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On Fatou maps into compact complex manifolds

KAZUTOSHI MAEGAWA

Graduate School of Human and Environmental Studies, Kyoto University,
Yoshida Nihonmatsu-cho, Sakyo-ku, Kyoto, Japan, 606-8501
(e-mail: km@math.h.kyoto-u.ac.jp)

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Abstract. We will deal with two topics about normality of the iterates of holomorphic self-maps in compact varieties. First, for a holomorphic self-map f of an irreducible compact variety Z, we show that if at least one subsequence of $\{f^n\}_{n\geq 0}$ converges uniformly, then the full sequence $\{f^n\}_{n\geq 0}$ itself is a normal family and the full set of the limit maps is finite or has the structure of a compact commutative Lie group. Second, in the case when Z is non-singular, we deal with the dynamics on forward-invariant compact subsets outside the closures of the post-critical sets. We will describe the semi-repelling structure of the dynamics in terms of repelling points and neutral Fatou discs (center manifolds). In particular, in the case of holomorphic self-maps of projective spaces, we will obtain a stronger result.

1. *Introduction*

In this paper, we deal with two topics on the iterates of a holomorphic self-map f of a compact variety Z.

First, we focus on the structure of the full set of the limit maps of $\{f^n\}_{n\geq 1}$ in the space $\mathrm{Hol}(Z)$ of holomorphic self-maps of Z endowed with the compact–open topology. In the case of a (possibly non-compact) taut manifold Z, Abate investigated the full set of the limit maps when $\{f^n\}_{n\geq 1}$ is not compactly divergent; equivalently, at least one subsequence of $\{f^n\}_{n\geq 1}$ converges uniformly on compact sets. He showed the following theorem.

THEOREM 1.1. **[Ab]** *Let Z be a taut complex manifold, and $f \in \mathrm{Hol}(Z)$. Assume that $\{f^n\}$ is not compactly divergent. Then, there exists a submanifold S which is the image of a holomorphic retraction $\rho : Z \to Z$ such that every limit map $h \in \mathrm{Hol}(Z)$ of $\{f^n\}$ is of the form $h = \gamma \circ \rho$, where γ is an automorphism of S. Moreover, ρ is a limit point of $\{f^n\}$.*

In the case when Z is a compact variety, by the compactness of Z, compact divergence is impossible, but, in general, $\{f^n\}_{n\geq 1}$ does not necessarily have limit maps; at least in the case of degree of $f \geq 2$, there can be no limit maps. So, in our study, we assume that

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at least one subsequence of $\{f^n\}_{n\geq 1}$ converges uniformly on compact sets. Then, we can show that the full sequence $\{f^n\}_{n\geq 1}$ is itself a normal family and the full set of the limit maps of $\{f^n\}_{n\geq 1}$ is finite or has the structure of a compact commutative Lie group whose identity element corresponds to a holomorphic retraction.

Second, we deal with the dynamics on forward-invariant compact subsets outside the closures of the post-critical sets. Let f be a surjective holomorphic self-map of a compact complex Hermitian manifold M. We will consider the possibilities of the dynamics on forward-invariant compact subsets outside D_f (the closure of the orbit of the critical locus C_f). Our theorem is the following.

THEOREM 1.2. *Let M be a compact complex Hermitian manifold of dimension ≥ 1. Let $f : M \to M$ be a surjective holomorphic self-map. Assume that $M \setminus D_f$ is hyperbolically imbedded in M. Let E be a compact subset in M such that $f(E) \subset E$ and $E \cap D_f = \emptyset$. Then, there are two subsets $E^u, E^c \subset E$ which have the following properties:*

(i) $E^u \cup E^c = E$, $E^u \cap E^c = \emptyset$;

(ii) $f(E^u) \subset E^u$, $f(E^c) \subset E^c$;

(iii) *each point in E^u is repelling;*

(iv) *for each $p \in E^c$, there is a neutral Fatou disc $\psi : \Delta \to M$ such that $\psi(0) = p$, $D\psi(0) \neq 0$.*

Moreover, if $f(E) = E$ and $E^c = \emptyset$, then E is a repeller with respect to the Hermitian metric in M.

