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RESTRICTED SIMULTANEOUS DIOPHANTINE APPROXIMATION

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Abstract. We study the problem of Diophantine approximation on lines in $\mathbb{R}^d$ under certain primality restrictions.

§1. *Introduction.* The subject of *metric Diophantine approximation of manifolds* studies Diophantine approximation of typical points on submanifolds in $\mathbb{R}^d$ by rational points in $\mathbb{R}^d$. This subject has received considerable attention in the last two decades, leading to dramatic progress using methods arising from the ergodic theory of flows on homogeneous spaces as well as analytic methods. If one puts further restrictions on the approximating rationals, then the situation is much less well understood. A very natural class of problems arises by imposing primality restrictions on the approximating rationals. In this paper, we study the problem of Diophantine approximation for vectors on lines in $\mathbb{R}^d$ with additional primality restrictions. Thus we combine the themes of simultaneous metric Diophantine approximation *on affine subspaces* and Diophantine approximation with restrictions. Both problems have their own substantial complications and have separately received considerable attention (cf. [**1**, **5**, **6**, **15**] for Diophantine approximation on affine subspaces and [**9**–**11**, **13**, **14**, **16**–**18**] for Diophantine approximation with primality constraints). The only previous works on the combined theme that we are aware of are the work of Harman and Jones [**12**], regarding Diophantine approximation on curves in $\mathbb{R}^2$ with constraints, and our previous work [**2**], where we addressed the problem of Diophantine approximation with constraints on lines in $\mathbb{R}^2$. The main result of the present paper generalises [**2**] to arbitrary dimensions and is the first such result in this generality. While the broad strategy in the present paper is similar to that of [**2**], the greater generality makes the problem significantly more complicated. In studying Diophantine approximation on lines and, more generally, affine subspaces, it is natural and indeed imperative to impose some Diophantine condition on the line or subspace itself. In [**2**], we had assumed that the slope of the line in $\mathbb{R}^2$ is irrational. In higher dimensions, the lack of a suitable continued fraction algorithm makes it necessary to replace irrationality with a suitable Diophantine condition, which we now introduce. Let $\| \ \|$ denote

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the distance to the nearest integer of a real number, let $\| \ \|_\infty$ denote the supremum norm of a vector and, for vectors $\mathbf{v}, \mathbf{c} \in \mathbb{R}^d$, denote by $\mathbf{v}\cdot\mathbf{c} = v_1c_1 + \cdots + v_dc_d$ the inner product of $\mathbf{v}$ and $\mathbf{c}$. Recall that $\mathbf{c} \in \mathbb{R}^d$ is called k-Diophantine ($\mathbf{c} \in D_k(\mathbb{R}^d)$) if there exists a constant $C > 0$ such that

$$\|\mathbf{v} \cdot \mathbf{c}\| > \frac{C}{\|\mathbf{v}\|_\infty^k} \quad \text{for every } \mathbf{v} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}. \tag{1}$$

Our main result is the following theorem.

THEOREM 1.1. *Let d be a positive integer and let $k \geqslant d$ be a positive real number. Define*

$$\gamma_{d,k} := \frac{1}{d(3k+2)} \tag{2}$$

and suppose that $0 < \varepsilon < \gamma_{d,k}$. Let $c_1, \ldots, c_d$ be positive irrational numbers such that the vector $\mathbf{c} = (c_1, \ldots, c_d)$ is k-Diophantine. Then, for almost all positive real α, with respect to the Lebesgue measure, there are an infinite number of $(d+2)$-tuples $(p, q_1, \ldots, q_d, r)$ with p and r prime and $q_1, \ldots, q_d$ positive integers such that, simultaneously,

$$\begin{aligned} &0 < p\alpha - r < p^{-\gamma_{d,k}+\varepsilon}, \\ &0 < pc_i\alpha - q_i < p^{-\gamma_{d,k}+\varepsilon} \quad \textit{for all } i \in \{1, \ldots, d\}. \end{aligned} \tag{3}$$

Remarks. (1) It is well known that $D_d(\mathbb{R}^d)$ is a non-empty set of zero Lebesgue measure and full Hausdorff dimension. These comprise the set of *badly approximable* vectors. Moreover, $D_k(\mathbb{R}^d)$ has full measure whenever $k > d$ (see [4], for example).

