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Results on Selmer groups over the PGL(2) extension

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Abstract. In this article, we study Iwasawa theory for the Selmer group of an elliptic curve over a certain non-commutative p-adic Lie extension whose Galois group is isomorphic to an open subgroup of $\mathrm{PGL}_2(\mathbb{Z}_p)$. Unlike the classes of extension usually considered in non-commutative Iwasawa theory, this extension does not contain the cyclotomic $\mathbb{Z}_p$-extension. We give a formula for its Euler characteristic in this case. We also show that the dual Selmer group contains no nonzero pseudonull submodules, a result analogous to a theorem of Ochi and Venjakob for GL_2-extensions. Finally, we provide an explicit formula computing the change in Iwasawa μ-invariants of dual Selmer groups when two elliptic curves are isogenous; this in the same spirit as that of Howson's result for the GL_2-extension which extends results in the commutative case by Schneider and Perrin-Riou.

1 Introduction

So far, non-commutative Iwasawa theory has only been developed for p-adic Lie extensions of a number field F containing the cyclotomic $\mathbb{Z}_p$-extension F^{cyc} with Galois group $\Gamma = \mathrm{Gal}(F^{cyc}/F) \cong \mathbb{Z}_p$. In particular, the definition of the category $\mathfrak{M}_H(G)$ (see [CFK+05]) and the construction of characteristic elements via the K-theory localization sequence in *(loc.cit)* make crucial use of the cyclotomic $\mathbb{Z}_p$-extension and the structure theory of finitely generated $\Lambda(\Gamma)$-modules, and it is not known what to expect for p-adic Lie extensions not containing F^{cyc}.

In this article, we study the Selmer group of an elliptic curve over a so-called PGL_2-extension, defined as follows. Let F be a number field, E and E' elliptic curves defined over F and p a prime $\geqslant 5$. Assume that both E and E' do not have complex multiplication. Let $F_\infty = F(E'_{p^\infty})$, so as shown in [Ser72] $\Sigma = \mathrm{Gal}(F_\infty/F)$ is an open subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$. It follows that if C denotes the center of Σ, then C is isomorphic to an open subgroup of $\mathbb{Z}_p^*$. Let $K_\infty = F_\infty^C$, so $G = \mathrm{Gal}(K_\infty/F)$ is a three-dimensional p-adic Lie group with no element of order p, isomorphic to an open subgroup of $\mathrm{PGL}_2(\mathbb{Z}_p)$. It is an easy consequence of the Weil pairing that K_∞ does not contain the cyclotomic $\mathbb{Z}_p$-extension of F.

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1.1 Result in Section 2

Let S be a finite set of primes of F containing all the primes above p and all the primes where either E or E' has bad reduction. Let $\mathrm{Sel}(E/K_\infty)$ be the Selmer group of E over K_∞ as in Section 2.2. The first main result of this article gives an explicit formula for the G-Euler characteristic of $\mathrm{Sel}(E/K_\infty)$ in terms of the arithmetic invariants of E.

Theorem 1.1 *Let E and E' be non-isogenous elliptic curves defined over a finite extension F of $\mathbb{Q}$, and assume that both E and E' do not have complex multiplication. Let p be a prime $\geqslant 5$, and let $F_\infty = F(E'_{p^\infty})$, and $\Sigma = \mathrm{Gal}(F_\infty/F)$. Let C be the center of $\mathrm{Gal}(F_\infty/F)$, and let $K_\infty = F_\infty^C$ and $G = \mathrm{Gal}(K_\infty/F) \cong \Sigma/C$. Assume that*

(i) *both E and E' have good ordinary reduction at all primes of F dividing p;*
(ii) *$\mathrm{Sel}(E/F)$ is finite;*
(iii) *$\mathcal{C}(E/K_\infty) = \mathrm{Sel}(E/K_\infty)^\vee$ is $\Lambda(G)$-torsion;*
(iv) *hypothesis (S_Σ) in Section 2.2 is satisfied.*

Then $\mathrm{Sel}(E/K_\infty)$ has finite G-Euler characteristic, and

$$\chi(G, \mathrm{Sel}(E/K_\infty)) = \chi(\Gamma, \mathrm{Sel}(E/F^{cyc})) \times \left| \prod_{v \in \mathfrak{M}} L_v(E,1) \times \prod_{v \in \mathfrak{N}} c_v(E) \right|_p . \tag{1}$$

Here, $c_v(E)$ is the Tamagawa factor of E at v, which is defined as $c_v = [E(F_v) : E_0(F_v)]$, where $E_0(F_v)$ is the subgroup of $E(F_v)$ consisting of the points of non-singular reduction modulo v. The Euler factor at v of the complex L-function of E over F is $L_v(E,s)$ and $|\cdot|_p$ is the p-adic valuation of $\mathbb{Q}$, normalized such that $|\,p\,|_p = p^{-1}$. The set $\mathfrak{M}$ is the set of primes of F not dividing p where $\mathrm{ord}_v(j_{E'}) < 0$, and $\mathfrak{N}$ is the set of primes of F not dividing p such that $\mathrm{ord}_v(j_{E'}) \geqslant 0$, $\mathrm{ord}_v(j_E) < 0$ and for which there exists a finite extension $F_{v'}$ of F_v contained in $F_{\infty,w}$ such that $|a_{v'}(E')| = 2\sqrt{q}$, where q is the cardinality of the residue field of $F_{v'}$.

Remark 1.2 Comparing the expression for $\chi(G, \mathrm{Sel}(E/K_\infty))$ with the Euler characteristic formula for $\mathrm{Sel}(E/F_\infty)$ given in [Coa99, Conjecture 1.13 and Theorem 1.14] shows that some of the Tamagawa factors disappear when one passes from F_∞ to K_∞, which indicates that the Iwasawa theory of p-adic Lie extensions not containing F^{cyc} might be very different from the Iwasawa theory of strongly admissible p-adic Lie extensions.

1.2 Result in Section 3

The definition of a pseudonull module over a non-commutative algebra was introduced by Venjakob in [Ven02]. These modules occur naturally in the theory of modules over Iwasawa algebras of p-adic Lie groups of dimension > 1, generalizing the role played by finite modules in the one-dimensional case. In particular, if one defines a *pseudo-isomorphism* of $\Lambda(G)$-modules to be a morphism whose kernel and cokernel are pseudonull, then the main result of [CSS03b] classifies finitely generated modules over non-commutative Iwasawa algebras up to pseudo-isomorphism, generalizing the well-known structure theorem for finitely-generated $\Lambda(\Gamma)$-modules.

The second main result of this article shows that some general results about the algebraic structure of the Selmer group remain true when one passes from the GL_2- to

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the PGL_2-extension. The main result of [OV02] shows that $\mathcal{C}(E/F_\infty)$ has no nontrivial pseudonull submodule. The following theorem shows that the same is true over K_∞.

Theorem 1.3 *Assume that both E and E' have good ordinary reduction at any prime of F dividing p. Further assume that for all $v \in S$, $v \nmid p$, $\mathrm{ord}_v(j_{E'}) < 0$. If $\mathcal{C}(E/K_\infty)$ is $\Lambda(G)$-torsion, then it has no nontrivial pseudonull submodule.*

1.3 Result in Section 4

The final main result of this article is a formula regarding the change in the Iwasawa μ-invariants (as $\Lambda(G)$-modules) of Selmer groups (over the PGL_2-extension) of elliptic curves with an isogeny between them. There are some extra terms appearing in the formula for the PGL_2-case compared to the GL_2-case by [How02, Theorem 3.1]. We also note that similar formulas in the commutative cases were treated by Schneider (see [Sch87]) and Perrin-Riou (see [PR87, Appendix] and [PR89]).

Let E_1 and E_2 be elliptic curves without complex multiplication defined over a number field F with good ordinary reduction at all places of F above p, and they are isogenous via $\varphi : E_1 \to E_2$. We assume that $F_\infty = F(E_{1,p^\infty}) = F(E_{2,p^\infty})$ is a pro-p extension of F. Let A be the p-part of the kernel of φ and let $\tilde{A}_v$ be the image of A under the reduction map $v : E_1 \to \tilde{E}_1$. Let S be a finite set of primes of F including all the primes which divide p and places, where E_1 (hence, E_2) has bad reduction.

Theorem 1.4 *Let $p \geq 5$. Let $\varphi : E_1 \to E_2$ be an isogeny between two elliptic curves without complex multiplication. Assume that the dual Selmer groups $\mathcal{C}(E_i/K_\infty)$ are $\Lambda(G)$-torsion for $i = \{1, 2\}$. Then*

$$\begin{aligned}\mu(\mathcal{C}(E_2/K_\infty)) - \mu(\mathcal{C}(E_1/K_\infty)) = &\sum_{v|\infty} ord_p\left(\#(A(F_v))\right) - |F:\mathbb{Q}|ord_p(\#A)\\ &- \sum_{v|p} ord_p(|\#\tilde{A}_v|_v)\\ &+ \sum_{\substack{v\in S, v\nmid p\\ ord_v(j_E)\geq 0}} \left(ord_p(\#(E_{1,p^\infty}(F_v)) - ord_p(\#(E_{2,p^\infty}(F_v)))\right).\end{aligned}$$

2 Euler characteristic formula for the PGL_2-extension

2.1 Strategy

Recall that E and E' are elliptic curves without complex multiplication defined over a number field F and p a prime $\geqslant 5$. We assume throughout that both E and E' have good ordinary reduction at all primes of F dividing p. Let $F_\infty = F(E'_{p^\infty})$, so as shown in [Ser72], $\Sigma = \mathrm{Gal}(F_\infty/F)$ is an open subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$. Let C denote the center of Σ, so C consists of scalar multiples of the identity and is isomorphic to an open subgroup of $\mathbb{Z}_p^\times$. Let $K_\infty = F_\infty^C$ and $G = \mathrm{Gal}(K_\infty/F) \cong \Sigma/C$. It follows from the previous remarks that K_∞ is a p-adic Lie extension of F of dimension 3, and $E_{p^\infty}(K_\infty)$ is finite as $E_{p^\infty}(F_\infty)$ is finite by [Zer04, Proposition 10] (see also [LM14, Lemma 6.2]).

In Section 2.2, we construct a commutative diagram relating $\mathrm{Sel}(E/K_\infty)$ to $\mathrm{Sel}(E/F_\infty)^C$, analogous to the "fundamental diagram" relating $\mathrm{Sel}(E/F_\infty)$ to $\mathrm{Sel}(E/F^{cyc})$ appearing in [CSS03a]. We then analyze the local extensions

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corresponding to the extension K_∞ of F in Section 2.3, and we use these results in Sections 2.4 and 2.5 to determine the kernels and cokernels of the local restriction maps of the fundamental diagram. Finally, in Section 2.6, we calculate the G-Euler characteristic of $\mathrm{Sel}(E/K_\infty)$, thereby completing the proof of Theorem 1.1.