In the case when $M = \mathbb{P}^k$ and $\deg(f) \geq 2$, without assuming that $\mathbb{P}^k \setminus D_f$ is hyperbolically imbedded in $\mathbb{P}^k$, the same assertion holds.

In a sense, this theorem is a generalization of the result that any fixed point outside D_f is a repelling fixed point or the center of a Siegel disc for a rational function f of degree≥ 2 in the Riemann sphere. Let us note that under a small perturbation of f in the family of holomorphic self-maps in M, the repeller E moves holomorphically and remains a repeller. (See [**J**].)

The definitions of neutral Fatou discs, repelling points and repellers appear in Definitions 3.11, 3.18 and 3.19. To prove the theorem, first, we will remove the possibility of the existence of contracting subsets, in particular, contracting Fatou discs attached to E. Next, we will consider the limit maps of locally defined inverse branches of the iterates of f. (In this paper, an 'inverse branch' of g means a single-valued branch of g^{-1}.) We will show that the limit maps are Fatou maps. Let us denote a complex projective space of dimension k (≥ 1) by $\mathbb{P}^k$. Fatou maps were originally introduced and studied by Ueda in the case of $\mathbb{P}^k$ in [**U1**]. They are generalizations of Fatou sets, i.e. domains of normality for the iterates of f. Ueda showed that the limit maps of the sequence of all inverse branches defined in a ball outside D_f are Fatou maps in the case of holomorphic endomorphisms of degree ≥ 2 in $\mathbb{P}^k$. (See [**U1, U2**].) Our result in the case of surjective holomorphic self-maps in any compact complex manifold is not a complete generalization of the original, but it is sufficient for the purpose here.

We will also show that Theorem 1.2 does not hold in general unless we assume the condition that $M \setminus D_f$ is hyperbolically imbedded in M.

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When the draft of this paper was in preparation, I communicated with J. Robertson and was told that he had showed independently a theorem which is similar to the last part of Theorem 1.2 in the case when E is equal to the Julia set, but he assumed that $\mathbb{P}^k \setminus D_f$ is hyperbolically imbedded in $\mathbb{P}^k$ [**R**].

2. *Stable dynamics in the whole space*

Let Z be an irreducible compact variety and $\mathrm{Sing}(Z)$ be the set of singular points of Z. Denote the space of holomorphic self-maps of Z endowed with the compact–open topology by $\mathrm{Hol}(Z)$ and the group of holomorphic automorphisms of Z by $\mathrm{Aut}(Z)$, viewed as a subspace of $\mathrm{Hol}(Z)$.

To show the main theorem in this section, we need the following theorem which is a generalization of a famous theorem due to Bochner and Montgomery. See [**Ak**] and [**BM**] for details.

THEOREM 2.1. (Gunning, Kerner) *The group* $\mathrm{Aut}(Z)$ *has the structure of a complex Lie group whose topology is compatible with the compact–open topology and whose group operator is compatible with the composition of maps.*

Since Z is compact, the Remmert proper mapping theorem implies that $f^n(Z)$ is an irreducible subvariety of Z for every $n \geq 0$ and there exists a number $m_0 \geq 0$ such that

$$Z \supsetneq f(Z) \supsetneq \cdots \supsetneq f^{m_0}(Z) = f^{m_0+1}(Z) = \cdots .$$

Let us call $S := f^{m_0}(Z)$ the *minimal image* of Z. Let g be the restriction of f on S. Then, g is a surjective holomorphic self-map of S.

LEMMA 2.2. *If the topological degree of* g *is* 1*, then* $S \setminus \mathrm{Sing}(Z)$ *is smooth.*

To show this, we need the following lemma (see [**Ab**]). We say that a holomorphic map of $\rho : Z \to Z$ is a *holomorphic retraction* if $\rho \circ \rho = \rho$.

LEMMA 2.3. (Rossi) *Let* M *be a complex manifold and* $\rho : M \to M$ *be a holomorphic retraction. Then,* $\rho(M)$ *is a closed submanifold of* M.