(2) In [2], we proved the analogue of Theorem 1.1 for lines in $\mathbb{R}^2$ under the assumption that the slope c of the line is irrational. In fact, what was used was the following Diophantine property of irrational numbers: there exists an infinite set $\mathcal{S}$ of integers and a positive constant $D > 0$ such that

$$\min_{\substack{\mathbf{v}\in\mathbb{Z}^d\setminus\{\mathbf{0}\} \\ \|\mathbf{v}\|_\infty \leqslant N}} \|\mathbf{v} \cdot \mathbf{c}\| \geqslant N^{-D} \quad \text{if } N \in \mathcal{S}. \tag{4}$$

For $d = 1$, this holds for $D = 1$ and for $\mathcal{S}$ being the set of numbers $[N/2]$, where the N are the denominators in the continued fraction approximants of $c = c_1$. This can be seen by approximating c by its continued fraction approximant a/N and using the fact that $|c - a/N| \leqslant 1/N^2$. In the case $d = 1$, a sequence $\mathcal{S}$ with the above property can also be constructed using the Dirichlet approximation theorem and the condition that c is irrational. There is a d-dimensional version of Dirichlet's approximation theorem, but for $d \geqslant 2$, a statement such as (4) does not follow from it. It is likely that the conclusion of Theorem 1.1 would also hold in arbitrary dimension by assuming (4) rather than the Diophantine condition that we have assumed, with perhaps a different exponent. We note, however, that, for $d = 1$, by choosing $k = 1$ we recover the exponent in [2]. The condition (4) is

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also an interesting Diophantine property, and it can be shown to include the class of *non-singular* vectors.

(3) It is an interesting problem to consider analogues of Theorem 1.1 for affine subspaces of lower codimension and, indeed, for manifolds not contained in affine subspaces, that is, the *non-degenerate* manifolds. We will consider these in a forthcoming work.

§2. *A metrical approach.* Our method is based on the following lemma in [12].

LEMMA 2.1 [12, Lemma 1]. *Let A and B be reals with $B > A > 0$. Let $F_N(\alpha)$ be a non-negative valued function of N (an integer) and α (a real variable) and let G_N, V_N be functions of N such that:*

(i) $G_N \to \infty$ *as* $N \to \infty$;

(ii) $V_N = o(G_N)$ *as* $N \to \infty$;

(iii) *for all a, b with $A \leqslant a < b \leqslant B$,*

$$\limsup_{N\to\infty} \int_a^b \frac{F_N(\alpha)}{G_N}\, d\alpha \geqslant b - a;$$

(iv) *there is a positive constant K such that, for any measurable set $\mathcal{C} \subseteq [A, B]$,*

$$\int_{\mathcal{C}} F_N(\alpha)\, d\alpha \leqslant K G_N \lambda(\mathcal{C}) + V_N.$$

Then, for almost all $\alpha \in [A, B]$,

$$\limsup_{N\to\infty} \frac{F_N(\alpha)}{G_N} \geqslant 1. \tag{5}$$

In our application, $F_N(\alpha)$ will be the number of solutions to (3) with $p < N$. Further, for given $0 < A < B$ we set

$$G_N = G_N(A, B) = \frac{A^2}{B^2} \cdot \frac{\min(c_1, \dots, c_d, d)^{d-1}}{2^{d+1}} N^{1-(d+1)(\gamma_{d,k}-\varepsilon)} (\log N)^{-2}.$$

We will prove the following theorem.

THEOREM 2.2. *The following holds for every natural number N.*

(i) *Let $0 < A < B$. Then, for all a, b with $A \leqslant a < b \leqslant B$,*

$$\int_a^b F_N(\alpha)\, d\alpha \geqslant (b - a) G_N(A, B)(1 + o(1)) \tag{6}$$

if $N \in \mathcal{S}$ and $N \to \infty$.

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(ii) *Let $0 < A < B$ and $\varepsilon > 0$. Then there exists a constant $K = K(A, B, \varepsilon)$ such that, for $\alpha \in [A, B]$,*

$$F_N(\alpha) \leqslant K G_N(A, B) + J_N(\alpha)$$

with

$$\int_A^B |J_N(\alpha)|\, d\alpha = o(G_N(A, B)) \quad \text{as } N \to \infty$$

if $N \in \mathcal{S}$ and $N \to \infty$.

Theorem 2.2(i) corresponds to Lemma 2 and Theorem 2.2(ii) corresponds to Lemma 3 in [**12**]. Now it follows that conditions (i) to (iv) in Lemma 2.1 are satisfied for

$$V_N = \int_A^B |J_N(\alpha)|\, d\alpha.$$

The claim in Theorem 1.1 follows from (5).