2.2 The fundamental diagram

In order to calculate the G-Euler characteristic of $\mathrm{Sel}(E/K_\infty)$, we compare $\mathrm{Sel}(E/K_\infty)$ with the C-invariants of $\mathrm{Sel}(E/F_\infty)$. Let S be a finite set of primes of F containing all the primes above p and all the primes where either E or E' has bad reduction. For any extension M/F, we write $G_S(M)$ for the Galois group of the maximal algebraic extension of M unramified outside the primes above S. Then we may define $\mathrm{Sel}(E/K_\infty)$ as the kernel of the natural map $\lambda_S(K_\infty) : H^1(G_S(K_\infty), E_{p^\infty}) \to \bigoplus_{v\in S} J_v(K_\infty)$, where $J_v(K_\infty)$ is the usual local cohomology term as defined in [Zer09, Section 2.1].

By definition of Selmer group, we have a short exact sequence

$$0 \to \mathrm{Sel}(E/K_\infty) \to H^1(G_S(K_\infty), E_{p^\infty}) \to \bigoplus_{v\in S} J_v(K_\infty). \tag{2}$$

Taking C-invariants of the exact sequence

$$0 \to \mathrm{Sel}(E/F_\infty) \to H^1(G_S(F_\infty), E_{p^\infty}) \xrightarrow{\lambda_S(F_\infty)} \bigoplus_{v\in S} J_v(F_\infty) \tag{3}$$

and comparing it with (2), we obtain the following fundamental diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Sel}(E/K_\infty) & \longrightarrow & H^1(G_S(K_\infty), E_{p^\infty}) & \xrightarrow{\lambda_S(K_\infty)} & \oplus_{v\in S} J_v(K_\infty) \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \eta=\oplus_{v\in S}\eta_v} \\ 0 & \longrightarrow & \mathrm{Sel}(E/F_\infty)^C & \longrightarrow & H^1(G_S(F_\infty), E_{p^\infty})^C & \xrightarrow{\psi_S(F_\infty)} & \oplus_{v\in S} J_v(F_\infty)^C. \end{array} \tag{4}$$

We will make the hypotheses as and when needed.
(WLC_G): $H^2(G_S(K_\infty), E_{p^\infty}) = 0$.
(S_Σ): The map $\lambda_S(F_\infty)$ in (3) is surjective.
(S_G): The map $\lambda_S(K_\infty)$ is surjective.

Remark 2.1 Let $R = \mathrm{Gal}(F_\infty/F(A[p]))$. If $\mathrm{Sel}(E/F_\infty)$ is $\Lambda(R)$-cotorsion, then by [Zer04, Corollary 1, p. 603], hypothesis (S_Σ) is satisfied. We also note that since $\mathrm{Gal}(F(A[p])/F)$ is a finite group, $\mathrm{Sel}(E/F_\infty)$ is $\Lambda(R)$-cotorsion, if and only if $\mathrm{Sel}(E/F_\infty)$ is $\Lambda(\Sigma)$-cotorsion (see [OV02, after Definition 3.2]).

Similarly, if $\mathrm{Sel}(E/K_\infty)$ is $\Lambda(G)$-cotorsion, (or equivalently $\mathrm{Sel}(E/K_\infty)$ is $\Lambda(\tilde{G})$-cotorsion, after replacing G by a finite index pro-p subgroup $\tilde{G}$) then by [HV03, Theorem 7.2], hypotheses (WLC_G) and (S_G) are satisfied.

2.3 Decomposition groups

We now describe the decomposition groups in K_∞ of primes of F. For a prime v of F, let w denote a fixed prime of K_∞ above v. The decomposition group G_v of w in G can be canonically identified with the local Galois group $\mathrm{Gal}(K_{\infty,w}/F_v)$. Note that

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$G_v \cong \Sigma_w/C_w$, where $C_w = C \cap \Sigma_w$. The following two propositions describe the structure of these local Galois groups.

Proposition 2.2 *Let v be a finite prime of F which does not divide p. If $\mathrm{ord}_v(j_{E'}) < 0$, then $K_{\infty,w} = F_v(E'_{p^\infty})$ and $\dim(G_v) = 2$. If $\mathrm{ord}_v(j_{E'}) \geqslant 0$, then G_v is finite of order prime to p if and only if there exists a finite extension $F_{v'}$ of F_v contained in $F_{\infty,w}$ such that $|a_{v'}(E')| = 2\sqrt{q}$. Otherwise, G_v is an open subgroup of $\mathbb{Z}_p^*$.*

Proof Suppose that $\mathrm{ord}_v(j_{E'}) < 0$. Then

$$F_v(E'_{p^\infty}) = F_v(\mu_{p^\infty}, \sqrt[p^\infty]{q_{E'}^{(v)}}),$$

where $q_{E'}^{(v)}$ is the Tate period of E' over F_v, and we have an exact sequence of $\mathrm{Gal}(F_v(E'_{p^\infty})/F_v)$-modules

$$0 \to \mu_{p^\infty} \to E'_{p^\infty} \to \mathbb{Q}_p/\mathbb{Z}_p \to 0.$$

It follows that every element that acts on E'_{p^∞} by scalar multiplication must be the identity, so $C_w = 1$ and hence,

$$G_w = \mathrm{Gal}(F_v(\mu_{p^\infty}, \sqrt[p^\infty]{q_{E'}^{(v)}})/F_v).$$

This is isomorphic to an open subgroup of $\mathbb{Z}_p^\times \rtimes \mathbb{Z}_p(1)$ and hence has p-cohomological dimension 2. Assume now that $\mathrm{ord}_v(j_{E'}) \geqslant 0$. Now $F_v(E'_{p^\infty})$ is a finite extension of order prime to p of the unramified $\mathbb{Z}_p$-extension of F_v, so it follows that $\mathrm{Gal}(K_{\infty,w}/F_v)$ is either finite or an open subgroup of $\mathbb{Z}_p^*$. Let $F_{v'}$ be an extension of F_v contained in $F_{\infty,w}$ such that $\mathrm{Gal}(F_{\infty,w}/F_{v'})$ is pro-p. Let φ be a topological generator of $\mathrm{Gal}(F_{\infty,w}/F_{v'})$, and let Φ be the matrix representing φ with respect to some fixed $\mathbb{Z}_p$-basis of E'_{p^∞}. Then Φ satisfies the polynomial

$$F_{v'}(T) = T^2 + a_{v'}(E')T + q,$$

where q is the cardinality of the residue field of $F_{v'}$. It follows that Φ is scalar if and only if $F_{v'}(T)$ has a double root, which happens precisely when $|a_{v'}(E')| = 2\sqrt{q}$. ∎

Corollary 2.3 *If v is a prime of F not dividing p where $\mathrm{ord}_v(j_{E'}) < 0$, then $H^1(K_{\infty,w}, E_{p^\infty}) = 0$.*

Proof This follows since the profinite degree of $\mathrm{Gal}(\overline{F}_v/K_{\infty,w})$ is prime to p, as in [Zer11, Lemma 4.2]. ∎

For the primes of F above p, the situation is more subtle.

Proposition 2.4 *Let v be a prime of F dividing p. Then $\dim G_v = 2$, and up to a finite extension, the extension $K_{\infty,w}$ of F_v is totally ramified.*

Proof We need to show that $C \cap \Sigma_w \cong \mathbb{Z}_p$. Now as shown in [Coa99], Σ_w is a p-adic Lie group of dimension 3, and by choosing a suitable $\mathbb{Z}_p$-basis of E'_{p^∞}, it can be represented by upper triangular matrices. Suppose now that $C \cap \Sigma_w = 1$. Then $C \times \Sigma_w$ is a subgroup of Σ of dimension 4, so it must be of finite index in Σ. However, with respect to the chosen basis every matrix in $C \times \Sigma_w$ is upper triangular, which gives a contradiction. Hence, $C_w \cong \mathbb{Z}_p$. Let k_v be the residue field of F_v. Now, upto a finite

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extension, the maximal unramified pro-p extension of k_v contained in $k_v(E'_{p^\infty})$ is the unique unramified $\mathbb{Z}_p$-extension, which is the field of definition of $\tilde{E}'_{v,p^\infty}$. It is well-known that we have an exact sequence of Σ_w-modules

$$0 \to \hat{E}'_{v,p^\infty}(\mathfrak{M}_\infty) \to E'_{p^\infty} \to \tilde{E}'_{v,p^\infty} \to 0,$$

where $\mathfrak{M}_\infty$ is the maximal ideal of the ring of integers of $F_v(E'_{p^\infty})$ and $\hat{E}'_v$ is the formal group of the elliptic curve over the ring of integers of $F_v(E'_{p^\infty})$. So if an element of Σ_w acts on E'_{p^∞} by scalar multiplication, then it must also act by scalar multiplication on $\tilde{E}'_{v,p^\infty}$. It follows that if $k_{K_\infty,w}$ denotes the residue field of $K_{\infty,w}$, then $\tilde{E}'_{v,p^\infty}(k_{K_\infty,w})$ is finite, or equivalently, the extension $K_{\infty,w}$ of F_v is totally ramified upto a finite extension. ∎

2.4 Local cohomology

Lemma 2.5 *Let $v \in S$, w be a prime of K_∞ lying above v, and let $k_{\infty,w}$ be the residue field of $F_{\infty,w}$. Let*

$$X_w = \begin{cases} H^1(C_w, E_{p^\infty}(F_{\infty,w})), & \text{if } v \nmid p, \\ H^1(C_w, \tilde{E}_{v,p^\infty}(k_{\infty,w})), & \text{if } v \mid p. \end{cases}$$

If $v \nmid p$, then $X_w^{G_v} \neq 0$ if and only if (i) $\mathrm{ord}_v(j_{E'}) \geqslant 0$, (ii) there exists a finite extension $F_{v'}$ of F_v contained in $F_{\infty,w}$ such that $|a_{v'}(E')| = 2\sqrt{q}$, and (iii) $\mathrm{ord}_v(j_E) < 0$. If this is the case, then

$$\chi(G_v, X_w) = \#X_w^{G_v} = |c_v(E)|_p^{-1}.$$

If $v \mid p$, then $\chi(G_v, X_w) = 1$.

