) = \varprojlim_n H^1(E, I/I^n).$$

Since $H^2(E, -) = 0$, from the short exact sequence

$$0 \to I^n/I^{n+1} \to I/I^{n+1} \to I/I^n \to 0$$

we see that for all $n \geq 1$,

$$H^1(E, I/I^n) = 0 \implies H^1(E, I/I^{n+1}) = 0.$$

It follows that

$$\begin{aligned} X \text{ is Du Bois} &\iff R^1 f_*\mathcal{O}_{\tilde{X}}(-E) = \varprojlim_n H^1(E, I/I^n) = 0 \\ &\iff H^1(E, I/I^n) = 0 \quad \text{for sufficiently large } n \\ &\iff H^1(E, I/I^n) = 0 \quad \text{for all } n. \end{aligned}$$

I am grateful to Sung Gi Park for telling me about this argument.

Remark 4.9. For a curve C, the works of Traverso [Tra70], Vorst [Vor79] and Weibel [Wei01] show that the following assertions hold:

- C is always K_{-1}-regular;
- C is K_0-regular $\iff$ C is K_1-regular $\iff$ C is Pic-regular $\iff$ C is seminormal. This fits into our Hodge theory picture, as a curve is seminormal if and only if it has Du Bois singularities;
- if C is K_2-regular, then it is smooth.

With a bit more work, one can extend Corollary 4.6 beyond the bound $m \leq (d-3)/2$ for local complete intersections of dimension greater than or equal to 2. We will need the smoothness criterion given by Theorem 2.3, as well as the following result about the torsion and cotorsion of the Kähler differentials. See [Gra15, Theorem 1.11] for the case of hypersurfaces, and [MV19, Corollary 3.1] for local complete intersections.

Theorem 4.10 (Graf, Miller–Vassiliadou). *Let X be a local complete intersection over F. Then the Kähler differential $\Omega^p_{X/F}$ is:*

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(1) *torsion-free if and only if* $p \leq \operatorname{codim} X_{\mathrm{sing}} - 1$;

(2) *reflexive, if and only if* $p \leq \operatorname{codim} X_{\mathrm{sing}} - 2$.

If it is reflexive, then we have $\Omega^p_{X/F} \simeq \mathcal{H}^0 \underline{\Omega}^p_{X/F}$.

Corollary 4.11. *Let X be a local complete intersection of dimension d with isolated singularities. Then*

$$\begin{aligned} X \text{ has } m\text{-Du Bois singularities} &\iff X \text{ is } K_{-d+2m+1}\text{-regular} \\ &\iff X \text{ is } K_{-d+2m+2}\text{-regular.} \end{aligned}$$

Proof. First, we show that

$$m\text{-Du Bois} \implies K_{-d+2m+2}\text{-regular.}$$

We check this case by case.

- If $-d+2m+2<0$, this follows by Proposition 4.3(1).
- If $-d+2m+2=0$, this follows by Proposition 4.3(2), because local complete intersections of dimension greater than or equal to 2 with isolated singularities are normal.
- If $-d+2m+2=1$, this follows by Proposition 4.3(3), because the surjectivity of $\Omega^1_{X/F} \to \mathcal{H}^0\underline{\Omega}^1_{X/F}$ (where F is the field of definition of X) holds by Theorem 4.10, as long as $d \geq 3$.
- If $-d+2m+2 \geq 2$, then $d \leq 2m$. Since X is m-Du Bois, it is smooth by Theorem 2.3, hence $K_{-d+2m+2}$-regular.

Next, we show that

$$K_{-d+2m+1}\text{-regular} \implies m\text{-Du Bois.}$$

By Proposition 4.5 applied with $s=0$, we know X is pre-m-Du Bois. If $m \leq d-2$, then X is m-Du Bois by Theorem 4.10. Thus, we may assume $m \geq d-1$. We consider several cases.

- If $d \geq 4$, then one can check that
$$m \geq d-1 \geq \left\lfloor \frac{d+1}{2} \right\rfloor \quad \text{and} \quad \left\lfloor \frac{d+1}{2} \right\rfloor \leq d-2,$$
so X is $\lfloor (d+1)/2 \rfloor$-Du Bois, hence smooth by Theorem 2.3.
- If $d=3$, then $m \geq 2$. By assumption X is K_2-regular, with the following consequences.
  - The map $\Omega^2_{X/\mathbb{Q}} \to \mathcal{H}^0\underline{\Omega}^2_{X/\mathbb{Q}}$ is surjective by Theorem 2.10. Since X is 1-Du Bois, Lemma 2.11(1) implies the map $\Omega^2_{X/F} \to \mathcal{H}^0\underline{\Omega}^2_{X/F}$ is surjective,
  - The map $\Omega^2_{X/F} \to \mathcal{H}^0\underline{\Omega}^2_{X/F}$ is injective by Theorem 4.10.

  It follows that X is 2-Du Bois, hence smooth by Theorem 2.3.
- If $d=2$, then $m \geq 1$. By assumption X is K_1-regular. Similar to the case $d=3$, one checks that X is 1-Du Bois, hence smooth.
- If $d=1$, then $m \geq 0$. If $m=0$, then X is K_0-regular, hence Pic-regular because $K_0(X) = \mathbb{Z} \oplus \operatorname{Pic}(X)$ for curves. By Traverso's theorem [Tra70, Theorem 3.5], X is seminormal, hence has Du Bois singularities; when $m \geq 1$, X is K_2-regular, hence smooth by Vorst's Theorem [Vor79, Theorem 3.4]. □

Corollary 4.12 ($\Longrightarrow$ Theorem A). *Let X be a complex local complete intersection with isolated singularities of dimension d. Let r be the codimension of X in its embedding, and $\tilde{\alpha}_X$*

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be the minimal exponent of X (which depends on the embedding). Then

$$X \text{ is } K_m\text{-regular} \iff \tilde{\alpha}_X \geq \left\lceil \frac{m+d}{2} \right\rceil + r - 1.$$

Proof. Combining Corollary 4.11 with the main theorems of [JKS+22, MOP+23, CDM24], stating that

$$X \text{ has } m\text{-Du Bois singularities} \iff \tilde{\alpha}_X \geq m + r,$$

we obtain the result. □

Remark 4.13. Similar arguments to Corollary 4.11 show that for an affine local complete intersection X with $\dim X_{\mathrm{sing}} = 1$, one has the following.

$$\begin{array}{ccc} K_{-d+1}\text{-regular} & & \\ \Uparrow & & \\ K_{-d+2}\text{-regular} & \Longleftrightarrow & \text{Du Bois} \\ \Uparrow & & \\ K_{-d+3}\text{-regular} & & \\ \Uparrow & & \\ K_{-d+4}\text{-regular} & \Longleftrightarrow & \text{1-Du Bois} \\ \Uparrow & & \\ \cdots & & \end{array}$$

In particular,

$$\tilde{\alpha}_X \geq m + r \iff X \text{ is } K_{-d+2m+2}\text{-regular}.$$

More generally, suppose X is an affine local complete intersection $\dim X_{\mathrm{sing}} = s$. Then

$$\begin{aligned} X \text{ is } K_{-d+2m+1+s}\text{-regular} &\Longrightarrow X \text{ has } m\text{-Du Bois singularities} \\ &\Longrightarrow X \text{ is } K_{-d+2m+2}\text{-regular}. \end{aligned}$$

Thus, K_m-regularity can be read off, at least partially, from the minimal exponent $\tilde{\alpha}_X$ and the codimension r of X in its embedding:

(1) X is K_m-regular for $m \leq -d + 2\tilde{\alpha}_X - 2r + 2$;
(2) X is not K_m-regular for $m = -d + 2\lceil \tilde{\alpha}_X \rceil - 2r + s$.