Proof of Lemma 2.2. Set $\rho := g^{-m_0} \circ f^{m_0}$. Then, $\rho : M \to M$ is a holomorphic retraction whose image is S. For any point $p \in S \setminus \mathrm{Sing}(Z)$, there is a neighborhood U_p of p such that $\rho(U_p)$ is contained in $S \setminus \mathrm{Sing}(Z)$ because $\rho(p) = p$. Set $V := \bigcup_{p \in S} U_p$. Then, V is a complex manifold and $\rho_{|V} : V \to V$ is a holomorphic retraction whose image is $S \setminus \mathrm{Sing}(Z)$. By Lemma 2.3, it follows that $S \setminus \mathrm{Sing}(Z)$ is smooth. □

THEOREM 2.4. *Let* Z *be an irreducible compact variety and* $f \in \mathrm{Hol}(Z)$*. If at least one subsequence of* $\{f^n\}_{n \geq 0}$ *which converges in* $\mathrm{Hol}(Z)$*, then the following (a)–(c) hold.*

(a) *The full sequence* $\{f^n\}_{n \geq 0}$ *is a normal family.*

(b) *The set* $\mathcal{L}$ *of all the limit maps of* $\{f^n\}_{n \geq 0}$ *is finite or has the structure of a compact commutative Lie group. (Here the topology is compatible with the compact–open topology and the group operator is compatible with the composition of maps. The identity element corresponds to a holomorphic retraction.)*

(c) *The minimal image* S *is smooth outside of* $\mathrm{Sing}(Z)$.

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Proof. (a) Suppose that a subsequence $\{f^{n_k}\}_{k\geq 0}$ of $\{f^n\}_{n\geq 0}$ converges in $\mathrm{Hol}(Z)$. Let g be the restriction of f on the minimal image S of Z. Then, $\{g^{n_k}\}_{k\geq 0}$ converges in $\mathrm{Hol}(S)$. It is sufficient to show that $\{g^n\}_{n\geq 0}$ is a normal family. By taking the limits on both sides of the equation

$$g^{n_{k+1}} = g^{n_{k+1}-n_k} \circ g^{n_k},$$

it follows that

$$\lim_{k\to\infty} g^{n_{k+1}-n_k} = \mathrm{id}_S.$$

In particular, g is injective, hence $g \in \mathrm{Aut}(S)$. (Precisely speaking, this is true only in the case when S is a normal variety. In the case when S is not normal, we consider the normalization of S instead.)

By Theorem 2.1, $\mathrm{Aut}(S)$ is a (complex) Lie group. Set $G := \overline{\{g^n\}}_{n\in\mathbb{Z}}$ (take the closure in $\mathrm{Aut}(S)$). Since G is a closed subgroup of $\mathrm{Aut}(S)$, it follows that G is a Lie group. Obviously, G is commutative. Let G_0 denote the connected component of G which contains id_S. Since G_0 is a connected commutative Lie group, it follows that G_0 is isomorphic to $\mathbb{T}^u \times \mathbb{R}^v, u, v \geq 0$, where $\mathbb{T}^u$ is a torus and $\mathbb{R}^v$ is the Euclidean space. Since $\lim_{k\to\infty} g^{n_{k+1}-n_k} = \mathrm{id}_S$, there exists an integer $p \geq 1$ such that g^p is in G_0. Assume that p is the smallest number. Then, it follows that $G_0 = \overline{\{g^{mp}\}}_{m\in\mathbb{Z}}$, i.e. g^p generates G_0. If $v > 1$, it is impossible that g^p generates G_0. Hence, G_0 is isomorphic to $\mathbb{T}^u$ (possibly a point, i.e. $u = 0$). In particular, G_0 is compact. This is equivalent to that $\{g^{mp}\}_{m\in\mathbb{Z}}$ is a normal family. Thus, it follows that $\{g^n\}_{n\geq 0}$ is also a normal family.

(b) Recall $G := \overline{\{g^m\}}_{m\in\mathbb{Z}}$. For any $m \in \mathbb{Z}$, it follows that

$$g^m = \lim_{k\to\infty} g^{m+(n_{k+1}-n_k)}$$

because $\lim_{k\to\infty} g^{n_{k+1}-n_k} = \mathrm{id}_S$. So, any $h \in G$ is a limit map of $\{g^n\}_{n\geq 0}$. Hence, it follows that $\mathcal{L} = \{h \circ f^{m_0} \mid h \in G\}$. From this, we have that $\mathcal{L}$ is finite or has the structure of a compact commutative Lie group. The identity element of $\mathcal{L}$ is a holomorphic retraction $g^{-m_0} \circ f^{m_0}$.