Proof Suppose $v \mid p$. Let $M = \tilde{E}_{v,p^\infty}(k_{\infty,w})$. We have the Hochschild–Serre spectral sequence

$$H^r(G_v, H^s(C_w, M)) \implies H^{r+s}(\Sigma_w, M).$$

From this, we obtain that $\chi(\Sigma_w, M) = \prod \chi(G_v, H^i(C_w, M))^{(-1)^i}$. Since, C_w has p-cohomological dimension at most one, one obtains

$$\chi(\Sigma_w, M) = \frac{\chi(G_v, M^{C_w})}{\chi(G_v, X_w)}.$$

But by [Zer04, Proposition 4], $\chi(\Sigma_w, M) = 1$. Hence, $\chi(G_v, X_w) = \chi(G_v, M^{C_w})$. But by Proposition 2.4, the residue field of $K_{\infty,w}$ is a finite extension of the residue field of F_v and hence $M^{C_w} = \tilde{E}_{v,p^\infty}(k_{K_{\infty,w}})$ is finite and so $\chi(G_v, M^{C_w}) = 1$.

Now, let $v \in S$ and $v \nmid p$. If $\mathrm{ord}_v(j_{E'}) < 0$, then by the proof of Proposition 2.2, $C_w = 1$ and hence $X_w = 0$. Hence, $\mathrm{ord}_v(j_{E'}) \geqslant 0$ and under this hypothesis there are two possible cases. Either G_v is finite of order prime to p which holds if and only if (ii) holds or G_v is an open subgroup of $\mathbb{Z}_p^*$. If G_v is finite of order prime to p, by inflation restriction exact sequence $X_w^{G_v} \cong H^1(\Sigma_w, E_{p^\infty}(F_{\infty,w}))$ with cardinality equal to $|c_v(E)|_p^{-1}$ for $\mathrm{ord}_v(j_E) < 0$ and $H^1(\Sigma_w, E_{p^\infty}(F_{\infty,w}))$ is zero otherwise by [Zer04, Proposition 2]. Suppose now that G_v is an open subgroup of $\mathbb{Z}_p^*$, then as shown in

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the proof of [Zer04, Proposition 2], $\dim \Sigma_w = 1$, and Σ_w has no element of order p (see the line after eq. (2.5) of [Zer04]). Hence, p-cohomological dimension of Σ_w is 1. This implies that p-cohomological dimension of C_w is zero and since the module $E_{p^\infty}(F_{\infty,w})$ is p-primary, we have $X_w = H^1(C_w, E_{p^\infty}(F_{\infty,w})) = 0$. ∎

2.5 Analysis of the local restriction maps

Recall that we have defined the local restriction maps

$$\eta = \bigoplus_{v \in S} \eta_v.$$

Here,

$$\eta_v : \prod_{w|v} \left(H^1(K_{\infty,w}, E)(p) \to H^1(F_{\infty,w}, E)(p)^{C_w} \right),$$

where w runs over all primes of K_∞ above v.

Lemma 2.6 *Let $v \in S$. Then,* $\operatorname{coker}(\eta_v) = 0$.

Proof By Kummer theory and applying inflation-restriction sequence, we have

$$\operatorname{coker}(\eta_v) = \begin{cases} \prod_{w|v} H^2(C_w, E_{p^\infty}(F_{\infty,w})), & \text{if } v \nmid p, \\ \prod_{w|v} H^2(C_w, \tilde{E}_{v,p^\infty}(k_{\infty,w})), & \text{if } w \mid p. \end{cases}$$

But $\operatorname{cd}_p(C_w) \leqslant 1$ for all w, so $\operatorname{coker}(\eta_v) = 0$ for all $v \in S$. ∎

2.6 Calculation of the Euler characteristic

To simplify notation, define the following conditions:

- (Tors_G) The Pontryagin dual $\mathcal{C}(E/K_\infty)$ is $\Lambda(G)$-torsion.
- (Tors_Σ) The Pontryagin dual $\mathcal{C}(E/F_\infty)$ is $\Lambda(\Sigma)$-torsion.

Also, we need the following definition.

Definition 2.7 Let $\mathfrak{R}$ be the set of primes of F such that

(i) $v \nmid p$,
(ii) $\operatorname{ord}_v(j_{E'}) \geqslant 0$,
(iii) there exists a finite extension $F_{v'}$ of F_v contained in $F_{\infty,w}$ such that $|a_{v'}(E')| = 2\sqrt{q}$, and
(iv) $\operatorname{ord}_v(j_E) < 0$.

Proposition 2.8 *Suppose that* ($\mathrm{S_G}$) *is satisfied. Then* $\mathrm{Sel}(E/K_\infty)$ *has finite G-Euler characteristic if and only if so does* $\mathrm{Sel}(E/F_\infty)^C$*, and if this is the case, then*

$$\chi(G, \mathrm{Sel}(E/K_\infty)) = \chi(G, \mathrm{Sel}(E/F_\infty)^C) \times \prod_{v \in \mathfrak{R}} |c_v|_p.$$

Proof Note that we are assuming that E and E' does not have complex multiplication and hence by the proof of [Zer04, Proposition 10] $E_{p^\infty}(F_\infty)$ is finite; therefore, $\ker(\beta)$ and $\operatorname{coker}(\beta)$ are both finite. (In fact, we have $\operatorname{coker}(\beta) = 0$ since $\operatorname{coker}(\beta) \cong H^2(C, E_{p^\infty}(F_\infty))$ and C has p-cohomological dimension 1.) It is shown in [Ser98]

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that $\chi(G, M) = 1$ for every finite p-primary G-module M. Applying the snake lemma to the fundamental diagram 4, we therefore deduce that $\mathrm{coker}(\alpha)$ has finite G-Euler characteristic if and only if so does $\ker(\eta)$, and if this is the case, then

$$\begin{aligned}\chi(G, \mathrm{coker}(\alpha)) &= \chi(G, \ker(\eta)) \\ &= \prod_{v \in \mathfrak{R}} \chi(G, \ker(\eta_v)).\end{aligned}$$

Here, the second equality is a direct consequence from Lemma 2.5 and the fact that

$$\ker(\eta_v) = \prod_{w|v} X_w,$$

where w runs over all primes of K_∞ above v. It follows from Shapiro's lemma that

$$\chi(G, \ker(\eta_v)) = \chi(G_v, X_w),$$

where w is some fixed prime of K_∞ above v. But as shown in Proposition 2.2, for any prime $v \in \mathfrak{R}$, the local cohomology group G_v is finite of order prime to p, so by Lemma 2.5, we have

$$\chi(G, \ker(\eta_v)) = |c_v(E)|_p^{-1}$$

and hence,

$$\chi(G, \mathrm{coker}(\alpha)) = \prod_{v \in \mathfrak{R}} |c_v(E)|_p^{-1}. \tag{5}$$

Now, $\ker(\alpha)$ is finite since it is contained in $\ker(\beta)$, so

$$\chi(G, \mathrm{Sel}(E/K_\infty)) = \chi(G, \mathrm{Sel}(E/K_\infty)/\ker(\alpha)). \tag{6}$$

The proposition therefore follows from (5) and the exact sequence

$$0 \to \mathrm{Sel}(E/K_\infty)/\ker\alpha \to \mathrm{Sel}(E/F_\infty)^C \to \mathrm{coker}(\alpha) \to 0. \qquad \blacksquare$$

It now remains to relate the G-Euler characteristic of $\mathrm{Sel}(E/F_\infty)^C$ to the Σ-Euler characteristic of $\mathrm{Sel}(E/F_\infty)$. We first need the following lemma.

Lemma 2.9 *If* (S_G) *holds, then we have a short exact sequence*

$$0 \to \mathrm{Sel}(E/F_\infty)^C \to H^1(G_S(F_\infty), E_{p^\infty})^C \to \bigoplus_{v \in S} J_v(F_\infty)^C \to 0.$$

Proof The proof follows from Lemma 2.6. $\blacksquare$

Proposition 2.10 *If hypotheses* (S_G), (S_Σ), *and* (WLC_G) *are satisfied, then*

$$H^i(C, \mathrm{Sel}(E/F_\infty)) = 0$$

for all $i \geqslant 1$.

Proof We have $\mathrm{cd}_p(C) = 1$, so $H^i(C, \mathrm{Sel}(E/F_\infty)) = 0$ for all $i \geqslant 2$. We first show that $H^1(C, H^1(G_S(F_\infty), E_{p^\infty})) = 0$. The Hochschild–Serre spectral sequence gives an exact

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sequence

$$H^2(G_S(K_\infty), E_{p^\infty}) \to H^1(C, H^1(G_S(F_\infty), E_{p^\infty})) \to H^3(C, E_{p^\infty}(F_\infty)),$$

where the term on the right is zero since $\mathrm{cd}_p(C) = 1$. The triviality of $H^1(C, H^1(G_S(F_\infty), E_{p^\infty}))$ now follows from hypothesis (WLC_G).

By assumption, we have a short exact sequence

$$0 \to \mathrm{Sel}(E/F_\infty) \to H^1(G_S(F_\infty), E_{p^\infty}) \to \bigoplus_{v \in S} J_v(F_\infty) \to 0.$$

Taking C-cohomology and using that $H^1(C, H^1(G_S(F_\infty), E_{p^\infty})) = 0$, we obtain

$$0 \to \mathrm{Sel}(E/F_\infty)^C \to H^1(G_S(F_\infty), E_{p^\infty})^C \to \bigoplus_{v \in S} J_v(F_\infty)^C \to H^1(C, \mathrm{Sel}(E/F_\infty)) \to 0.$$

As shown in Lemma 2.9, the map

$$H^1(G_S(F_\infty), E_{p^\infty})^C \to \bigoplus_{v \in S} J_v(F_\infty)^C$$

is surjective, which finishes the proof. ∎

Corollary 2.11 *We have canonical isomorphisms*

$$H^i(G, \mathrm{Sel}(E/F_\infty)^C) \cong H^i(\Sigma, \mathrm{Sel}(E/F_\infty))$$

for all $i \geqslant 0$.

Proof The result is an immediate consequence of Proposition 2.10 and the Hochschild–Serre spectral sequence. ∎

Combining Corollary 2.11 with Proposition 2.8, we get the following result.

Proposition 2.12 *Assume hypotheses* (S_Σ) *and (Tors* $_G$*) are satisfied. Then,* $\mathrm{Sel}(E/K_\infty)$ *has finite G-Euler characteristic if and only if* $\mathrm{Sel}(E/F_\infty)$ *has finite* Σ*-Euler characteristic, and if this is the case, then*

$$\chi(G, \mathrm{Sel}(E/K_\infty)) = \chi(\Sigma, \mathrm{Sel}(E/F_\infty)) \times \prod_{v \in \mathfrak{R}} |c_v(E)|_p.$$

Combining Proposition 2.12 with the formula for $\chi(\Sigma, \mathrm{Sel}(E/F_\infty))$ given by [Zer04, Theorem 1] gives Theorem 1.1.