5. Application to regularity criteria

In this section, we prove the following regularity criterion. Recall that K_m-regularity implies K_{m-1}-regularity for all m (see [Vor79, DW80]), hence it immediately implies Theorem B.

THEOREM 5.1 ($\Longrightarrow$ Theorem B). *Let X be an affine local complete intersection, with singular locus X_{sing}. Then the following assertions hold.*

(1) *If $\operatorname{codim} X_{\mathrm{sing}}$ is odd, and X is K_2-regular, then it is regular.*
(2) *If $\operatorname{codim} X_{\mathrm{sing}}$ is even, and X is K_1-regular, then it is regular.*

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The proof is similar to that of Corollary 4.11, and relies on three main ingredients:

(1) a regularity criterion due to Mustaţă–Popa (Theorem 2.3) – the relevance of this result in relation to extensions of Vorst's conjecture is observed in [HW25];

(2) results due to Graf and to Miller and Vassiliadou about the torsion and cotorsion of the Kähler differentials of local complete intersections (Theorem 4.10);

(3) the relationship between K_m-regularity and pre-k-Du Bois singularities (Proposition 4.5), which in turn relies on a vanishing theorem of Mustaţă–Popa (Theorem 3.3).

Proof of Theorem B. Let F be the field of definition of X. By Theorem 2.3, regularity of X is implied by having m-Du Bois singularities for $m > (d-1)/2$.

(1) Suppose $\operatorname{codim} X_{\mathrm{sing}} = 2m+1$ is odd. To show X is smooth, it suffices to show X has $(m+1)$-Du Bois singularities, that is, the following two conditions hold.

– The intersection X has pre-$(m+1)$-Du Bois singularities. By Proposition 4.5, this is implied by $K_{-d+2(m+1)+1+s} = K_2$-regularity.

– The maps $\Omega^p_{X/F} \to \mathcal{H}^0\underline{\Omega}^p_{X/F}$ are isomorphisms for $p \le m+1$. By Theorem 4.10, this is the case when $p \le \operatorname{codim} X_{\mathrm{sing}} - 2$. Thus, we have the desired isomorphism when $m+1 \le \operatorname{codim} X_{\mathrm{sing}} - 2 = (2m+1) - 2$, that is, when $m \ge 2$.

When $m = 1$, we need to show $\Omega^2_{X/F} \to \mathcal{H}^0\underline{\Omega}^2_{X/F}$ is an isomorphism. By Theorem 2.8, K_2-regularity implies that this map is surjective with F replaced by $\mathbb{Q}$, and Lemma 2.11(2) implies the surjectivity over F; on the other hand, Theorem 4.10 implies that $\Omega^2_{X/F}$ is torsion-free, so the map is an isomorphism.

Finally, it is impossible that $m = 0$, as K_2-regularity implies X has 1-Du Bois singularities, so $\operatorname{codim} X_{\mathrm{sing}} \ge 3$.

Thus, X has $(m+1)$-Du Bois singularities, hence is smooth.

(2) Suppose $\operatorname{codim} X_{\mathrm{sing}} = 2m$ is even. To show X is smooth, it suffices to show X has m-Du Bois singularities, that is, the following two conditions hold.

– The intersection X has pre-m-Du Bois singularities. By Proposition 4.5, this is implied by $K_{-d+2m+1+s} = K_1$-regularity.

– The maps $\Omega^p_{X/F} \to \mathcal{H}^0\underline{\Omega}^p_{X/F}$ are isomorphisms for $p \le m$. By Theorem 4.10, this is the case when $p \le \operatorname{codim} X_{\mathrm{sing}} - 2$. Thus, we have the desired isomorphism when $m \le \operatorname{codim} X_{\mathrm{sing}} - 2 = 2m - 2$, that is, when $m \ge 2$.

When $m = 1$, we need to show $\Omega^1_{X/F} \to \mathcal{H}^0\underline{\Omega}^1_{X/F}$ is an isomorphism. By Theorem 2.8 and Lemma 2.11(1), K_1-regularity implies that this map is surjective; on the other hand, the results [Gra15, MV19] show that Ω^1_X is torsion-free, so the map is injective.

Thus, X has m-Du Bois singularities, hence is smooth. □

Remark 5.2. The proof above can be used to show the following fact: if X is a local complete intersection that is pre-m-Du Bois for all m, then either $\operatorname{codim} X_{\mathrm{sing}} \le 3$ or X is smooth.

The result above can be improved for local complete intersections of minimal embedding codimension $r \ge 2$.

Definition 5.3. The *minimal embedding codimension* of a local complete intersection X is the minimal integer r such that for every $x \in X$, there exist a regular local ring R, and a regular sequence $(f_1, \dots, f_r)$ in R, such that

$$\mathcal{O}_{X,x} \simeq R/(f_1, \dots, f_r).$$

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The following theorem, which I learned from B. Dirks, shows that the codimension bound in Theorem 2.3 can be improved when X is a local complete intersection of minimal embedding codimension r. The proof follows as in [CDM24, Corollary 1.3]. See [CHW+13, Theorem 3.5] for a detailed argument.

Theorem 5.4 (Bradley Dirks). *Let X be a local complete intersection of minimal embedding codimension r. Suppose X has m-Du Bois singularities. Then*

$$\operatorname{codim} X_{\mathrm{sing}} \geq 2m + r.$$

As a consequence, we have the following corollary.

Corollary 5.5. *Let X be an affine local complete intersection of minimal embedding codimension r. Then the following assertions hold.*

(1) *If $\operatorname{codim} X_{\mathrm{sing}} - r \geq 6 - 2r$ is even, and X is K_{3-r}-regular, then X is smooth.*
(2) *If $\operatorname{codim} X_{\mathrm{sing}} - r \geq 5 - 2r$ is odd, and X is K_{2-r}-regular, then X is smooth.*

Proof. By Theorem 5.4, a variety X with m-Du Bois singularities satisfies $\operatorname{codim} X_{\mathrm{sing}} \geq 2m + r$. Thus, (1) when $\operatorname{codim} X_{\mathrm{sing}} = 2m + r$, to show smoothness we need to show X is $(m+1)$-Du Bois, which is the case if the following conditions hold:

- X is pre-$(m+1)$-Du Bois, implied by $K_{-d+2(m+1)+1+s} = K_{3-r}$-regularity;
- Ω^p_X is reflexive for $p \leq m+1$; this is the case if $m+1 \leq \operatorname{codim} X_{\mathrm{sing}} - 2 = 2m + r - 2$, that is, when $m \geq 3 - r$.

The proof of (2) is similar. □

Remark 5.6. It is desirable to know about the implications of K_2-regularity beyond affine local complete intersections. For local complete intersections with isolated singularities (that are not necessarily affine), Theorem B and Corollary 5.5 continue to hold, thanks to the degeneration of the local-to-global spectral sequence

$$E_2^{pq} = H^p(X, \mathcal{NK}_q(X)) \implies NK_{q-p}(X).$$

When the singular locus has higher dimension, it is not known whether K_2-regular implies smooth; see Question 8.2.

Remark 5.7. By Theorem 2.8, if an affine variety X is K_2-regular, then the natural map

$$\Omega^1_X \to \mathcal{H}^0 \underline{\Omega}^1_X$$

is an isomorphism. In the study of higher Du Bois singularities beyond the local complete intersection case, one difficulty is that there is no known variety satisfying this condition; see Remark 2.5. This suggests also that K_2-regular varieties that are not local complete intersections (if there are any) are tricky to find.