(c) The map g is in $\mathrm{Aut}(S)$. By Lemma 2.2, $S \setminus \mathrm{Sing}(Z)$ is smooth. □

3. *Fatou maps outside the post-critical set*

Let M be a compact complex Hermitian manifold of dimension $k \geq 1$. Let d_M be the distance from the Hermitian metric $|\cdot|$ on M. Let $f : M \to M$ be a surjective holomorphic map. The maximal rank of the Jacobian matrix for f is k. We denote by C_f the set of critical points. By the proper mapping theorem of Remmert, $f(C_f)$ is an analytic subset of M. The restriction $f : f^{-1}(M \setminus f(C_f)) \to M \setminus f(C_f)$ is a finite-to-one topological covering. We set $D_f := \bigcup_{n\geq 1} f^n(C_f)$. Let us note that $\bigcup_{1\leq n\leq k} f^n(C_f)$ is the set of critical values of f^k.

Throughout this section, we often consider whether a family which consists of inverse branches defined in a ball outside D_f is normal. In the case of holomorphic endomorphisms of degree ≥ 2 in $\mathbb{P}^k$, by Ueda's result, the normality follows without additional assumptions. (See [**U1, U2**].) Let us keep this in mind in the sequel.

We say that a topological space X is a *continuum* if X contains at least two points and is connected.

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Definition 3.1. (Contracting sets) Let A be a subset in M and f be a surjective holomorphic self-map of M. We say that A is a contracting set for f if A is a continuum and there is a subsequence of $\{f^n\}_{n\geq 0}$ which converges to a constant uniformly in A. We set

$$\mathcal{L}(A) := \{p \in M \mid \exists\{n_j\} \text{ such that } f^{n_j} \to p \text{ as } j \to \infty \text{ uniformly in } A\}.$$

Remark 3.2. From the definition, it follows immediately that $\mathcal{L}(A)$ is forward-invariant and compact.

We will introduce the following condition in order to prove the main theorem later.

Definition 3.3. Let f be a surjective holomorphic self-map of M. Let $\mathbf{T}(M \setminus D_f)$ be the holomorphic tangent bundle over $M \setminus D_f$. The condition (C) is as follows.

(C) There is a constant $c > 0$ such that $|v|_K \geq c|v|$ for any $v \in \mathbf{T}(M \setminus D_f)$ where $|\cdot|_K$ is the infinitesimal Kobayashi metric in $M \setminus D_f$ (i.e. $M \setminus D_f$ is hyperbolically imbedded in M).

Remark 3.4. Since M is compact, (C) implies that the family of all holomorphic maps from any complex manifold N to $M \setminus D_f$ is a normal family as a family of maps from N to M. (See [**K**].)

Remark 3.5. $M \setminus D_f$ is backward-invariant for f.

THEOREM 3.6. *Suppose that f satisfies (C). Let A be a contracting set for f. Then, $\mathcal{L}(A) \subset D_f$.*

Proof. Let $p \in \mathcal{L}(A)$. Suppose that $p \notin D_f$. Then, there is a simply connected neighborhood U of p which is outside D_f. Let $\{f^{n_j}\}_{j\geq 0}$ be the subsequence such that f^{n_j} converges to p uniformly in A as $j \to \infty$. There is an integer m such that $f^{n_j}(A) \subset U$ for all $j \geq m$. Let us take an inverse branch g_j of f^{n_j} defined in U for each $j \geq m$ such that $g_j(U)$ contains at least one point in A. Then, $A \subset g_j(U)$ because A is connected. Obviously, $g_j(U) \subset M \setminus D_f$ for all $j \geq m$. From (C), it follows that $\{g_j\}_{j\geq m}$ is a normal family in U. But this is impossible since $\{g_j\}_{j\geq m}$ is not equicontinuous at p. □

Remark 3.7. In the case when $M = \mathbb{P}^k$ and $\deg(f) \geq 2$, without the condition (C), it follows that $\{g_j\}_{j\geq m}$ is normal by a general result in [**U2**] (see the beginning of this section). So, the theorem above holds without the condition (C).