Remark 2.13 It follows from the equation in Proposition 2.12 that in theory $\chi(G, \mathrm{Sel}(E/K_\infty))$ could be non-integral (which cannot happen over strongly admissible p-adic Lie extensions!), but unfortunately we do not have the numerical data to construct an example.

3 Results on non-existence of pseudonull submodules

Throughout this section, let G be a compact p-adic Lie group without p-torsion and $\Lambda(G)$ its Iwasawa algebra. Let $d = \dim G$. It is shown in [GW04] that $\Lambda(G)$ is a Noetherian ring. In the following sections, we collect some results from the theory of $\Lambda(G)$-modules that will be needed in the proof of Theorem 1.3. Since we restrict our discussions to the absolute minimum, we give references where a detailed treatment of the topics can be found.

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3.1 Pseudonull modules

In the following, we recall the definition of pseudonull modules as given in [Ven02] and state one of its basic properties. In the presentation, we follow Section 2 in [OV02], but we restrict the discussion to the Iwasawa algebras of p-adic Lie groups. A general treatment can be found in [Ven03].

Convention 3.1 *For a finitely generated $\Lambda(G)$-module M, write*

$$E^i(M) = \mathrm{Ext}^i_{\Lambda(G)}(M, \Lambda(G))$$

for any $i \geqslant 0$.

Recall the following definition.

Definition 3.2 (i) If $M \neq 0$ is a $\Lambda(G)$-module, then

$$j(M) = \min\{i \mid E^i(M) \neq 0\}$$

is called the *grade* of M.

(ii) The Noetherian ring $\Lambda(G)$ is an *Auslander regular ring* if it has finite global homological dimension and if the following condition holds: For any $\Lambda(G)$-module M, any $m \geqslant 0$ and any submodule N of $E^m(M)$, we have $j(N) \geqslant m$.

In [Ven02], Theorem 3.26, it is shown that for any p-adic Lie group G without p-torsion, its Iwasawa algebra $\Lambda(G)$ is Auslander regular. In particular, this implies that any finitely generated $\Lambda(G)$-module M has a canonical filtration

$$T_0(M) \subset T_1(M) \subset \cdots \subset T_{d-1}(M) \subset T_d(M) = M,$$

which allows us to make the following definition.

Definition 3.3 A finitely generated $\Lambda(G)$-module M is *pseudonull* if $M = T_{d-2}$.

The following lemma, which characterizes pseudonull $\Lambda(G)$-modules in terms of their Ext-groups, is very useful in practice.

Lemma 3.4 *A finitely generated $\Lambda(G)$-module is pseudonull if and only if*

$$E^i E^i(M) = 0 \text{ for all } i \geqslant 2.$$

Proof See the proof of Proposition 2.4 in [OV02]. ∎

3.2 Duality groups

One of the fundamental results of Lazard is that every p-adic Lie group with no element of order p is a Poincar é duality group (cf. [SW00]), so in particular, it is a duality group in the sense of [NSW13, Chapter III, Section 4].

For us, this fact is particularly useful because of the following result.

Proposition 3.5 *Let G be a duality group at p of dimension n, and let M be a $\Lambda(G)$-module which has a resolution by finitely generated projectives.*

(i) *If M is a free finitely generated $\mathbb{Z}_p$-module, then*

$$E^k M = 0 \text{ if } k \neq n.$$

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(ii) *If M is a finite p-primary G-module, then*

$$E^k M = 0 \text{ if } k \neq n+1.$$

3.3 Two useful exact sequences

In this section, we recall the definition of two exact sequences of $\Lambda(G)$-modules, which were first defined in Venjakob's generalization of Jannsen's homotopy theory (cf. [Jan89]) in [OV02, Section 3], and which will be crucial in the proof of Theorem 1.3. We follow the presentation given in [OV02].

Let $\mathcal{G}$ be a profinite group which is topologically of finite type of rank d. As shown in [NSW13, Chapter V.6], there is a surjective homomorphism from a free profinite group $\mathcal{F}$ of rank d to $\mathcal{G}$, which gives an exact sequence

$$0 \to \mathcal{N} \to \mathcal{F} \to \mathcal{G} \to 0.$$

The module $\mathcal{N}^{ab}(p)$ fits into the following exact sequence, which is called the Fox–Lyndon resolution associated with the above free presentation of $\mathcal{F}$:

$$0 \to \mathcal{N}^{ab}(p) \to \Lambda(\mathcal{G})^d \to \Lambda(\mathcal{G}) \to \mathbb{Z}_p \to 0$$

(see [NSW13, Theorem 5.6.6]). We extract from it the short exact sequence

$$0 \to \mathcal{I} \to \Lambda(\mathcal{G}) \to \mathbb{Z}_p \to 0 \tag{7}$$

and one can show that the augmentation ideal $\mathcal{I} = \ker(\Lambda(\mathcal{G}) \to \mathbb{Z}_p)$ is a free $\Lambda(\mathcal{F})$-module of rank d.

To simplify notation, we make the following definition.

Definition 3.6 Let A be a p-divisible p-primary abelian group of finite $\mathbb{Z}_p$-corank r with a continuous action of $\mathcal{G}$. For a finitely generated $\Lambda(\mathcal{G})$-module M, define

$$M[A] = M \otimes_{\mathbb{Z}_p} A^\vee,$$

where $A^\vee$ denotes the Pontryagin dual of A. Here, $M[A]$ is equipped with the usual diagonal action of $\mathcal{G}$.

Note that $\otimes_{\mathbb{Z}_p} A^\vee$ is an exact functor on the category of finitely generated $\Lambda(\mathcal{G})$-modules.

Lemma 3.7 *The module $\Lambda(\mathcal{G})[A]$ is a free $\Lambda(\mathcal{G})$-module of rank $r = \operatorname{cork}_{\mathbb{Z}_p} A$.*

Proof See the proof of Lemma 4.2 in [OV02]. ∎

We now specialize to the case when $\mathcal{G}$ is the absolute Galois group of some finite extension K of $\mathbb{Q}$ or $\mathbb{Q}_p$; that is, let $\bar{K}$ be an algebraic closure of K, and let $\mathcal{G} = \mathrm{Gal}(\bar{K}/K)$. Note that $\mathcal{G}$ is profinite by definition. Let K_∞ be a p-adic Lie extension of K contained in $\bar{K}$, and let $\mathcal{H} = \mathrm{Gal}(\bar{K}/K_\infty)$ and $G = \mathrm{Gal}(K_\infty/K)$. Again, let A be a p-divisible p-primary abelian group of finite $\mathbb{Z}_p$-corank r with a continuous action of $\mathcal{G}$. Tensoring (7) by $A^\vee$ gives an exact sequence

$$0 \to \mathcal{I}[A] \to \Lambda(\mathcal{G})[A] \to A^\vee \to 0, \tag{8}$$

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where $\Lambda(\mathcal{G})[A]$ is a projective $\Lambda(\mathcal{G})$-module by Lemma 3.7. Taking $\mathcal{H}$-homology of (8) therefore gives an exact sequence

$$0 \to H_1(\mathcal{H}, A^\vee) \to (\mathcal{I}[A])_{\mathcal{H}} \to (\Lambda(\mathcal{G})[A])_{\mathcal{H}} \to (A^\vee)_{\mathcal{H}} \to 0, \tag{9}$$

from which we extract the short exact sequence

$$0 \to X \to Y \to I \to 0, \tag{10}$$

where

$$X = H_1(\mathcal{H}, A^\vee), \tag{11}$$

$$Y = (\mathcal{I}[A])_{\mathcal{H}}, \tag{12}$$

and I is defined by the exactness of the exact sequence

$$0 \to I \to (\Lambda(\mathcal{G})[A])_{\mathcal{H}} \to (A^\vee)_{\mathcal{H}} \to 0. \tag{13}$$

Equations (10) and (13) are the *fundamental exact sequences*.

3.4 Strategy of proof

Let us return to the setup and notations in Theorem 1.3. Let us assume (S_G) and (WLC_G). We also note that if we assume $(Tors_G)$, then it implies (S_G) and (WLC_G) by Remark 2.1.

Note that neither Σ nor G have elements of order p since $p \geqslant 5$. Recall that the Selmer group of E over K_∞ is given by the exact sequence

$$0 \to \mathrm{Sel}(E/K_\infty) \to H^1(G_S(K_\infty), E_{p^\infty}) \to \bigoplus_{v \in S} J_v(K_\infty), \tag{14}$$

where S is a finite set of primes of F containing those above p and the ones where at least one of E or E' has bad reduction. For the proof of Theorem 1.3, it will be more convenient to rewrite the local term as follows. Let $v \in S$, and denote by G_w the decomposition group of a fixed prime of K_∞ above v. Then

$$J_v(K_\infty) = \mathrm{CoInd}_{G_w}^G H^1(K_{\infty,w}, E)(p),$$

so that (14) can be rewritten as

$$0 \to \mathrm{Sel}(E/K_\infty) \to H^1(G_S(K_\infty), E_{p^\infty}) \xrightarrow{\lambda_S(K_\infty)} \bigoplus_{v \in S} \mathrm{CoInd}_{G_w}^G H^1(K_{\infty,w}, E)(p). \tag{15}$$

By assumption (S_G), the map $\lambda_S(K_\infty)$ is surjective, so dualizing (15) gives the short exact sequence

$$0 \to \bigoplus_{v \in S} U_v \to X_S \to \mathcal{C}(E/K_\infty) \to 0, \tag{16}$$

where

$$\mathcal{C}(E/K_\infty) = \mathrm{Sel}(E/K_\infty)^\vee, \tag{17}$$

$$X_S = H^1(G_S(K_\infty), E_{p^\infty})^\vee, \text{ and} \tag{18}$$

$$U_v = \mathrm{Ind}_{G_w}^G H^1(K_{\infty,w}, E)(p)^\vee. \tag{19}$$

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Here, we quote the following result from [OV02].

Proposition 3.8 *Assume* (WLC_G). *Then,* X_S *has no nontrivial pseudonull submodule. Equivalently, we have*

$$E^i E^i X_S = 0 \text{ for all } i \geqslant 2.$$

By Lemma 3.4, proving that $\mathcal{C}(E/K_\infty)$ has no nontrivial pseudonull submodule is equivalent to showing that

$$E^i E^i \mathcal{C}(E/K_\infty) = 0 \text{ for all } i \geqslant 2. \tag{20}$$

We split the proof of this result in two steps: In Section 3.5, we prove a couple of results about the vanishing of the Ext-groups of the local terms U_v. We then finish the proof of Theorem 1.3 in Section 3.6 by imitating the arguments in [OV02, Section 5].