Remark 5.8. By Theorem B, a normal K_1-regular surface is smooth if it is a local complete intersection. It is natural to ask if there is any non-local-complete-intersection surface that is K_1-regular but not smooth. By results in [Sri86] and [CHW+13], one can show that no affine cone over a smooth curve can be K_1-regular. I thank V. Srinivas for telling me about these references.

Note that the normality assumption is necessary: the surface $(xyz = 0) \subset \mathbb{A}^3$ is K_1-regular. In fact, any affine simple normal crossings divisor is K_1-regular.

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6. Bass's question and the Du Bois invariants

Let X be a variety over F. The Du Bois invariants, defined as

$$b^{p,q} = \dim_F \mathcal{H}^q \underline{\Omega}^p_{X/F}$$

for $q > 0$, are introduced by Steenbrink [Ste97] when X has isolated singularities. It follows from Proposition 2.2 that $b^{p,q} = 0$ for all $q \geq d := \dim X$. We arrange the invariants $b^{p,q}$ for $0 \leq p \leq d$ and $1 \leq q \leq d-1$ into a $(d-1) \times (d+1)$ table, and refer to it as the *Du Bois table*.

For surfaces over $\mathbb{Q}$, we have a 1×3 table of the following form.

$b^{0,1}$	$b^{1,1}$	$b^{2,1}$

In fact, it is well known that for surfaces, the only interesting invariants are $b^{0,1}$ and $b^{1,1}$.

Lemma 6.1 [CHW+11, Proposition 3.2]. *Let X be a surface over F. For $q > 0$ and $p \geq 0$, we have*

$$\mathcal{H}^q \underline{\Omega}^p_{X/F} = 0$$

unless $(p, q) = (0, 1)$ or $(1, 1)$.

Proof. By Steenbrink's vanishing theorem (see Theorem 3.1), we have $\mathcal{H}^q \underline{\Omega}^p_{X/F} = 0$ when $p + q > 2$. The vanishing when $(p, q) = (0, 2)$ follows from Proposition 2.2(1). □

Applications of the Du Bois invariants to K-theory started in [CHW+11], where the authors used an example of a surface with $b^{0,1} = 1$ and $b^{1,1} = 0$ to give negative answers to Bass's question [Bas73], which asks whether

$$K_m(X) \simeq K_m(X \times \mathbb{A}^1) \implies K_m(X) \simeq K_m(X \times \mathbb{A}^2).$$

Indeed, for surfaces, the main theorem of [CHW+10] implies that Bass's question for the negative K-groups has an affirmative answer if and only if $b^{0,1}$ and $b^{1,1}$ are both zero, or both non-zero. In view of the potential applications of these invariants to K-theory, it is desirable to have a better understanding of them. In this section, we give a complete classification of these invariants for surfaces.

First, along the lines of [PSV, Proposition C], we show that $b^{0,1}$ and $b^{1,1}$ are constrained by the inequality $b^{0,1} \geq b^{1,1}$.

Lemma 6.2. *Let X be a surface over F. Then the map*

$$d : \mathcal{H}^1 \underline{\Omega}^0_{X/F} \to \mathcal{H}^1 \underline{\Omega}^1_{X/F}$$

is surjective. In particular, we have $b^{0,1} \geq b^{1,1}$ when X has isolated singularities.

Proof. By Lemma 2.11(2), it suffices to prove the result when $F = \mathbb{C}$. In this case, we have by the Poincaré lemma that

$$\mathbb{C} \xrightarrow{\sim} (\underline{\Omega}^\bullet_{X/\mathbb{C}})^{\mathrm{an}}$$

as sheaves in the analytic topology on $X(\mathbb{C})$.

By construction of the Du Bois complex (in the analytic category, see [Du81]), we have a spectral sequence

$$E_1^{pq} = \mathcal{H}^q (\underline{\Omega}^p_{X/\mathbb{C}})^{\mathrm{an}} \mapsto \mathcal{H}^{p+q} (\underline{\Omega}^\bullet_{X/\mathbb{C}})^{\mathrm{an}}$$

converging to $(\underline{\Omega}^\bullet_{X/\mathbb{C}})^{\mathrm{an}} \simeq \mathbb{C}[0]$ supported in cohomological degree zero. Analyzing the maps in the spectral sequence shows that there is a surjection

$$d^{\mathrm{an}} : \mathcal{H}^1 (\underline{\Omega}^0_{X/\mathbb{C}})^{\mathrm{an}} \to \mathcal{H}^1 (\underline{\Omega}^1_{X/\mathbb{C}})^{\mathrm{an}}.$$

Using Serre's GAGA theorem, we then conclude that the map d is surjective. □

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In fact, the invariants $b^{0,1}$ and $b^{1,1}$ can take on arbitrary values.

Proposition 6.3. *Let X be a normal surface over $\mathbb{Q}$.*

(1) *If X is an affine cone over a projective curve, or a quasi-homogeneous hypersurface, then $b^{0,1} = b^{1,1}$; in these cases, Bass's question for K_m has positive answer when $m \leq 0$.*

(2) *Let $m \geq n \geq 0$ be any pair of non-negative integers. The hypersurface $(x^3 + y^3 + z^{3m} + xyz^{m+n} = 0)$ has Du Bois invariants*

$$b^{0,1} = m \quad \text{and} \quad b^{1,1} = n.$$

Proof. (1) When X is an affine cone over a projective curve C associated to a line bundle L on C, explicit calculation as in [CHW+13] and [PS] shows that

$$\mathcal{H}^1\underline{\Omega}^0_{X/\mathbb{Q}} = \mathcal{H}^1\underline{\Omega}^1_{X/\mathbb{Q}} \simeq \bigoplus_{m\geq 0} \mathbb{H}^1(C, \underline{\Omega}^0_{C/\mathbb{Q}} \otimes L^m).$$

For quasi-homogeneous hypersurfaces, we use van Straten and Steenbrink's computation of the geometric genus $p_g \leq b^{0,1}$ and the *alpha invariant* $\alpha := b^{0,1} - b^{1,1}$: by [vSS85, Theorem 3.1, Corollary 6.3], for a surface $X = (f = 0) \subset \mathbb{A}^3 = Y$ with isolated singularities at 0, we have

$$p_g = \dim Q^f/V_{>0}$$

and

$$p_g + \alpha = \dim Q^f/\overline{K} \cap V_{>0},$$

where $Q^f = \Omega^3_{Y,0}/df \wedge \Omega^2_{Y,0}$ and $\overline{K}$ is the kernel of the multiplication by f on Q^f. If f is quasi-homogeneous, multiplication by f is zero on Q^f, so $\overline{K} = Q^f$, and it follows that $\alpha = b^{0,1} - b^{1,1} = 0$, as claimed.

(2) For semi-quasi-homogeneous hypersurfaces of weight (w_x, w_y, w_z) with isolated singularities, the V-filtration on Q^f is computed in [Sai88]. One has that $V_{>l}Q^f$ is generated by monomials $x^a y^b z^c$ satisfying

$$w_x(a+1) + w_y(b+1) + w_z(c+1) - 1 > l.$$

In particular, for $f = x^3 + y^3 + z^{3m} + xyz^{m+n}$, we have

$$V_{>0} = (x, y, z^m),$$

and a Gröbner basis computation shows that

$$\overline{K} = (x, y, z^{2m-n}).$$

This gives that

$$p_g = \dim Q^f/V_{>0} = m \geq b^{0,1}$$

and

$$p_g + \alpha = \dim Q^f/\overline{K} \cap V_{>0} = 2m - n.$$

Next, using the deformation argument in [CHW+11] (see Theorem 2.14 and the proof of Proposition 4.3 in loc. cit.), we see that $b^{0,1}$ is independent of n. When $n = 0$, we have $\alpha = m$, and since $0 \leq b^{1,1} = b^{0,1} - \alpha = b^{0,1} - m \leq 0$, we have $b^{0,1} = m$ and $b^{1,1} = 0$. It follows that for $n \geq 1$, we have

$$b^{0,1} = m \quad \text{and} \quad b^{1,1} = n,$$

as required. □

Moving to higher dimensions, we show that for homogeneous hypersurfaces with isolated singularities, Bass's question has positive answer for the non-positive K-groups. The proof relies

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on the computation of the cohomology groups $H^q(S, \Omega^p_S(l)) = 0$ for smooth hypersurfaces in projective spaces. Complete computations can be found in [Vo24]. We state part of the theorem below in the form of what we need.