Let us recall the following concept which is a generalization of Fatou set. This was introduced by Ueda in [**U1**] and he studied its properties in the case of holomorphic endomorphisms of projective spaces. The study in the case of birational maps is found in [**M**].

Definition 3.8. (Fatou maps) Let N be a complex manifold and $\psi : N \to M$ be a holomorphic map such that $\{f^n \circ \psi\}_{n\geq 0}$ is a normal family. Then, we call ψ a *Fatou map*. Particularly, in the case when ψ is a holomorphic disc, i.e. $N = \Delta$, the unit disc in $\mathbb{C}$, and ψ is not constant, we call it a *Fatou disc*. We say that a map $\phi : N \to M$ is a *limit map* of $\{f^n \circ \psi\}_{n\geq 0}$ if there is a subsequence of $\{f^n \circ \psi\}_{n\geq 0}$ which converges uniformly to ϕ locally uniformly in N.

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Remark 3.9. Sometimes, we call the image $\psi(\Delta)$ a Fatou disc too.

Further, here we shall introduce a concept which is a mixture of contracting set and Fatou disc.

Definition 3.10. (Contracting Fatou discs) Let $\psi : \Delta \to M$ be a Fatou disc. We say that ψ is *contracting* if at least one of the limit maps of $\{f^n \circ \psi\}_{n\geq 0}$ is a constant.

Definition 3.11. (Neutral Fatou discs) Let $\psi : \Delta \to M$ be a Fatou disc. We say that ψ is *neutral* if ψ is not contracting.

The theorem below is a direct application of Theorem 3.6.

THEOREM 3.12. *Suppose that f satisfies (C). Let Λ be a hyperbolic set for f such that the (complex) dimension of the stable direction at each point in Λ is at least 1. Then, $\Lambda \subset D_f$.*

Proof. Let $p \in \Lambda$ and the dimension of the stable direction at p be $m \geq 1$. Then, there is a holomorphic map $\psi : \Delta^m \to M$ such that $\psi(\Delta^m) = W^s_{\mathrm{loc}}(p)$, where $W^s_{\mathrm{loc}}(p)$ is the local stable manifold for p. It is obvious that ψ is a contracting Fatou map. So, from Theorem 3.6, the argument follows. □

PROPOSITION 3.13. *Suppose that f satisfies (C). Let E be a compact subset in M such that $f(E) \subset E$, $E \cap D_f = \emptyset$. Then, all Fatou discs which intersect E are neutral.*

Proof. Suppose that a Fatou disc $\psi(\Delta)$ which intersects E is contracting. There exist $p \in M$ and a subsequence $\{f^{n_j}\}_{j\geq 1}$ such that $f^{n_j} \circ \psi(z) \to p$ as $j \to \infty$ locally uniformly in Δ. By the forward invariance of E, the point p is located in E. From Theorem 3.6, it follows that $p \in D_f$. This contradicts that $E \cap D_f = \emptyset$. □

Remark 3.14. In the case when $M = \mathbb{P}^k$ and $\deg(f) \geq 2$, by using the fact in Remark 3.7 instead of Theorem 3.6, this proposition holds without the condition (C).

Suppose (C). Let E be a compact subset in M such that $f(E) \subset E$ and $E \cap D_f = \emptyset$. Let $q \in E$. Set $B_r := \{z \in \mathbb{C}^k \mid \|z\| < r\}$ for $0 < r \leq 1$, where $\|\cdot\|$ is the Euclidean norm in $\mathbb{C}^k$. Let U be a neighborhood of q such that $U \cap D_f = \emptyset$. Assume that U can be identified with B_1 in a local chart of M and q is the center O of B_1 under the identification. Set

$$S := \{h \mid h \text{ is an inverse branch of } f^n \text{ for some } n \geq 0 \text{ in } U(= B_1)\}.$$

By (C), it follows that S is a normal family.