3.5 Local results

To analyze the Ext-groups of the local terms U_v for $v \in S$, we use the following lemma, which we quote from [OV02].

Lemma 3.9 *Let* $v \in S$, *and denote by* G_v *the decomposition group in* G *of some finite prime of* K_∞ *above* v. *Then for any finitely generated* $\Lambda(G_v)$*-module* M, *we have canonical isomorphisms*

$$E^i_{\Lambda(G)}(\mathrm{Ind}_{G_v}^G M, \Lambda(G)) \cong \mathrm{Ind}_{G_v}^G E^i_{\Lambda(G_v)}(M, \Lambda(G_v))$$

for all $i \geqslant 0$.

The structure of the local extensions $K_{\infty,w}$ over F_v is given by Propositions 2.2 and 2.4. For a prime v of F, let w denote a fixed prime of K_∞ above v. Recall that the decomposition group G_v of w in G can be canonically identified with the local Galois group $\mathrm{Gal}(K_{\infty,w}/F_v)$. Also recall that $G_v \cong \Sigma_w/C_w$, where $C_w = C \cap \Sigma_w$.

In the following, for $v \in S$, we denote $\mathcal{H}_v = \mathrm{Gal}(\overline{F}_v/K_{\infty,w})$, $G_v = \mathrm{Gal}(K_{\infty,w}/F_v)$, $\mathcal{G}_v = \mathrm{Gal}(\overline{F}_v/F_v)$.

The next result is the heart of the proof of Theorem 1.3.

Proposition 3.10 *For all* $v \in S$, *we have*

$$E^i U_v = 0$$

for all $i \geqslant 2$.

Proof Let $v \in S$. Let $X_v = H^1(K_{\infty,w}, E)(p)^\vee$. By Lemma 3.9, it is sufficient to show that

$$E^i_{\Lambda(G_v)} X_v = 0 \tag{21}$$

for all $i \geqslant 2$ and for all $v \in S$. We have to consider different cases.

Assume first that $v \nmid p$. If $\mathrm{ord}_v(j_{E'}) < 0$, then is shown in [Coa99] we have $X_v = 0$, so (21) is clear. Similarly, if $\mathrm{ord}_v(j_{E'}) \geqslant 0$ and G_v is trivial, then

$$X_v = H^1(F_v, E_{p^\infty})^\vee,$$

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which is finite by [Zer04, Lemma 1], so $E^i_{\Lambda(G_v)}X_v = 0$ for all $i \geqslant 2$ by Proposition 3.5. The same argument also works if G_v is finite of order prime to p. Assume now that $\mathrm{ord}_v(j_{E'}) \geqslant 0$ and $G_v \cong \mathbb{Z}_p$. The fundamental exact sequences (10) and (13), applied to $A = E_{p^\infty}$, give

$$0 \to I_v \to \Lambda(G_v)^{n_v} \to W_v \to 0, \tag{22}$$

$$0 \to X_v \to Y_v \to I_v \to 0, \tag{23}$$

where $n_v \geqslant 1$, $\mathrm{pd}\, Y_v \leqslant 1$ (cf. [OV02, Lemma 4.5]), and $W_v = E_{p^\infty}(K_{\infty,w})^\vee$. Here, $Y_v = (\mathcal{I}[E_{p^\infty}])_{\mathcal{H}_v}$, $X_v = H_1(\mathcal{H}_v, E^\vee_{p^\infty})$, and I_v is defined using (13) with $\mathcal{G}$, $\mathcal{H}$, and A replaced by $\mathcal{G}_v$, $\mathcal{H}_v$, and E_{p^∞}, respectively. We have to distinguish between two different cases:

(1) $\mathrm{ord}_v(j_E) \geqslant 0$: It follows from [Zer04, Lemma 1] that if E_{p^∞} is not rational over $K_{\infty,w}$, then it will become rational over some finite extension of $K_{\infty,w}$ of order prime to p. We can therefore without loss of generality assume that E_{p^∞} is rational over $K_{\infty,w}$. It follows from (22) and (23) that for all $i \geqslant 2$ we have canonical isomorphisms

$$E^i_{\Lambda(G_v)}X_v \cong E^{i+1}_{\Lambda(G_v)}I_v \cong E^{i+2}_{\Lambda(G_v)}W_v.$$

But $E^{i+2}_{\Lambda(G_v)}W_v$ is zero by Proposition 3.5 which leads to the conclusion.

(2) $\mathrm{ord}_v(j_E) < 0$: From the Tate parameterization, we get a short exact sequence of G_v-modules

$$0 \to \mu_{p^\infty} \to E_{p^\infty}(K_{\infty,w}) \to T \to 0, \tag{24}$$

where T is a finite p-group on which G_v acts trivially. Dualizing (24) and using Proposition 3.5 shows that

$$E^i_{\Lambda(G_v)}W_v = 0$$

for all $i \geqslant 3$. Arguing as above shows that we have canonical isomorphisms $E^i_{\Lambda(G_v)}X_v \cong E^{i+2}_{\Lambda(G_v)}W_v$ for all $i \geqslant 2$, which finishes the prove in this case.

Assume now that $v \mid p$. In complete analogy to the above situation, we have short exact sequences

$$0 \to I_v \to \Lambda(G_v)^{n_v} \to W_v \to 0, \tag{25}$$

$$0 \to X_v \to Y_v \to I_v \to 0, \tag{26}$$

where $n_v \geqslant 1$, $\mathrm{pd}\, Y_v \leqslant 1$, and

$$W_v = \tilde{E}_{v,p^\infty}(k_{K_{\infty,w}})^\vee.$$

Here, we recall that $\tilde{E}_v$ denotes the reduction of E mod v and $k_{K_{\infty,w}}$ is the residue field of $K_{\infty,w}$. Now $\dim G_v = 2$ and $\tilde{E}_{v,p^\infty}(k_{K_{\infty,w}})$ is finite, so we can argue as above. ∎

The following result will also be crucial in the proof.

Proposition 3.11 *For all $v \in S$, we have*

$$E^2_{\Lambda(G_v)}E^1_{\Lambda(G_v)}X_v = 0. \tag{27}$$

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Proof From the fundamental exact sequences (10) and (13), for all $v \in S$, we have short exact sequences

$$0 \to I_v \to \Lambda(G_v)^{n_v} \to W_v \to 0, \tag{28}$$

$$0 \to X_v \to Y_v \to I_v \to 0, \tag{29}$$

where $n_v \geqslant 1$, $\operatorname{pd} Y_v \leqslant 1$, and

$$W_v = \begin{cases} \tilde{E}_{v,p^\infty}(k_{K_{\infty,w}})^\vee & \text{if } v \mid p, \\ E_{p^\infty}(K_{\infty,w})^\vee & \text{if } v \nmid p. \end{cases}$$

Suppose that $v \mid p$. Using (28) and (29), we deduce that we have the short exact sequence

$$0 \to E^1_{\Lambda(G_v)} Y_v \to E^1_{\Lambda(G_v)} X_v \to E^3_{\Lambda(G_v)} W_v \to 0.$$

Now, by Auslander regularity in Definition 3.2(ii), $E^2_{\Lambda(G_v)} E^3_{\Lambda(G_v)} W_v = 0$, so we have an injection

$$E^2_{\Lambda(G_v)} E^1_{\Lambda(G_v)} X_v \hookrightarrow E^2_{\Lambda(G_v)} E^1_{\Lambda(G_v)} Y_v.$$

But since $\operatorname{pd} Y_v \leqslant 1$, it follows that $E^1_{\Lambda(G_v)} Y_v \cong DY_v$ (see [OV02, discussion before Lemma 3.1]), where by Lemma 4.9 in [OV02] DY_v is given by

$$DY_v \cong (\varinjlim \operatorname{Hom}(\tilde{E}_{v,p^i}, \mu_{p^\infty})(K_{\infty,w}))^\vee.$$

But this is finite since $K_{\infty,w}$ is totally ramified up to a finite extension, so we deduce from Lemma 3.5 that $E^2_{\Lambda(G_v)} DY_v = 0$, which proves (27) when $v \mid p$.

Suppose now that $v \nmid p$. If $\dim G_v = 2$ (or equivalently if $\operatorname{ord}_v(j_{E'}) < 0$), then we see from Lemma 5.4 in [OV02] that $X_v = 0$, so we can assume that $\dim G_v \leqslant 1$. If $\dim G_v = 0$, then clearly the result is trivial. Assume, therefore, that $\dim G_v = 1$, that is, $K_{\infty,w} = F_v^{cyc}$. We have to distinguish between two different cases:

(1) $\operatorname{ord}_v(j_E) \geqslant 0$: In this case, E_{p^∞} is rational over $K_{\infty,w}$ (or maybe over a finite extension of $K_{\infty,w}$ of order prime to p, which does not really make any difference), so $W_v \cong \mathbb{Z}_p^2$ as an abelian group. Using (28) and Proposition 3.5, we deduce that $E^i_{\Lambda(G_v)} I_v = 0$ for all $i \geqslant 1$, so it follows from (29) that we have a canonical isomorphism

$$E^1_{\Lambda(G_v)} X_v \cong E^1_{\Lambda(G_v)} Y_v.$$

Now $\operatorname{pd} Y_v \leqslant 1$, so arguing as above, we have

$$E^1_{\Lambda(G_v)} Y_v \cong \varinjlim \operatorname{Hom}(E_{p^i}, \mu_{p^\infty})^\vee,$$

which is congruent to $\mathbb{Z}_p^2$ as an abelian group. Proposition 3.5, therefore, implies that $E^2_{\Lambda(G_v)} E^1_{\Lambda(G_v)} X_v = 0$, as required.