Theorem 6.4. *Let S be a smooth hypersurface in $\mathbb{P}^n$ of degree d. Then the following assertions hold.*

(1) *For $l \geq 0$, we have $H^q(S, \Omega^p_S(l)) = 0$ if $p + q \neq \dim S$.*
(2) *For $p + q = \dim S$,*

$$H^q(S, \Omega^p_S(l)) = 0 \text{ for all } l \geq 1 \iff (p+1)d \leq n+1.$$

Proof. The result follows from [Vo24, Theorem D, Proposition 1, Remark 1], and also the paragraphs before the proof of Theorem D, when $d \geq 2$ and $n \geq 4$.

When $d = 1$, the result follows from Bott vanishing for projective spaces; when $n = 1$, the statement is vacuous; when $n = 2, 3$, one can compute $H^q(S, \Omega^p_S(l))$ using the long exact sequence associated to the following short exact sequences (see [Vo24, Proposition 1] for one implication in (2) when $n = 3$):

$$\begin{aligned}
&0 \to \mathcal{O}_{\mathbb{P}^n}(-d+l) \to \mathcal{O}_{\mathbb{P}^n(l)} \to \mathcal{O}_S(l) \to 0, \quad n = 2, 3,\\
&0 \to \Omega^1_{\mathbb{P}^3}|_S(l) \to \mathcal{O}_S(l-1)^{\oplus 4} \to \mathcal{O}_S(l) \to 0,\\
&0 \to \mathcal{O}_S(l-d) \to \Omega^1_{\mathbb{P}^3}|_S(l) \to \Omega^1_S(l) \to 0,
\end{aligned}$$

and verify that the results hold true. □

Proposition 6.5. *Let X be a homogeneous hypersurface with at worst isolated singularities. Then for every m,*

$$K_m(X) \xrightarrow{\sim} K_m(X \times \mathbb{A}^1) \implies K_m(X) \xrightarrow{\sim} K_m(X \times \mathbb{A}^r)$$

for all $r \geq 1$. In particular, if $K_2(X) \simeq K_2(X \times \mathbb{A}^1)$, then X is smooth.

Proof. Since X has isolated singularities, we may assume it is affine. Let F be the field of definition of X. If F has infinite transcendence degree over $\mathbb{Q}$, then by [CHW+10, Theorem 0.2(b)], Bass's question has a positive answer. Thus, we may assume F is algebraic over $\mathbb{Q}$.

By [CHW+10, Theorem 0.1], the implication we need to show is equivalent to

$$NK_m(X) = 0 \implies NK_i(X) = 0 \quad \text{for } i < m.$$

Applying [CHW+10, Theorem 0.4], this is in turn equivalent to

$$\mathcal{H}^q C^p = 0 \text{ for } q - p = -m \implies \mathcal{H}^q C^p = 0 \text{ for } q - p = -m - 1,$$

where $C^p = \operatorname{Cone}(\mathbb{L}^p_{X/\mathbb{Q}} \to \underline{\Omega}^p_{X/\mathbb{Q}})$. Since F is algebraic over $\mathbb{Q}$, we have $\Omega^p_{F/\mathbb{Q}} = 0$ for all $p \geq 1$, so the spectral sequence in Lemma 2.11(1) degenerates, giving that

$$\mathcal{H}^q \underline{\Omega}^p_{X/F} \simeq \mathcal{H}^q \underline{\Omega}^p_{X/\mathbb{Q}}$$

and

$$\mathcal{H}^q \mathbb{L}^p_{X/F} \simeq \mathcal{H}^q \mathbb{L}^p_{X/\mathbb{Q}}.$$

Note that since X is a hypersurface with isolated singularities, as in the proof of Lemma 2.4, we have $\mathbb{L}^p_{X/F} \simeq \Omega^p_{X/F}$ for $p \leq \dim X$. Moreover, by Theorem 4.10, we have $\mathcal{H}^0 C^p = 0$ for $p \leq \dim X - 2$ and $\mathcal{H}^{-1} C^p = 0$ for $p \leq \dim X - 1$. Consequently, the exact sequences in [CHW+13, Proposition 1.9] give the isomorphisms

$$\operatorname{tors} \Omega^n_{X/F} \simeq \operatorname{tors} \Omega^{n+1}_{X/F}$$

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and

$$\mathcal{H}^0\underline{\Omega}^{n-1}_{X/F}/\Omega^{n-1}_{X/F} \simeq \mathcal{H}^0\underline{\Omega}^{n}_{X/F}/\Omega^{n}_{X/F},$$

where $n = \dim X$. Therefore, it suffices for us to check

$$\mathcal{H}^q\underline{\Omega}^p_{X/F} = 0 \text{ for } q-p=-m \implies \mathcal{H}^q\underline{\Omega}^p_{X/F} = 0 \text{ for } q-p=-m-1.$$

Since X is a homogeneous hypersurface with isolated singularities, we can view it as an affine cone over a smooth projective hypersurface S. Using the computation of $\mathcal{H}^q\underline{\Omega}^p_{X/F}$ for cones [CHW+13], as well as Theorem 6.4, we see that the Du Bois table of X takes the following form.

$$\begin{array}{lll} & b^{0,d}=0 & \\ b^{0,d-1}=\bigoplus_{m\geq 1} H^{d-1}(\mathcal{O}_S(m)) & b^{1,d-1}=\bigoplus_{m\geq 1} H^{d-1}(\mathcal{O}_S(m)) & \\ & b^{1,d-2}=\bigoplus_{m\geq 1} H^{d-2}(\Omega^1_S(m)) & b^{2,d-2}=\bigoplus_{m\geq 1} H^{d-2}(\Omega^1_S(m)) \\ & & b^{2,d-3}=\bigoplus_{m\geq 1} H^{d-3}(\Omega^2_S(m)) \\ & & \cdots \end{array}$$

Thus, we only need to check

$$H^q(\Omega^p_S(m)) = 0 \text{ for all } m \geq 1 \implies H^{q+1}(\Omega^{p-1}_S(m)) = 0 \text{ for all } m \geq 1$$

for $p+q = \dim S$. This follows from Theorem 6.4(2). □

Remark 6.6. With notation as in the proof above, we have that for $p \leq \dim X - 2$, the following assertions are equivalent:

(1) X has (pre-)p-Du Bois singularities;
(2) X is $K_{-d+2+2p}$-regular;
(3) $NK_{-d+2+2p}(X) = 0$;
(4) $\deg f \leq (d+1)/(p+1)$.

For X to be singular, one needs $2 \leq \deg f \leq (d+1)/(p+1)$, in which case $-d+2+2p \leq 1$, so X can be at most K_1-regular, as predicted by Theorem B.

7. K-regularity for projective varieties

In this section, we apply the main theorem of [CHS+08] to characterize K-regularity for projective varieties. I thank Dori Bejleri, Anh Duc Vo, and especially Elden Elmanto for discussions that shaped the contents presented here.