LEMMA 3.15. *For $0 < r < 1$, set $S_r := \{h \in S \mid h(\overline{B_r}) \cap E \neq \emptyset\}$. Then, for some sufficiently small $0 < r_0 < 1$,*

$$\inf_{h\in S_{r_0}} d_M(h(\overline{B_{r_0}}), D_f) > 0.$$

Proof. Let r_0 be a constant such that $0 < r_0 < 1$. Then, $S_{r_0} \subset S$. Set $m := d_M(E, D_f) > 0$. S is equicontinuous at O. So, if r_0 is sufficiently small, then $d_M(h(\overline{B_{r_0}}), D_f) > m/2$ for all $h \in S_{r_0}$. □

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Set $\epsilon := \inf_{h \in S_{r_0}} d_M(h(\overline{B_{r_0}}), D_f) > 0$. Let us note that if $h \in S_{r_0}$, then $f \circ h \in S_{r_0}$ since $f(E) \subset E$.

THEOREM 3.16. *Let g be a holomorphic map from $B_1 (= U)$ to M. Suppose that there exist $g_j \in S_{r_0}, j = 1, 2, 3, \ldots$ such that $g_j \to g, j \to \infty$ locally uniformly in B_1. Then, g is a Fatou map in B_{r_0}.*

Proof. From (C), it is sufficient to show that $f^i \circ g(\overline{B_{r_0}}) \subset M \setminus D_f$ for any $i \geq 1$. Fix i. Since f^i is uniformly continuous in M, there is a constant $\delta > 0$ such that $d_M(f^i(s), f^i(t)) < \epsilon$ if $s, t \in M$ and $d_M(s, t) < \delta$. If j is sufficiently large, then $d_M(g_j(z), g(z)) < \delta$ for any $z \in \overline{B_{r_0}}$. So, it follows that $d_M(f^i \circ g_j(z), f^i \circ g(z)) < \epsilon$ in $\overline{B_{r_0}}$. Let j be sufficiently greater than i. Since $f^i \circ g_j$ is an element in S_{r_0}, it follows that $d_M(f^i \circ g_j(\overline{B_{r_0}}), D_f) \geq \epsilon$. Hence, it follows that $d_M(f^i \circ g(\overline{B_{r_0}}), D_f) > 0$. □

Remark 3.17. In the case when $M = \mathbb{P}^k$ and $\deg(f) \geq 2$, without the condition (C), Ueda showed that the map g is a Fatou map. He dealt with a more general situation. See **[U1]**.

Let us define repelling points and repellers.

Definition 3.18. (Repelling points) Let $p \in M$. Let $\mathbf{T}_p$ be the holomorphic tangent space at p. We say that p is *repelling* for f if $\min_{v \in \mathbf{T}_p, |v|=1} |D(f^j)(v)| \to +\infty$ as $j \to +\infty$.

Definition 3.19. (Repellers) Let E be a compact subset in M such that $f(E) = E$. Let $\mathbf{T}(E)$ be the holomorphic tangent bundle over E. We say that E is a *repeller* if there are a constant $\rho > 1$ and an integer $m \geq 1$ such that $|D(f^m)(v)| \geq \rho|v|$ for all $v \in \mathbf{T}(E)$.

Remark 3.20. Let us note that E is a repeller if and only if there are constants $c > 0, \lambda > 1$ such that $|D(f^n)(v)| \geq c\lambda^n|v|$ for all $v \in \mathbf{T}(E)$ and all $n \geq 0$.

Remark 3.21. Under the small perturbation of f in the family of holomorphic self-maps in M, the repeller E moves holomorphically remaining a repeller. See **[J]** for details.

The theorem below is regarded as a generalization of the fact that any fixed point p outside D_f is repelling or has a center manifold. (In the case of rational functions in the Riemann sphere, such a p is a repelling fixed point or the center of a Siegel disc.)

THEOREM 3.22. *Let M be a compact complex Hermitian manifold of dimension ≥ 1. Let $f : M \to M$ be a surjective holomorphic self-map satisfying (C). Let E be a compact subset in M such that $f(E) \subset E$ and $E \cap D_f = \emptyset$. Then, there are two subsets E^u, $E^c \subset E$ which have the following properties:*

(i) $E^u \cup E^c = E, E^u \cap E^c = \emptyset$;

(ii) $f(E^u) \subset E^u, f(E^c) \subset E^c$;

(iii) *each point in E^u is repelling;*

(iv) *for each $p \in E^c$, there is a neutral Fatou disc $\psi : \Delta \to M$ such that $\psi(0) = p$, $D\psi(0) \neq 0$.*

Moreover, if $f(E) = E$ and $E^c = \emptyset$, then E is a repeller with respect to the Hermitian metric in M.