(2) $\operatorname{ord}_v(j_E) < 0$: From the Tate parameterization, it is clear that we have an isomorphism of abelian groups

$$E_{p^\infty}(K_{\infty,w}) \cong \mathbb{Z}_p \times \Delta,$$

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where Δ is a finite p-group. Let $F_{\infty,w} = F_v(E_{p^\infty})$, so if we define $\Omega_w = \mathrm{Gal}(F_{\infty,w}/K_{\infty,w})$, then $\Omega_w \cong \mathbb{Z}_p$. By Hochschild–Serre, we have an exact sequence

$$0 \to H^1(\Omega_w, E_{p^\infty}) \to H^1(K_{\infty,w}, E_{p^\infty}) \to H^1(F_{\infty,w}, E_{p^\infty})^{\Omega_w}.$$

As shown in [Coa99, Lemma 3.3], we have $H^1(F_{\infty,w}, E_{p^\infty}) = 0$, so we can canonically identify X_v with $H^1(\Omega_w, E_{p^\infty})^\vee$. Now, E_{p^∞} fits into a short exact sequence of Galois modules

$$0 \to \mu_{p^\infty} \to E_{p^\infty} \to \mathbb{Q}_p/\mathbb{Z}_p \to 0. \tag{30}$$

Taking Ω_w-cohomology of (30) and observing that Ω_w acts trivially on μ_{p^∞} gives

$$\begin{aligned} 0 \to \mu_{p^\infty} \to E_{p^\infty}(K_{\infty,w}) \to \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\delta} \mathrm{Hom}(\Omega_w, \mu_{p^\infty}) \\ \to H^1(\Omega_w, E_{p^\infty}) \to \mathbb{Q}_p/\mathbb{Z}_p \to 0. \end{aligned}$$

By explicit calculation, it is easy to see that δ is surjective, so

$$H^1(\Omega_w, E_{p^\infty}) \cong \mathbb{Q}_p/\mathbb{Z}_p,$$

and hence, $X_v \cong \mathbb{Z}_p$. By Proposition 3.5, we have

$$(E^1_{\Lambda(G_v)}\mathbb{Z}_p)^\vee = \varinjlim_i \varinjlim_{U \leqslant G_v} H^1(U, \mathbb{Z}/p^i\mathbb{Z})^*,$$

where U runs over the open normal subgroups of G_v, the inner direct limit is taken with respect to the corestriction maps and * means $\mathrm{Hom}(\sim, \mathbb{Q}/\mathbb{Z})$.

Claim. $E^1_{\Lambda(G_v)}\mathbb{Z}_p \cong \mathbb{Z}_p$.

Proof of claim. The claim easily follows from [NSW13, Corollary 5.5.7] but we also add another proof.

Clearly, every open normal subgroup of G_v is isomorphic to $\mathbb{Z}_p$, and there is a unique one of index p^n for all $n \geqslant 1$; denote this subgroup by U_n, and set $U_0 = G_v$. For $n \geqslant 0$ and $a \geqslant 1$, and denote by

$$\mathrm{cor}_{n,a} : H^1(U_{n+a}, \mathbb{Z}/p^i\mathbb{Z}) \to H^1(U_n, \mathbb{Z}/p^i\mathbb{Z}), \tag{31}$$

$$\mathrm{res}_{n,a} : H^1(U_n, \mathbb{Z}/p^i\mathbb{Z}) \to H^1(U_{n+a}, \mathbb{Z}/p^i\mathbb{Z}) \tag{32}$$

the respective (co)restriction maps. Let

$$f_{n+a} \in H^1(U_{n+a}, \mathbb{Z}/p^i\mathbb{Z}) \cong \mathrm{Hom}(U_{n+a}, \mathbb{Z}/p^i\mathbb{Z}),$$

and choose some lift

$$\tilde{f}_n \in \mathrm{Hom}(U_n, \mathbb{Z}/p^{i+a}\mathbb{Z})$$

of f_{n+a}. Note that this lift is certainly not unique! We then have

$$\begin{aligned} \mathrm{cor}_{n,a}(f_{n+a}) &= \mathrm{cor}_{n,a} \circ \mathrm{res}_{n,a}(\tilde{f}_n) \\ &= p^a \tilde{f}_n, \end{aligned}$$

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which is independent of the chosen lift. It follows immediately from this equation that $\mathrm{cor}_{n,a}$ gives an isomorphism

$$\mathrm{cor}_{n,a} : H^1(U_n, \mathbb{Z}/p^i\mathbb{Z}) \xrightarrow{\cong} H^1(U_{n+a}, \mathbb{Z}/p^i\mathbb{Z}),$$

and so

$$\varinjlim_n H^1(U_n, \mathbb{Z}/p^i\mathbb{Z}) \cong \mathbb{Z}/p^i\mathbb{Z},$$

which proves the claim.

Since $\dim G_v = 1$, it therefore follows from Lemma 3.5 that

$$E^2_{\Lambda(G_v)} E^1_{\Lambda(G_v)} X_v = 0,$$

which finishes the proof. ∎

3.6 Algebraic structure of the dual Selmer group

As a consequence of Propositions 3.10 and 3.11, we get the following partial proof of (20).

Proposition 3.12 *$E^i E^i \mathcal{C}(E/K_\infty) = 0$ for all $i \geqslant 3$.*

Proof Let $U = \bigoplus_{v \in S} U_v$. From (16), for all $i \geqslant 2$, we obtain exact sequences

$$E^i U \to E^{i+1}\mathcal{C}(E/K_\infty) \to E^{i+1} E_S \to E^{i+1} U,$$

so by Proposition 3.10, for all $i \geqslant 3$, we have canonical isomorphisms

$$E^i \mathcal{C}(E/K_\infty) \cong E^i X_S.$$

The proposition is now an immediate consequence of Proposition 3.8. ∎

Proof of Theorem 1.3. It remains to show that $E^2 E^2 \mathcal{C}(E/K_\infty) = 0$. From the long Ext-sequence of (16), we get the exact sequence

$$E^1 X_S \to E^1 U \to E^2 \mathcal{C}(E/K_\infty) \to E^2 X_S \to 0,$$

which we can split up into the short exact sequences

$$0 \to B \to E^2 \mathcal{C}(E/K_\infty) \to E^2 X_S \to 0, \tag{33}$$

$$0 \to C \to E^1 U \to B \to 0, \tag{34}$$

$$0 \to D \to E^1 X_S \to C \to 0. \tag{35}$$

In complete analogy to the GL_2-situation analyzed in [OV02, Proof of Lemma 5.6], we have a canonical isomorphism $E^2 X_S \cong E^4 W$, where $W = E_{p^\infty}(K_\infty)^\vee$. We therefore have $E^3 E^2 X_S = 0$, and so from the first sequence we get an isomorphism

$$E^2 E^2 \mathcal{C}(E/K_\infty) \cong E^2 B.$$

From the second sequence (34), we get an exact sequence

$$E^1 E^1 U \to E^1 C \to E^2 B \to E^2 E^1 U.$$

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Combining Lemma 3.9 with Proposition 3.11, we have $E^2E^1U = 0$. Proposition 2.4 with [OV02, Lemma 5.4(i)] shows that for all $v \in S$, $v \mid p$, X_v is torsionfree. Note that we are assuming that for all $v \in S$, $v \nmid p$, $\mathrm{ord}_v(j_{E'}) < 0$; this implies, by [OV02, Lemma 5.4(ii)], that $X_v = 0$. Since $\mathrm{pd}\, X_v \leqslant 1$, we have $E^1_{\Lambda(G_v)}E^1_{\Lambda(G_v)}X_v \cong E^1_{\Lambda(G_v)}(DX_v) = \mathrm{tor}_{\Lambda(G_v)}X_v = 0$ (cf. [NSW13, Proposition 5.5.3(i) and Definition 5.4.10]), and so $E^1E^1U = 0$. Hence, we have $E^1C \cong E^2B$. Now, D is a quotient of $E^1\mathcal{C}(E/K_\infty)$, so clearly

$$E^0D \hookrightarrow E^0E^1\mathcal{C}(E/K_\infty) = 0.$$

From the third sequence, we therefore get an injection

$$E^1C \hookrightarrow E^1E^1X_S.$$

Claim $E^1E^1X_S = \mathrm{tor}_{\Lambda(G)}X_S$.

Proof of Claim In the notation of [OV02], we have a short exact sequence

$$0 \to I \to \Lambda(G)^2 \to W \to 0.$$

Using Proposition 3.5 and the fact that $\dim(G) = 3$, we deduce that

$$E^0I \cong E^0\Lambda(G)^2, \tag{36}$$

$$E^3I \cong E^4W, \tag{37}$$

$$E^iI = 0 \text{ otherwise.} \tag{38}$$

From the short exact sequence (which is a special case of (10)),

$$0 \to X_S \to Y \to I \to 0,$$

we therefore get an isomorphism $E^1X_S \cong E^1Y$. But $\mathrm{pd}\, Y \leqslant 1$, so $DY \cong E^1Y$, and hence,

$$\mathrm{tor}_{\Lambda(G)}X_S \cong \mathrm{tor}_{\Lambda(G)}Y = E^1(DY) \cong E^1E^1Y \cong E^1E^1X_S,$$

which proves the claim (the second equality follows from [NSW13, Proposition 5.5.3(i) and Definition 5.4.10]).

So E^2B is a pseudonull submodule of X_S, which implies that $E^2B = 0$, and hence $E^2E^2\mathcal{C}(E/K_\infty) = 0$. ∎

4 Variation of Iwasawa μ-invariants under isogeny

Let $p \geq 5$ be a fixed prime as before and E_1 and E_2 be elliptic curves without complex multiplication defined over a number field F with good ordinary reduction at all places of F above p, and they are isogenous via $\varphi : E_1 \to E_2$. We assume that $F_\infty = F(E_{1,p^\infty}) = F(E_{2,p^\infty})$ is a pro-p extension of F. Let A be the p-part of the kernel of φ. Throughout this section, S is a finite set of primes of F including all the primes which divide p and places where E_1 (hence E_2) has bad reduction. Let F^S be the maximal extension of F unramified outside S. Similar to an assumption on the surjectivity of the global to local maps over the $GL(2)$-extension F_∞ (see [How02, Assumption L, Equation (72)]), we assume the corresponding condition over the PGL_2-extension K_∞.

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Assumption K The localization maps

$$\lambda_{i,K_\infty} : H^1(F^S/K_\infty, E_{i,p^\infty}) \to \bigoplus_{v\in S} J_v(E_i/K_\infty)$$

are surjective for both $i = 1, 2$.

Remark 4.1 Assume the weak Leopoldt's conjecture over K_∞, that is, $H^2(F_S/K_\infty, E_{i,p^\infty}) = 0$. Then Assumption K is satisfied if and only if the Selmer group $\mathcal{C}(E_i/K_\infty)$ is $\Lambda(G)-$torsion (see [RS22, Theorem 3.4]).

Remark 4.2 We also note that if we assume that $\mathcal{C}(E_i/K_\infty)$ is $\Lambda(G)-$torsion then [HV03, Theorem 7.2] shows that $H^2(F_S/K_\infty, E_{i,p^\infty}) = 0$ and Assumption K is satisfied.