Our main result is the following characterization of K_m-regularity for projective varieties over a field F of characteristic zero.

Theorem 7.1 ($\implies$ Theorem D). *Let X be a projective variety over F. Then X is K_m-regular if and only if*

$$\mathbb{H}^i(X, \mathbb{L}^p_{X/F}) \xrightarrow{\sim} \mathbb{H}^i(X, \underline{\Omega}^p_{X/F})$$

are isomorphisms for $i - p \geq -m+1$.

Remark 7.2. Let X be a projective variety of dimension d. It follows from Theorem 7.1 that

$$X \text{ is } K_{-d+1}\text{-regular} \iff \mathbb{H}^d(X, \mathcal{O}_X) \to \mathbb{H}^d(X, \underline{\Omega}^0_X) \text{ is an isomorphism.}$$

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In particular, if X has Du Bois singularities then it is K_{-d+1}-regular, which is consistent with Proposition 4.1(1).

By [CHS+08, Theorem 5.5], there is a homotopy fibration sequence

$$\tilde{C}_j\mathbf{HC} \to \tilde{C}_j L_{\mathrm{cdh}}\mathbf{HC} \to \tilde{C}_j\mathcal{K},$$

where $\mathcal{K}$ denotes the non-connective K-theory spectrum of perfect complexes on Sch/F, $\tilde{C}_j\mathcal{E}$ denotes the cofiber of the map $\mathcal{E} \to \mathcal{E}(- \times \mathbb{A}^j)$ for a presheaf of spectra on Sch/F, and $\mathbf{HC}$ is the presheaf of complexes computing cyclic cohomologies over $\mathbb{Q}$. Following loc. cit., we write H^i_{cdh} for the cohomology taken in the cdh topology, and H^i for the cohomology taken in the Zariski topology. By definition, a scheme X is K_m-regular if and only if $\pi_m\tilde{C}_j\mathcal{K}(X) = \ker(K_m(X) \to K_m(X \times \mathbb{A}^j)) = 0$ for all $j \geq 0$.

Taking cohomology, we obtain a long exact sequence

$$\begin{aligned}\cdots \to & H^{d-1}(\tilde{C}_j\mathbf{HC}(X)) \to H^{d-1}_{\mathrm{cdh}}(\tilde{C}_j\mathbf{HC}(X)) \to \pi_{-d+1}\tilde{C}_j\mathcal{K}(X)\\ & \to H^d(\tilde{C}_j\mathbf{HC}(X)) \to H^d_{\mathrm{cdh}}(\tilde{C}_j\mathbf{HC}(X)) \to \pi_{-d}\tilde{C}_j\mathcal{K}(X) \to H^{d+1}(\tilde{C}_j\mathbf{HC}(X)). \qquad (2)\end{aligned}$$

Thus, Theorem 7.1 can be proved by analyzing the surjectivity of the maps

$$H^i(\tilde{C}_j\mathbf{HC}(X)) \to H^i_{\mathrm{cdh}}(\tilde{C}_j\mathbf{HC}(X)).$$

The main Hodge-theoretic input for the proof is the following result, whose statement goes back to at least Du Bois [Du81] when $F = \mathbb{C}$ and $p = 0$.

Proposition 7.3. *Let X be a projective variety over F. Then there is a surjection*

$$\mathbb{H}^i(X, L\Omega^{\leq p}_{X/F}) \to \mathbb{H}^i(X, \underline{\Omega}^{\leq p}_{X/F})$$

for all integers i and $p \geq 0$.

Proof. By standard base change arguments (see the footnote to the proof of Lemma 2.11), we reduce to the case when $F = \mathbb{C}$. When X is a complex projective variety, the Hodge-to-de-Rham spectral sequence

$$E_1^{p,q} = H^q(X, \underline{\Omega}^p_{X/\mathbb{C}}) \implies \mathbb{H}^{p+q}(X, \mathbb{C})$$

degenerates at E_1, giving a surjection

$$H^i(X, \mathbb{C}) \to H^i(X, \mathbb{C})/F^{p+1}H^i(X, \mathbb{C}) = \mathbb{H}^i(X, \underline{\Omega}^\bullet_{X/\mathbb{C}}/F^{p+1}\underline{\Omega}^\bullet_{X/\mathbb{C}}) =: \mathbb{H}^i(X, \underline{\Omega}^{\leq p}_{X/\mathbb{C}})$$

for all integers i and $p \geq 0$.

By [Bha12], the derived de Rham complex $\widehat{\mathrm{dR}}_{X/\mathbb{C}}$ computes the Betti cohomology of $X(\mathbb{C})$, so we have the following commutative diagram.

$$\begin{array}{ccc} H^i(X, \mathbb{C}) & \twoheadrightarrow & H^i_{\mathrm{cdh}}(X, \Omega^{\leq p}_{X/\mathbb{C}}) \simeq \mathbb{H}^i(X, \underline{\Omega}^{\leq p}_{X/\mathbb{C}}) \\ \sim\big\downarrow & & \big\uparrow \\ H^i(X, \widehat{\mathrm{dR}}_{X/\mathbb{C}}) & \longrightarrow & H^i(X, L\Omega^{\leq p}_{X/\mathbb{C}}) \end{array}$$

Since the top horizontal row is a surjection, we have that the maps

$$H^i(X, L\Omega^{\leq p}_{X/\mathbb{C}}) \to H^i_{\mathrm{cdh}}(X, \Omega^{\leq p}_{X/\mathbb{C}}) \simeq \mathbb{H}^i(X, \underline{\Omega}^{\leq p}_{X/\mathbb{C}})$$

are surjective, for all integers i and $p \geq 0$. □

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We are now ready to prove the main theorem of this section.

Proof of Theorem 7.1. First assume X is K_m-regular. We want to show

$$\mathbb{H}^i(X, \mathbb{L}^p_{X/F}) \xrightarrow{\sim} \mathbb{H}^i(X, \underline{\Omega}^p_{X/F})$$

are isomorphisms when $i - p \geq -m+1$. By (2), we have that

$$H^i(X, \tilde{C}_j \mathbf{HC}(X)) \xrightarrow{\sim} H^i_{\mathrm{cdh}}(X, \tilde{C}_j \mathbf{HC}(X))$$

are isomorphisms for $i \geq -m+1$. Using the decomposition of cyclic homology

$$\mathbf{HC}(X) = \bigoplus_{p \geq 0} L\Omega^{\leq p}_{X/\mathbb{Q}}[2p],$$

and the fact that the cdh sheafification of $L\Omega^{\leq p}_{X/\mathbb{Q}}$ is $\underline{\Omega}^{\leq p}_{X/\mathbb{Q}}$, we see that

$$\mathbb{H}^{i+2p}(X, \tilde{C}_j L\Omega^{\leq p}_{X/\mathbb{Q}}) \xrightarrow{\sim} \mathbb{H}^{i+2p}_{\mathrm{cdh}}(X, \tilde{C}_j \underline{\Omega}^{\leq p}_{X/\mathbb{Q}})$$

are isomorphisms for all $i \geq -m+1$ and $p \geq 0$.[4]

Claim 1. *The maps* $\mathbb{H}^{i+2p}(X, \tilde{C}_j L\Omega^{\leq p}_{X/\mathbb{Q}}) \xrightarrow{\sim} \mathbb{H}^{i+2p}_{\mathrm{cdh}}(X, \tilde{C}_j \underline{\Omega}^{\leq p}_{X/\mathbb{Q}})$ *are isomorphisms for all* $i \geq -m+1$ *and* $p \geq 0$. *By induction, it suffices to show* $\mathbb{H}^{-m+1+2p}(X, \tilde{C}_j L\Omega^{\leq p}_{X/\mathbb{Q}}) \xrightarrow{\sim} \mathbb{H}^{-m+1+2p}_{\mathrm{cdh}}(X, \tilde{C}_j \underline{\Omega}^{\leq p}_{X/\mathbb{Q}})$ *are isomorphisms for all* $p \geq 0$.