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In the case when $M = \mathbb{P}^k$ and $\deg(f) \geq 2$, without assuming (C), the same assertion holds.

Proof. Set $E^u := \{p \in E \mid p \text{ is repelling}\}$ and $E^c := E \setminus E^u$. We will show that for any $p \in E^c$, there is a neutral Fatou disc $\psi : \Delta \to M$ such that $\psi(0) = p$, $D\psi(0) \neq 0$. We can take a sequence $\{f^{n_j}(p)\}_{j\geq 1}, n_j \in \mathbb{Z}_{\geq 0}, n_1 < n_2 < \cdots$ which have the following property $(*)$.

$(*)$ There is a constant $0 < C < +\infty$ and there is a sequence $\{v_j\}_{j\geq 1}, v_j \in \mathbf{T}_p, |v_j| = 1$ such that $w_j := D(f^{n_j})(v_j) \in \mathbf{T}_{f^{n_j}(p)}$ satisfy $|w_j| \leq C$ for $j = 1, 2, 3, \ldots$.

Set $p_j := f^{n_j}(p)$. By transferring to a subsequence, we can assume that $p_j \to q \in E$ as $j \to +\infty$. Let U be a neighborhood of q outside D_f. Suppose that U is identified with B_1 in a local chart of M and q is the center O of B_1 under the identification. Further, suppose that $p_j \in B_1$ for all $j \geq 1$. For each $j \geq 1$, let us take the inverse branch g_j in B_1 for f^j such that $g_j(p_j) = p$. Since $g_j(B_1) \subset M \setminus D_f$ for $j \geq 1$, it follows that $\{g_j\}_{j\geq 1}$ is a normal family in B_1. Let g be a limit map of $\{g_j\}_{j\geq 1}$. By transferring to a subsequence, we can assume that $g_j \to g$ as $j \to +\infty$ locally uniformly in B_1. It follows easily that $g(O) = p$ because $g_j(p_j) = p$ and $p_j \to O(= q)$ as $j \to +\infty$. By Theorem 3.16, g is a Fatou map in B_{r_0} for some $0 < r_0 < 1$. From $(*)$, it follows that $|D(g_j)(w_j/|w_j|)| \geq 1/C$ for all $j \geq 1$. This guarantees that g is not a constant map in B_{r_0}. In fact, there is a tangent vector $v \in \mathbf{T}_O$ such that $|v| = 1, |D(g)(v)| \geq 1/C$. We can take a holomorphic disc $\psi_1 : \Delta_t \to B_{r_0}, \psi_1(0) = O, D(\psi_1)((d/dt)_{t=0}) = av$ for some $a > 0$. Set $\psi := g \circ \psi_1$. Then, ψ is automatically a neutral Fatou disc by Proposition 3.13. Further, it follows that $\psi(0) = p, D\psi(0) \neq 0$.

(In the case of a holomorphic endomorphism f of degree ≥ 2 in $\mathbb{P}^k$, by a general result of Ueda in [**U2**], it follows that $\{g_j\}$ is normal, even if we do not assume the condition (C). By Remarks 3.14 and 3.17, it follows that g is a Fatou map and ψ is a neutral Fatou disc.)

Suppose that $f(E) = E$ and $E^c = \emptyset$. Let us show that E is a repeller. To do this, it is sufficient to show that

$$\min_{p\in E} \min_{v\in \mathbf{T}_p, |v|=1} |D(f^n)(v)| \to +\infty, n \to +\infty.$$