The isogeny φ induces homomorphisms on the cohomology groups and Selmer groups, and hence, we have the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Sel}(E_1/K_\infty) & \longrightarrow & H^1(F^S/K_\infty, E_{1,p^\infty}) & \xrightarrow{\lambda_{1,K_\infty}} & \oplus_{v\in S} J_v(E_1/K_\infty) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3=\oplus\varphi_{3,v}} & & \\
0 & \longrightarrow & \mathrm{Sel}(E_2/K_\infty) & \longrightarrow & H^1(F^S/K_\infty, E_{2,p^\infty}) & \xrightarrow{\lambda_{2,K_\infty}} & \oplus_{v\in S} J_v(E_2/K_\infty) & \longrightarrow & 0.
\end{array}$$

As in the proof of [How02, Theorem 3.1], for a discrete, cofinitely generated $\Lambda(G)$-module D, we let $\chi(G, D) = \prod_{i\geq 0} \#(H^i(G, D))^{(-1)^i}$. If M is a finitely generated $\Lambda(G)$-module then we have the μ-invariant $\mu(M) = ord_p(\chi(G, M(p)^\vee))$, where $M(p)$ is the submodule of M consisting of all the elements of M which are annihilated by some power of p. Our main theorem below is an analog of [How02, Theorem 3.1] but for the *non-admissible* PGL_2-extension K_∞.

Theorem 4.3 *Let $p \geq 5$. Let $\varphi : E_1 \to E_2$ be an isogeny between two elliptic curves without complex multiplication. Assume that $\mathcal{C}(E_i/K_\infty)$ is $\Lambda(G)-$torsion for both $i = \{1, 2\}$ Then as $\Lambda(G)$-modules*

$$\begin{aligned}
\mu(\mathrm{coker}(\varphi_1)^\vee) - \mu(\ker(\varphi_1)^\vee) = & \sum_{v|\infty} ord_p\left(\#(A(F_v))\right) - |F : \mathbb{Q}| ord_p(\#A) \\
& - \sum_{v|p} ord_p(|\#\tilde{A}_v|_v) \\
& + \sum_{\substack{v\in S, v\nmid p \\ ord_v(j_E)\geq 0}} \left(ord_p(\#(E_{1,p^\infty}(F_v)) - ord_p(\#(E_{2,p^\infty}(F_v)))\right).
\end{aligned}$$

Proof From the exact sequence,

$$0 \to \ker(\varphi_1) \to \ker(\varphi_2) \to \ker(\varphi_3) \to \mathrm{coker}(\varphi_1) \to \mathrm{coker}(\varphi_2) \to \mathrm{coker}(\varphi_3) \to 0$$

obtained using snake Lemma, since the Euler characteristic $\chi(G, -)$ is multiplicative, we obtain

$$\frac{\chi(G, \mathrm{coker}(\varphi_1))}{\chi(G, \ker(\varphi_1))} = \frac{\chi(G, \mathrm{coker}(\varphi_2))}{\chi(G, \ker(\varphi_2))} \times \frac{\chi(G, \ker(\varphi_3))}{\chi(G, \mathrm{coker}(\varphi_3))}. \tag{39}$$

Note that all the Euler characteristics above are defined because of [How02, Corollary 1.7] and [RS22, Lemma 2.1]. Since $\mathrm{coker}(\varphi_1)$ and $\ker(\varphi_1)$ are both annihilated by a

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finite power of p, the number we require is

$$\log_p\left(\frac{\chi(G,\mathrm{coker}(\varphi_1))}{\chi(G,\ker(\varphi_1))}\right).$$

■

4.1 Global calculations

We start with the exact sequence

$$0 \to A \to E_{1,p^\infty} \to E_{2,p^\infty} \to 0. \tag{40}$$

Since the p-cohomological dimension $\mathrm{cd}_p(\mathrm{Gal}(F^S/K_\infty)) \le \mathrm{cd}_p(\mathrm{Gal}(F^S/F)) = 2$ (by [NSW13, Proposition 10.9.3]), we have $H^j(F^S/K_\infty, E_{i,p^\infty})$ and $H^j(F^S/K_\infty, A) = 0$ for all $j \ge 3$.

Taking $\mathrm{Gal}(F_S/K_\infty)$-invariants of the terms in the sequence (40) and using the assumption that $H^2(F^S/K_\infty, E_{1,p^\infty}) = 0$ one obtains the exact sequence

$$\begin{aligned} 0 \to A^C \to E_{1,p^\infty}^C \to E_{2,p^\infty}^C \xrightarrow{\alpha} & H^1(F^S/K_\infty, A) \\ \to H^1(F^S/K_\infty, E_{1,p^\infty}) \xrightarrow{\varphi_2} & H^1(F^S/K_\infty, E_{2,p^\infty}) \to H^2(F^S/K_\infty, A) \to 0. \end{aligned}$$

Hence, it is easy to see that

$$\mathrm{coker}(\varphi_2) = H^2(F^S/K_\infty, A). \tag{41}$$

As C is a pro-p group, procyclic group of dimension one. We choose a generator σ of C. By [NSW13, Proposition 1.7.7], $H^1(C, E_{i,p^\infty}) = \frac{E_{i,p^\infty}}{p^{n_i} E_{i,p^\infty}} = 0$ by p-divisibility of E_{i,p^∞}. Here, σ acts as multiplication by $1 + p^{n_i}$ on E_{i,p^∞}. Taking C-invariants of (40), one obtains the following exact sequence:

$$0 \to A^C \to E_{1,p^\infty}^C \to E_{2,p^\infty}^C \to H^1(C, A) \to 0.$$

Thus, one obtains an exact sequence

$$0 \to H^1(C, A) = \mathrm{Im}(\alpha) \to H^1(F_S/K_\infty, A) \to \ker(\varphi_2) \to 0, \tag{42}$$

and this gives that

$$\ker(\varphi_2) = \frac{H^1(F^S/K_\infty, A)}{H^1(C, A)}.$$

Since $H^j(F^S/K_\infty, A) = 0$ for all $j \ge 3$, the Hochschild–Serre spectral sequence

$$H^i(G, H^j(F^S/K_\infty, A)) \implies H^{i+j}(F^S/F, A)$$

yields the following Euler characteristic formula:

$$\frac{\chi(G, H^0(F^S/K_\infty, A))\chi(G, H^2(F^S/K_\infty, A))}{\chi(G, H^1(F^S/K_\infty, A))} = \chi(\mathrm{Gal}(F^S/F), A). \tag{43}$$

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We have $H^0(F^S/K_\infty, A) = A^C$ which is finite and hence the Euler characteristic $\chi(G, A^C) = 1$. From (43), we have

$$\frac{\chi(G, H^2(F^S/K_\infty, A))}{\chi(G, H^1(F^S/K_\infty, A))} = \chi(\mathrm{Gal}(F^S/F), A). \tag{44}$$

By multiplicativity of the Euler characteristics along the exact sequence (42), one obtains

$$\chi(G, H^1(F^S/K_\infty, A)) = \chi(G, H^1(C, A)) \cdot \chi(G, \ker(\varphi_2)).$$

As C is procyclic, pro-p with p-cohomological dimension one, $H^1(C, A) = A_C$ and hence $H^1(C, A)$ is finite. Thus, $\chi(G, H^1(C, A)) = 1$ and hence,

$$\chi(G, \ker(\varphi_2)) = \chi(G, H^1(F^S/K_\infty, A)). \tag{45}$$

Combining (41), (44), and (45), we get

$$\frac{\chi(G, \mathrm{coker}(\varphi_2))}{\chi(G, \ker(\varphi_2))} = \chi(\mathrm{Gal}(F^S/F), A). \tag{46}$$

This completes the computation of Euler characteristics of the kernel and cokernel of the global map φ_2.

4.2 Local computations

Next, let us do the local calculations. As before, let us fix a prime v of F in S and w a prime of K_∞ above v. Let $\tilde{A}_v$ be the image of A under the reduction map $v : E_1 \to \tilde{E}_1$. We define

$$C_v^i(K_\infty) = \begin{cases} \prod_{\eta|v} H^i(K_{\infty,w}, A), & \text{if } v \nmid p, \\ \prod_{\eta|v} H^i(K_{\infty,w}, \tilde{A}_v), & \text{if } v \mid p. \end{cases}$$

As the inertia subgroup of K_∞ over F at the prime w over v is infinite for v a prime of F above p, we have that $K_{\infty,w}$ is a deeply ramified extension of F_v (cf. [RS22, p. 929]). Hence, by [CG96, Proposition 4.8 and Theorem 2.13], one gets

$$J_v(E_i/K_\infty) = \begin{cases} \prod_{w|v} H^1(K_{\infty,w}, E_{i,p^\infty}), & \text{if } v \nmid p, \\ \prod_{w|v} H^1(K_{\infty,w}, \tilde{E}_{i,p^\infty}), & \text{if } v \mid p. \end{cases}$$

Let X be A if $v \nmid p$ and $\tilde{A}_v$ if $v \mid p$. Let Y_i be E_{i,p^∞} if $v \nmid p$ and $\tilde{E}_{i,p^\infty}$ if $v \mid p$.

Now, let v be any prime of F in S, not necessarily above p. Then, from the proof of [CH01, Lemma 5.2], for each finite extension L of F_v, we have that $H^2(L, E_{i,p^\infty}) = 0$. The reduction map $E_{i,p^\infty} \to \tilde{E}_{i,p^\infty}$ is surjective and hence the map induced on the cohomologies $H^2(L, E_{i,p^\infty}) \to H^2(L, \tilde{E}_{i,p^\infty})$ is also surjective as $cd_p(G_L) = 2$ (see [Ser02, Corollary in Section 4.3]). Therefore, $H^2(L, \tilde{E}_{i,p^\infty}) = 0$. Passing to direct limit over all the finite extensions of F_v contained in $K_{\infty,w}$ we have that $H^2(K_{\infty,w}, Y_i) = 0$ and thus we have $\prod_{w|v} H^2(K_{\infty,w}, Y_i) = 0$.