Proof of Claim 1. Applying $\tilde{C}_j$ in Lemma 2.11(1), we obtain spectral sequences

$$E_1^{ij} = \Omega^i_{F/\mathbb{Q}} \otimes \mathbb{H}^j(X, L\Omega^{\leq p-i}_{X/F}) \implies \mathbb{H}^{i+j}(X, L\Omega^{\leq p}_{X/\mathbb{Q}}),$$
$$E_1^{ij} = \Omega^i_{F/\mathbb{Q}} \otimes \mathbb{H}^j(X, \underline{\Omega}^{\leq p-i}_{X/F}) \implies \mathbb{H}^{i+j}(X, \underline{\Omega}^{\leq p}_{X/\mathbb{Q}}).$$

Analyzing the spectral sequence, we see that it is enough to show the maps

$$\mathbb{H}^{-m+2p-1-i}(X, \tilde{C}_j L\Omega^{\leq p-i}_{X/F}) \to \mathbb{H}^{-m+2p-1-i}(X, \tilde{C}_j \underline{\Omega}^{\leq p-i}_{X/F})$$

are surjective for all $i \geq 0$. This is implied by the surjectivity of the maps

$$\mathbb{H}^{-m+2p-1-i}(X, L\Omega^{\leq p-i}_{X \times \mathbb{A}^j/F}) \to \mathbb{H}^{-m+2p-1-i}(X, \underline{\Omega}^{\leq p-i}_{X \times \mathbb{A}^j/F})$$

for $i \geq 0$. To prove the latter surjectivity, we analyze the spectral sequences, proved along the same lines as Lemma 2.11:

$$E_1^{ij} = \mathbb{H}^j(p^* L\Omega^i_{X/F} \otimes \Omega^{\leq p-i}_{X \times \mathbb{A}^j/X}) \implies \mathbb{H}^{i+j}(L\Omega^{\leq p}_{X \times \mathbb{A}^j/k}),$$
$$E_1^{ij} = \mathbb{H}^j(p^* \underline{\Omega}^i_{X/F} \otimes \Omega^{\leq p-i}_{X \times \mathbb{A}^j/X}) \implies \mathbb{H}^{i+j}(\underline{\Omega}^{\leq p}_{X \times \mathbb{A}^j/k}).$$

We need to check surjectivity of the maps $E_1^{ij} \to \tilde{E}_1^{ij}$ for $i + j = -m + 2p - 1 - i$. That is, we need to show the maps

$$\mathbb{H}^{-m+2p-1-i-k}(p^* L\Omega^k_{X/F} \otimes \Omega^{\leq p-i-k}_{X \times \mathbb{A}^j/X}) \to \mathbb{H}^{-m+2p-1-i-k}(p^* \underline{\Omega}^k_{X/F} \otimes \Omega^{\leq p-i-k}_{X \times \mathbb{A}^j/X})$$

are surjective for $0 \leq k \leq p - i$. By induction using long exact sequences associated to

$$\Omega^{p-i-k}_{X \times \mathbb{A}^j/X}[-p+i+k] \to \Omega^{\leq p-i-k}_{X \times \mathbb{A}^j/X} \to \Omega^{\leq p-i-k-1}_{X \times \mathbb{A}^j/X} \xrightarrow{+1},$$

[4] It is unclear that this directly implies

$$\mathbb{H}^{i+2p}(X, L\Omega^{\leq p}_{X/\mathbb{Q}}) \xrightarrow{\sim} \mathbb{H}^{i+2p}_{\mathrm{cdh}}(X, \underline{\Omega}^{\leq p}_{X/\mathbb{Q}})$$

are isomorphisms for all $i \geq -m+1$ and $p \geq 0$, so the argument that follows takes a somewhat roundabout path.

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it suffices to show the maps

$$\mathbb{H}^{-m+p-1}(L\Omega^k_{X/F}) \to \mathbb{H}^{-m+p-1}(\underline{\Omega}^k_{X/F})$$

are surjective, for all $p \geq 0$ and $k \leq p$. This follows by induction using

$$\mathbb{L}^k_{X/F}[-k] \to L\Omega^{\leq k}_{X/F} \to L\Omega^{\leq k-1}_{X/F} \xrightarrow{+1},$$
$$\underline{\Omega}^k_{X/F}[-k] \to \underline{\Omega}^{\leq k}_{X/F} \to \underline{\Omega}^{\leq k-1}_{X/F} \xrightarrow{+1}$$

and Proposition 7.3. This finishes the proof of Claim 1. □

Claim 2. *The maps* $\mathbb{H}^{i+p}(\tilde{C}_j\mathbb{L}^p_{X/F}) \xrightarrow{\sim} \mathbb{H}^{i+p}_{\mathrm{cdh}}(\tilde{C}_j\underline{\Omega}^p_{X/F})$ *are isomorphisms for all* $i \geq -m+1$ *and* $p \geq 0$.

Proof of Claim 2. Indeed, this follows by induction using

$$\tilde{C}_j\mathbb{L}^p_{X/F}[-p] \to \tilde{C}_jL\Omega^{\leq p}_{X/F} \to \tilde{C}_jL\Omega^{\leq p-1}_{X/F} \xrightarrow{+1},$$
$$\tilde{C}_j\underline{\Omega}^p_{X/F}[-p] \to \tilde{C}_j\underline{\Omega}^{\leq p}_{X/F} \to \tilde{C}_j\underline{\Omega}^{\leq p-1}_{X/F} \xrightarrow{+1}$$

and Proposition 7.3. □

Claim 3. *The maps* $\mathbb{H}^{i+p}(\mathbb{L}^p_{X/F}) \xrightarrow{\sim} \mathbb{H}^{i+p}_{\mathrm{cdh}}(\underline{\Omega}^p_{X/F})$ *are isomorphisms for all* $i \geq -m+1$ *and* $p \geq 0$.

Proof of Claim 3. We have a split exact triangle (the splitting is induced by the zero section $s : X \to X \times \mathbb{A}^j$)

$$p^*\Omega^1_{X/F} \to \Omega^1_{X\times\mathbb{A}^j/F} \to p^*\mathcal{O}_X \otimes \Omega^1_{X\times\mathbb{A}^j/X} \xrightarrow{+1}.$$

Taking the wedge product gives

$$\Omega^p_{X\times\mathbb{A}^j/F} \simeq \bigoplus_{i=0}^{p} p^*\Omega^i_X \otimes \Omega^{p-i}_{X\times\mathbb{A}^j/X}$$

(with the convention $\Omega^p_{X\times\mathbb{A}^j/X} = \mathcal{O}_{X\times\mathbb{A}^j}$ when $p = 0$).