Assume that the left-hand side does not go to infinity. Then, there is a constant $0 < C < \infty$ and there are sequences $\{p_l\}_{l\geq 1}, p_l \in E$ and $\{n_l\}_{l\geq 1}, n_l \in \mathbb{Z}_{\geq 0}, n_1 < n_2 < \cdots$ and $\{v_l\}_{l\geq 1}, v_l \in \mathbf{T}_{p_l}, |v_l| = 1$ such that $w_l := D(f^{n_l})(v_l) \in \mathbf{T}_{f^{n_l}(p_l)}$ satisfy $|w_l| \leq C$ for $l = 1, 2, 3, \ldots$. By the compactness of E, without loss of generality, we can assume that there is a point $q \in E$ and $f^{n_l}(p_l) \to q$ as $l \to \infty$. Take a small neighborhood U of q. In U, for every $l \geq 1$, take an inverse branch g_l of f^{n_l} such that $g_l(f^{n_l}(p_l)) = p_l$. This sequence $\{g^l\}$ is a normal family. We obtain Fatou discs attached to E from limit maps of $\{g_l\}_{l\geq 1}$. This is a contradiction to the fact that $E^c = \emptyset$.

(In the case of a holomorphic endomorphism f of degree ≥ 2 in $\mathbb{P}^k$, without the condition (C), it follows that $\{g_l\}$ is normal.) □

Remark 3.23. If we remove the condition (C), then a counterexample is found. Let us consider the case in which f is not invertible. Let $M = \hat{\mathbb{C}} \times \hat{\mathbb{C}}$ and $f(x, y) = (g(x), h(y))$, where $g(x)$ is a hyperbolic rational function of degree ≥ 2 and $h(y)$ is a Möbius transformation with an attracting fixed point p and a repelling fixed point q.

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Let J_g denote the Julia set of g. Set $E := J_g \times \hat{\mathbb{C}}$. If a Fatou disc $\psi(\Delta)$ intersects E, then it is contained in $E \setminus (J_g \times \{q\})$ and is contracting. Hence, the theorem does not hold in this case.

From Theorem 3.22, the following question naturally arises.

Question. Let f be a holomorphic self-map of $\mathbb{P}^k$ of degree at least 2. Suppose that D_f is disjoint from the support $\mathrm{Supp}(\mu)$ of the unique maximal entropy measure μ for f. Is $\mathrm{Supp}(\mu)$ a repeller?

Let us consider E^u, E^c for $\mathrm{Supp}(\mu)$. Since the Lyapunov exponents are strictly positive, it follows that $E^u \neq \emptyset$ (see [**BD**]). So, we can show the following.

PROPOSITION 3.24. *Suppose that E^c is non-empty. Then, E^c is not closed. Moreover, E^u is not closed.*

Proof. Since $\mathrm{Supp}(\mu)$ is totally invariant, both of E^u, E^c are also totally invariant. Suppose that E^c is closed. Since $\mathrm{Supp}(\mu) \cap C_f = \emptyset$, by a result in [**BD**], for any point $p \in E^c$,

$$\mu_n := \frac{1}{d^{nk}} \sum_{f^n(y)=p} \delta_y$$

converges to μ in the weak sense, where δ_y is Dirac measure at y. This implies that E^c contains $\mathrm{Supp}(\mu)$. This is a contradiction to $E^u \neq \emptyset$.

In the same manner, if we suppose that E^u is closed, we have a contradiction to E^c being non-empty. □

In [**U1**], it is shown that the family of all Fatou maps from the unit disc Δ to $\mathbb{P}^k$ is a normal family and every limit map of the family is also a Fatou map, possibly a constant map.

Suppose that E^c is non-empty. By the proposition above, it follows that there exists a sequence $\{p_n\}_{n\geq 1}$ such that $p_n \in E^c$ for all $n \geq 1$ and

$$\lim_{n\to\infty} p_n = q \in E^u.$$

Let $h_n : \Delta \to \mathbb{P}^k$ be neutral Fatou discs such that $h_n(0) = p_n$. Since the sequence $\{h_n\}$ is normal, without loss of generality, we may assume that it converges locally uniformly in Δ. If we can control the derivatives, for instance as follows,

$$\left| D(h_n)\left(\left(\frac{d}{dt}\right)_{t=0}\right)\right| > C > 0,$$

where C is a constant which does not depend on n, then the limit map h of $\{h_n\}$ is non-constant, that is, h is a neutral Fatou disc which is through $q \in E^u$. Thus, we have a contradiction.

Can we control the neutral Fatou discs to converge to a neutral Fatou disc?

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