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Next, we have the exact sequence

$$0 \to X(K_{\infty,w}) \to Y_1(K_{\infty,w}) \to Y_2(K_{\infty,w}) \to H^1(K_{\infty,w}, X) \\ \to H^1(K_{\infty,w}, Y_1) \xrightarrow{\varphi_{3,w}} H^1(K_{\infty,w}, Y_2) \to H^2(K_{\infty,w}, X) \to 0$$

from which it is easy to deduce that

$$\operatorname{coker}(\varphi_3) = \bigoplus_{v \in S} C_v^2(K_\infty). \tag{47}$$

From the above sequence, we obtain the exact sequences

$$0 \to X(K_{\infty,w}) \to Y_1(K_{\infty,w}) \to Y_2(K_{\infty,w}) \to H^1(K_{\infty,w}, X) \to \ker(\varphi_{3,w}) \to 0.$$

Taking product over all primes w lying above a fixed prime v, one gets that the sequence

$$0 \to \prod_{w|v} X(K_{\infty,w}) \to \prod_{w|v} Y_1(K_{\infty,w}) \to \prod_{w|v} Y_2(K_{\infty,w}) \to C_v^1(K_\infty) \to \prod_{w|v} \ker(\varphi_{3,w}) \to 0$$

is exact. The multiplicativity of Euler characteristics in exact sequences yields

$$\chi(G, \prod_{w|v} \ker(\varphi_{3,w})) = \frac{\chi(G, C_v^1(K_\infty)) \cdot \chi(G, \prod_{w|v} Y_1(K_{\infty,w}))}{\chi(G, \prod_{w|v} X(K_{\infty,w})) \cdot \chi(G, \prod_{w|v} Y_2(K_{\infty,w}))}. \tag{48}$$

We remark again that finiteness of these Euler characteristics follow from [How02, Corollary 1.7]. Applying Shapiro's lemma, one obtains

$$H^j(G, \prod_{w|v} M) = H^j(G_v, M)$$

for $M = H^1(K_{\infty,w}, X), Y_i(K_{\infty,w}),$ or $X(K_{\infty,w})$. From this, one gets

$$\chi(G, \prod_{w|v} M) = \chi(G_v, M). \tag{49}$$

Since X is finite, so is $X(K_{\infty,w})$ which gives that $\chi(G_v, X(K_{\infty,w})) = 1$.

Let $C_w = \mathrm{Gal}(F_{\infty,w}/K_{\infty,w})$ be the decomposition subgroup of C (by a slight abuse of notation, we are still denoting by w a prime of F_∞ over the prime w of K_∞). So,

$$\chi(G_v, Y_i(K_{\infty,w})) = \chi(G_v, Y_i^{\mathrm{Gal}(F_{\infty,w}/K_{\infty,w})})) = \chi(G_v, Y_i^{C_w})).$$

Now, let us analyze each of the cases: (i) $v \mid p$, (ii) $v \nmid p$ and $ord_v(j_E) < 0$, and (iii) $v \nmid p$ and $ord_v(j_E) \geq 0$ separately.

Case (I) $v \mid p$.

When $v \mid p$, we have $Y_i = \tilde{E}_{i,p^\infty}$. By [CH01, Lemma 5.1], Σ_w has p-cohomological dimension at most 3 and as $C_w = C \cap \Sigma_w$, by using [NSW13, Chapter 3, Section 3, Exercise 4], we obtain that G_v has p-cohomological dimension at most 2. Since $H^i(C_w, \tilde{E}_{i,p^\infty}) = 0$ for all $i \geq 2$, an application of Hochschild–Serre spectral sequence yields the exact sequence

$$0 \to H^1(G_v, \tilde{E}_{i,p^\infty}^{C_w}) \to H^1(\Sigma_w, \tilde{E}_{i,p^\infty}) \to H^1(C_w, \tilde{E}_{i,p^\infty})^{G_v} \to H^2(G_v, \tilde{E}_{i,p^\infty}^{C_w}) \\ \to H^2(\Sigma_w, \tilde{E}_{i,p^\infty}) \to H^1(G_v, H^1(C_w, \tilde{E}_{i,p^\infty})) \to 0. \tag{50}$$

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As E_{i,p^∞} is p-divisible, we have $H^1(C_w, E_{i,p^\infty}) = 0$ and as C_w is of p-cohomological dimension 1 and there is a surjection $E_{i,p^\infty} \to \tilde{E}_{i,p^\infty}$, so $H^1(C_w, \tilde{E}_{i,p^\infty})$ is also trivial. Hence, we get that

$$H^1(G_v, \tilde{E}_{i,p^\infty}^{C_w}) \cong H^1(\Sigma_w, \tilde{E}_{i,p^\infty}) \text{ and } H^2(G_v, \tilde{E}_{i,p^\infty}^{C_w}) \cong H^2(\Sigma_w, \tilde{E}_{i,p^\infty}).$$

We obviously have $H^0(G_v, \tilde{E}_{i,p^\infty}^{C_w}) = H^0(\Sigma_w, \tilde{E}_{i,p^\infty})$. By assumption, since E has good ordinary reduction at v, by [CH01, Corollary 5.26], we have $H^3(\Sigma_w, \tilde{E}_{i,p^\infty}) = 0$. Therefore, finally we deduce that $\chi(G_v, \tilde{E}_{i,p^\infty}^{C_w}) = \chi(\Sigma_w, \tilde{E}_{i,p^\infty})$ but this Euler characteristic is 1, again by *(loc.cit)*.

Case (II) $v \nmid p$ and $ord_v(j_E) < 0$.

When $v \nmid p$ and $ord_v(j_E) < 0$, we have $J_v(F_\infty) = 0$ by [CH01, Corollary 5.12]. Since by the proof of Theorem 3.1 of [RS22], $J_v(K_\infty) = J_v(F_\infty)^C$, we get that $J_v(K_\infty)$ is trivial.

Case (III) $v \nmid p$ and $ord_v(j_E) \geq 0$.

When $v \nmid p$ and $ord_v(j_E) \geq 0$, since $p \geqslant 5$, we have $H^j(\Sigma_w, E_{p^\infty}) = 0$ for $j \geq 1$ from [CH01, Proposition 5.12]. By an analogous Hochschild–Serre spectral sequence (see (50)) and [CH01, Lemma 5.1(ii)], we have $H^j(G_v, E_{i,p^\infty}^{C_w}) = 0$ for $j \geq 1$. So the Euler characteristics is $\chi(G_v, E_{i,p^\infty}^{C_w}) = \#H^0(G_v, E_{i,p^\infty}^{C_w}) = \#E_{i,p^\infty}(F_v)$.

Combining these cases, from (48), we obtain

$$\chi(G, \ker(\varphi_3)) = \prod_{v \in S} \chi(G, \ker(\varphi_{3,v})) = \prod_{v \in S, v|p} \chi(G_v, H^1(K_{\infty,w}, \tilde{A}_v)) \tag{51}$$

$$\times \prod_{v \in S, v \nmid p} \chi(G_v, H^1(K_{\infty,w}, A)) \times \prod_{\substack{v \in S, v \nmid p \\ ord_v(j_E) \geq 0}} \frac{\#(E_{1,p^\infty}(F_v))}{\#(E_{2,p^\infty}(F_v))}. \tag{51}$$

By (47) and applying Shapiro's lemma,

$$\chi(G, \operatorname{coker}(\varphi_3)) = \prod_{v \in S, v|p} \chi(G_v, H^2(K_{\infty,w}, \tilde{A}_v)) \times \prod_{v \in S, v \nmid p} \chi(G_v, H^2(K_{\infty,w}, A)). \tag{52}$$

By considering the Hochschild–Serre spectral sequence,

$$H^i(G_v, H^j(K_{\infty,w}, X)) \implies H^{i+j}(F_v, X)$$

we get

$$\frac{\chi(G, \ker(\varphi_3))}{\chi(G, \operatorname{coker}(\varphi_3))} = \frac{\prod_{v \in S, v \nmid p, ord_v(j_E) \geq 0} \frac{\#(E_{1,p^\infty}(F_v))}{\#(E_{2,p^\infty}(F_v))}}{\prod_{v \in S, v|p} \chi(\mathrm{Gal}(\overline{F}_v/F_v), \tilde{A}_v) \times \prod_{v \in S, v \nmid p} \chi(\mathrm{Gal}(\overline{F}_v/F_v), A)}. \tag{53}$$

Since A is a p-group, from [Mil06, Lemma I.2.9], we have that $\prod_{v \in S, v \nmid p} \chi(\mathrm{Gal}(\overline{F}_v/F_v), A) = 1$. Using (39) and recalling (46), one gets

$$\mu(\operatorname{coker}(\varphi_1)^\vee) - \mu(\ker(\varphi_1)^\vee) = \sum_{\substack{v \in S, v \nmid p \\ ord_v(j_E) \geq 0}} \left(ord_p(\#(E_{1,p^\infty}(F_v)) - ord_p(\#(E_{2,p^\infty}(F_v))) \right)$$

$$- \sum_{v \in S, v|p} ord_p(\chi(\mathrm{Gal}(\overline{F}_v/F_v), \tilde{A}_v)) + ord_p(\chi(\mathrm{Gal}(F^S/F), A).$$

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We have from [Mil06, Theorem I.5.1]

$$\chi(\mathrm{Gal}(F^S/F), A) = \prod_{v|\infty} \frac{\#H^0(\overline{F}_v/F_v, A)}{|\#A|_v}.$$

If v is a non-archimedean prime of F, then from [Mil06, Theorem I.2.8],

$$\chi(\mathrm{Gal}(\overline{F}_v/F_v), \tilde{A}_v) = |\#\tilde{A}_v|_v.$$

Substituting these above we get the formula stated in Theorem 4.3.

Let $\mathcal{C}(E_i/K_\infty)$ be the Pontryagin dual of the Selmer group $\mathrm{Sel}(E_i/K_\infty)$.

Corollary 4.4 *Let $p \geq 5$. Let $\varphi : E_1 \to E_2$ be an isogeny between two elliptic curves without complex multiplication. Assume that the dual Selmer groups $\mathcal{C}(E_i/K_\infty)$ are $\Lambda(G)$-cotorsion for both $i = \{1, 2\}$. Then*

$$\begin{aligned}\mu(\mathcal{C}(E_2/K_\infty)) - \mu(\mathcal{C}(E_1/K_\infty)) = &\sum_{v|\infty} ord_p\left(\#(A(F_v))\right) - |F:\mathbb{Q}|ord_p(\#A) \\ &- \sum_{v|p} ord_p(|\#\tilde{A}_v|_v) \\ &+ \sum_{\substack{v\in S, v\nmid p \\ ord_v(j_E)\geq 0}} \left(ord_p(\#(E_{1,p^\infty}(F_v)) - ord_p(\#(E_{2,p^\infty}(F_v)))\right).\end{aligned}$$

Proof The proof follows by combining Remark 4.1 and Theorem 4.3. ∎

Remark 4.5 Since $\ker(\varphi_1)$ and $\mathrm{coker}(\varphi_1)$ are killed by some finite power of p (cf. statement before Section 4.1), $\mathcal{C}(E_1/K_\infty) \otimes \mathbb{Q}_p \cong \mathcal{C}(E_2/K_\infty) \otimes \mathbb{Q}_p$ and hence $\mathcal{C}(E_1/K_\infty)$ and $\mathcal{C}(E_2/K_\infty)$ have the same $\Lambda(G)$-ranks.

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