Taking derived functor and cdh sheafification gives

$$\mathbb{L}^p_{X\times\mathbb{A}^j/F} \simeq \bigoplus_{i=0}^{p} p^*\mathbb{L}^i_X \otimes \Omega^{p-i}_{X\times\mathbb{A}^j/X},$$
$$\underline{\Omega}^p_{X\times\mathbb{A}^j/F} \simeq \bigoplus_{i=0}^{p} p^*\underline{\Omega}^i_X \otimes \Omega^{p-i}_{X\times\mathbb{A}^j/X}.$$

Thus,

$$\mathbb{H}^{i+p}(\tilde{C}_j\mathbb{L}^p_{X\times\mathbb{A}^j/F}) \simeq \mathbb{H}^{i+p}(\mathbb{L}^p_X \otimes V_j) \oplus \bigoplus_{i=0}^{p-1} \mathbb{H}^{i+p}(p^*\mathbb{L}^i_X \otimes \Omega^{p-i}_{X\times\mathbb{A}^j/X}),$$
$$\mathbb{H}^{i+p}(\tilde{C}_j\underline{\Omega}^p_{X\times\mathbb{A}^j/F}) \simeq \mathbb{H}^{i+p}(\underline{\Omega}^p_X \otimes V_j) \oplus \bigoplus_{i=0}^{p-1} \mathbb{H}^{i+p}(p^*\underline{\Omega}^i_X \otimes \Omega^{p-i}_{X\times\mathbb{A}^j/X}),$$

where $V_j = F[t_1, \ldots, t_j]/F$.

Since the maps in Claim 2 are compatible with this direct sum decomposition, the maps $\mathbb{H}^{i+p}(\mathbb{L}^p_{X/F}) \xrightarrow{\sim} \mathbb{H}^{i+p}_{\mathrm{cdh}}(\underline{\Omega}^p_{X/F})$ are isomorphisms for all $i \geq -m+1$ and $p \geq 0$. □

This completes the proof in one direction. For the converse, assume that the maps

$$\mathbb{H}^i(X, \mathbb{L}^p_{X/F}) \xrightarrow{\sim} \mathbb{H}^i(X, \underline{\Omega}^p_{X/F})$$

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are isomorphisms for $i-p \geq -m+1$. We will show X is K_m-regular. By (2) and the decomposition of cyclic homology, we need to show:

(1) $\mathbb{H}^{-m+1+2p}(X, \tilde{C}_j L\Omega^{\leq p}_{X/\mathbb{Q}}) \xrightarrow{\sim} \mathbb{H}^{-m+1+2p}_{\mathrm{cdh}}(X, \tilde{C}_j \underline{\Omega}^{\leq p}_{X/\mathbb{Q}})$ are isomorphisms for all $p \geq 0$;

(2) $\mathbb{H}^{-m+2p}(X, \tilde{C}_j L\Omega^{\leq p}_{X/\mathbb{Q}}) \xrightarrow{\sim} \mathbb{H}^{-m+2p}_{\mathrm{cdh}}(X, \tilde{C}_j \underline{\Omega}^{\leq p}_{X/\mathbb{Q}})$ are surjective for all $p \geq 0$.

Analyzing the spectral sequences

$$E_1^{ij} = \Omega^i_{F/k} \otimes \tilde{C}_j \mathbb{H}^j(X, L\Omega^{\leq p-i}_{X/F}) \Longrightarrow \tilde{C}_j \mathbb{H}^{i+j}(X, L\Omega^{\leq p}_{X/\mathbb{Q}}),$$

$$E_1^{ij} = \Omega^i_{F/k} \otimes \tilde{C}_j \mathbb{H}^j(X, \underline{\Omega}^{\leq p-i}_{X/F}) \Longrightarrow \tilde{C}_j \mathbb{H}^{i+j}(X, \underline{\Omega}^{\leq p}_{X/\mathbb{Q}})$$

shows that it is enough to show:

(1) $\mathbb{H}^{-m+1+2p}(X, \tilde{C}_j L\Omega^{\leq p}_{X/F}) \xrightarrow{\sim} \mathbb{H}^{-m+1+2p}_{\mathrm{cdh}}(X, \tilde{C}_j \underline{\Omega}^{\leq p}_{X/F})$ are isomorphisms for all $p \geq 0$;

(2) $\mathbb{H}^{-m+2p}(X, \tilde{C}_j L\Omega^{\leq p}_{X/F}) \xrightarrow{\sim} \mathbb{H}^{-m+2p}_{\mathrm{cdh}}(X, \tilde{C}_j \underline{\Omega}^{\leq p}_{X/F})$ are surjective for all $p \geq 0$.

By definition of $\tilde{C}_j$, it suffices to show:

(1) $\mathbb{H}^{-m+1+2p}(X, L\Omega^{\leq p}_{X/F}) \xrightarrow{\sim} \mathbb{H}^{-m+1+2p}(X, \underline{\Omega}^{\leq p}_{X/F})$ are isomorphisms for all $p \geq 0$;

(2) $\mathbb{H}^{-m+1+2p}(X \times \mathbb{A}^j, L\Omega^{\leq p}_{X\times\mathbb{A}^j/F}) \xrightarrow{\sim} \mathbb{H}^{-m+1+2p}(X \times \mathbb{A}^j, \underline{\Omega}^{\leq p}_{X\times\mathbb{A}^j/F})$ are isomorphisms for all $p, j \geq 0$;

(3) $\mathbb{H}^{-m+2p}(X \times \mathbb{A}^j, L\Omega^{\leq p}_{X\times\mathbb{A}^j/F}) \xrightarrow{\sim} \mathbb{H}^{-m+2p}(X \times \mathbb{A}^j, \underline{\Omega}^{\leq p}_{X\times\mathbb{A}^j/F})$ are surjective for all $p, j \geq 0$.

Part (1) follows by assumptions. For (2) and (3), analyzing the spectral sequences

$$E_1^{ij} = \mathbb{H}^j(p^* L\Omega^i_{X/F} \otimes \Omega^{\leq p-i}_{X\times\mathbb{A}^j/X}) \Longrightarrow \mathbb{H}^{i+j}(L\Omega^{\leq p}_{X\times\mathbb{A}^j/F}),$$

$$E_1^{ij} = \mathbb{H}^j(p^* \underline{\Omega}^i_{X/F} \otimes \Omega^{\leq p-i}_{X\times\mathbb{A}^j/X}) \Longrightarrow \mathbb{H}^{i+j}(\underline{\Omega}^{\leq p}_{X\times\mathbb{A}^j/F}),$$

we see that it suffices to show:

(1) $\mathbb{H}^{-m+1+2p-k}(p^* L\Omega^k_{X/F} \otimes \Omega^{\leq p-k}_{X\times\mathbb{A}^j/X}) \to \mathbb{H}^{-m+1+2p-k}(p^* \underline{\Omega}^k_{X/F} \otimes \Omega^{\leq p-k}_{X\times\mathbb{A}^j/X})$ are isomorphisms for all $p \geq 0$ and $k \leq p$;

(2) $\mathbb{H}^{-m+2p-k}(p^* L\Omega^k_{X/F} \otimes \Omega^{\leq p-k}_{X\times\mathbb{A}^j/X}) \to \mathbb{H}^{-m+2p-k}(p^* \underline{\Omega}^k_{X/F} \otimes \Omega^{\leq p-k}_{X\times\mathbb{A}^j/X})$ are surjective for all $p \geq 0$ and $k \leq p$.

By induction using long exact sequences associated to

$$\Omega^{p-k}_{X\times\mathbb{A}^j/X}[-p+k] \to \Omega^{\leq p-k}_{X\times\mathbb{A}^j/X} \to \Omega^{\leq p-k-1}_{X\times\mathbb{A}^j/X} \xrightarrow{+1},$$

it suffices to show:

(1) $\mathbb{H}^{-m+1+p}(L\Omega^k_{X/F}) \to \mathbb{H}^{-m+1+p}(\underline{\Omega}^k_{X/F})$ are isomorphisms for all $p \geq 0$ and $k \leq p$;

(2) $\mathbb{H}^{-m+p}(L\Omega^k_{X/F}) \to \mathbb{H}^{-m+p}(\underline{\Omega}^k_{X/F})$ are surjective for all $p \geq 0$ and $k \leq p$.

We have (1) by assumption, and (2) follows by Proposition 7.3.

As an application, we construct an example that shows local K_m-regularity does not imply K_m-regularity in general.

Example 7.4. For this example, we assume all varieties are defined over $\mathbb{C}$, and we write $\underline{\Omega}^p_X$ for $\underline{\Omega}^p_{X/\mathbb{C}}$.

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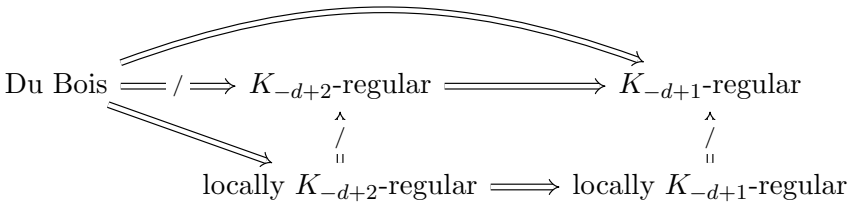
Let X be any projective surface with $\mathcal{H}^1\underline{\Omega}^0_X \simeq \mathbb{C}$ (e.g., the projective cone over a curve of genus 2), and Y be a curve of genus $g \geq 1$, so that $H^1(Y, \mathcal{O}_Y) \cong \mathbb{C}$. Take $Z = X \times Y$. Then there is an exact triangle

$$\mathcal{O}_Z \to \underline{\Omega}^0_Z \to \mathcal{O}_Y[-1] \xrightarrow{+1}.$$

Since $\mathcal{H}^2\underline{\Omega}^0_Z = 0$, we see that Z is locally K_{-2}-regular. However Z is not K_{-2}-regular by Theorem 7.1, because

$$\mathbb{H}^2(\mathcal{O}_Y[-1]) \simeq H^1(\mathcal{O}_Y) \neq 0.$$

For this reason, we do not expect pre-m-Du Bois varieties to be K_{-d+m+2}-regular in general. We summarize the relationships between these concepts when $m = 0$ as follows.



We conclude with the following example, which shows that Proposition 4.1(2) need not hold if X is not affine, and has higher-dimensional singular locus.

Example 7.5. We construct a 3-fold X that is K_{-1}-regular, but not pre-0-Du Bois. It is non-affine with one-dimensional singular locus. For this example, we assume all varieties are defined over $\mathbb{C}$, and we write $\underline{\Omega}^p_X$ for $\underline{\Omega}^p_{X/\mathbb{C}}$.

Let C be a genus 2 curve in $\mathbb{P}^2$, and let $S = C \times \mathbb{P}^1$. Denote by $\pi : S \to \mathbb{P}^1$ the projection map. The line bundle

$$\begin{aligned} L &= \omega_S \otimes (\pi^*\omega_{\mathbb{P}^1}(1))^\vee \\ &= \omega_{S/\mathbb{P}^1} \otimes (\pi^*\mathcal{O}_{\mathbb{P}^1}(1))^\vee \end{aligned}$$

is relatively ample over $\mathbb{P}^1$, so we can consider the relative affine cone

$$Y = \mathbf{Spec}_{\mathbb{P}^1} \bigoplus_{m \geq 0} \pi_* L^m$$

over S defined by L. One checks that the arguments in [CHW+13] computing the Du Bois complexes of cones carry over to the relative case, giving that

$$\mathcal{H}^1\underline{\Omega}^0_Y \simeq \bigoplus_{m \geq 1} R^1\pi_* L^m$$

is a sheaf supported on $Y_{\mathrm{sing}} \simeq \mathbb{P}^1$. We have $R^1\pi_* L \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$ by Grothendieck duality, and $R^1\pi_* L^m = 0$ by Kodaira vanishing. It follows that

$$\mathcal{H}^1\underline{\Omega}^0_Y \simeq \mathcal{O}_{\mathbb{P}^1}(-1).$$

Similarly, one checks that

$$\mathcal{H}^1\underline{\Omega}^1_Y \simeq \mathcal{O}_{\mathbb{P}^1}(-1).$$

Now, let X be a projectivization of Y that is smooth away from the singular points of Y. Then we have

$$\mathcal{H}^1\underline{\Omega}^0_X \simeq \mathcal{H}^1\underline{\Omega}^1_X \simeq \mathcal{O}_{\mathbb{P}^1}(-1).$$

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In particular, X is not pre-0-Du Bois. We claim that X is K_{-1}-regular. Indeed, by Theorem 7.1, this is equivalent to the following maps being isomorphisms:

$$\mathbb{H}^3(\mathcal{O}) \xrightarrow{\sim} \mathbb{H}^3(\underline{\Omega}^0_X),$$
$$\mathbb{H}^2(\mathcal{O}) \xrightarrow{\sim} \mathbb{H}^2(\underline{\Omega}^0_X),$$
$$\mathbb{H}^3(\mathbb{L}_X) \xrightarrow{\sim} \mathbb{H}^3(\underline{\Omega}^1_X).$$

Since the cone of the map $\mathcal{O}_X \to \underline{\Omega}^0_X$ is $\mathcal{O}_{\mathbb{P}^1}(-1)[-1]$, which has no cohomology, we have the two isomorphisms. The cone of the map $\mathbb{L}_X \to \underline{\Omega}^1_X$ is $\mathcal{O}_{\mathbb{P}^1}(-1)[-1] \oplus C$, where the complex C is supported on the one-dimensional subvariety X_{sing}, and in cohomological degrees less than or equal to 0. Analyzing the spectral sequence

$$E_2^{ij} = H^i(\mathcal{H}^j C) \implies \mathbb{H}^{i+j}(C)$$

gives the last isomorphism.

8. Further questions

(1) As noted in [HW25], being K_m-regular also implies *strict* $(m-1)$-*Du Bois*, that is, the maps

$$\Omega^p_{X/\mathbb{C}} \to \underline{\Omega}^p_{X/\mathbb{C}}$$

are isomorphisms for $p \leq m-1$. Since there is no known example of a variety with strict-1-Du Bois singularities that is not a local complete intersection, this leads to the following question.

Question 8.1. Is there a K_2-regular variety that is not a local complete intersection?

(2) We showed in Theorem B that K_2-regular affine local complete intersections are smooth. We do not know if this is the case when the variety is not necessarily affine.

Problem 8.2. Give examples of singular non-affine varieties that are K_2-regular.

One approach could be to construct projective varieties that are K_2-regular using Theorem 7.1.

(3) The notion of *higher-F-injectivity* is introduced in [KW24] as an analog of higher Du Bois singularities in positive characteristic, extending the well-known notion of F-injectivity (analogous to the Du Bois singularities for complex algebraic varieties). One can ask the following question.

Question 8.3. Is there any relationship between higher F-injectivity and K-regularity in positive characteristic?

(4) In § 6, we classified all possible Du Bois tables in dimension 2. We ask if one can extend this classification to higher dimensions.

Question 8.4. What are all possible constraints in the Du Bois table in higher dimensions?

For 3-folds over $\mathbb{Q}$, the Du Bois table takes the following form.

$$\begin{array}{|cccc} b^{0,2} & b^{1,2} & 0 & 0 \\ b^{0,1} & b^{1,1} & b^{2,1} & 0 \end{array}$$

Using the spectral sequence argument as in Lemma 6.2, one can show that

$$b^{0,2} \geq b^{1,2}, \quad b^{0,2} - b^{1,2} \geq b^{2,1} - b^{1,1}.$$

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The question amounts to asking whether there are further relations among the entries in the Du Bois table. By the main theorem of [CHW+10], answers to this question will in particular give answers to Bass's question.

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Conflicts of interest

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