§3. *Proof of Theorem 2.2(i).*

3.1. *Reduction to a counting problem.* To prove Theorem 2.2(i), we broadly follow the approach in [**12**, §3]. However, we use exponential sum estimates instead of zero density estimates for the Riemann zeta function since they turn out to be more suitable for our purposes. This is the content of the next subsection.

Throughout the following, we denote the set of primes by $\mathbb{P}$. Let

$$\mathcal{B}_p = \bigcup_{\substack{r \in \mathbb{P} \\ q_1, \dots, q_d \in \mathbb{N}}} \left[\frac{r}{p}, \frac{r+\tilde{\eta}}{p}\right) \cap \left[\frac{1}{c_1} \cdot \frac{q_1}{p}, \frac{1}{c_1} \cdot \frac{q_1+\tilde{\eta}}{p}\right) \cap \cdots \cap \left[\frac{1}{c_d} \cdot \frac{q_d}{p}, \frac{1}{c_d} \cdot \frac{q_d+\tilde{\eta}}{p}\right) \cap [a, b],$$

where $\tilde{\eta} = p^{\varepsilon - \gamma_{d,k}}$. Then

$$\int_a^b F_N(\alpha)\, d\alpha = \sum_{\substack{p \in \mathbb{P} \\ p \leqslant N}} \lambda(\mathcal{B}_p), \tag{7}$$

where λ is Lebesgue measure. Set

$$\mu := (a + b)/(2a). \tag{8}$$

Our strategy is to split the interval $[1, N]$ into subintervals $[P, P\mu]$ and sum up over the P. Accordingly, we restrict p to the interval $P \leqslant p < P\mu$ with $P\mu \leqslant N$. We then obtain a lower bound for (7) by replacing $\tilde{\eta}$ with

$$\eta = (\mu P)^{\varepsilon - \gamma_{d,k}}. \tag{9}$$

Clearly, $\tilde{\eta} \geqslant \eta$ if $P \leqslant p < P\mu$.

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We note that if

$$\frac{r}{p} \leqslant \frac{1}{c_i} \cdot \frac{q_i}{p} \leqslant \frac{r+\eta/2}{p} \quad \text{for } i = 1, \dots, d,$$

then

$$\lambda\left(\left[\frac{r}{p}, \frac{r+\eta}{p}\right) \cap \left[\frac{1}{c_1} \cdot \frac{q_1}{p}, \frac{1}{c_1} \cdot \frac{q_1+\eta}{p}\right) \cap \dots \cap \left[\frac{1}{c_d} \cdot \frac{q_d}{p}, \frac{1}{c_d} \cdot \frac{q_d+\eta}{p}\right)\right) \geqslant \nu,$$

where

$$\nu := \frac{\eta}{\mu P} \min\left(\frac{1}{2}, \frac{1}{c_1}, \dots, \frac{1}{c_d}\right) = (\mu P)^{-1-\gamma_{d,k}+\varepsilon} \min\left(\frac{1}{2}, \frac{1}{c_1}, \dots, \frac{1}{c_d}\right). \quad (10)$$

Also, for all $p \in [P, P\mu)$,

$$Pa\mu \leqslant r \leqslant bP \Longrightarrow a \leqslant \frac{r}{p} \leqslant b,$$

where r runs over the primes in an interval of length $(b-a)/2P$. Thus

$$\sum_{P \leqslant p < \mu P} \lambda(\mathcal{B}_p) \geqslant \nu N(P), \quad (11)$$

where $N(P)$ counts the number of solutions $(p, q_1, \dots, q_d, r) \in \mathbb{P} \times \mathbb{Z}^d \times \mathbb{P}$ to

$$q_i \in [c_i r, c_i r + \delta) \quad \text{for } i = 1, \dots, d, \qquad P \leqslant p < P\mu, \qquad Pa\mu \leqslant r \leqslant bP,$$

where

$$\delta := \frac{\min\{c_1, \dots, c_d\}\eta}{2} = \frac{\min\{c_1, \dots, c_d\}}{2(\mu P)^{\gamma_{d,k}-\varepsilon}}. \quad (12)$$

Note that, in contrast to the problem considered by Harman and Jones, the conditions on p and q are here independent, which simplifies matters to some extent. By the prime number theorem, the number $R(P)$ of prime solutions to

$$P \leqslant p < P\mu$$

satisfies

$$R(P) \sim (\mu - 1)P(\log 2P)^{-1} \quad \text{as } P \to \infty. \quad (13)$$

It remains to count the number of solutions $(q_1, \dots, q_d, r) \in \mathbb{N} \times \mathbb{P}$ to

$$q_i \in [c_i r, c_i r + \delta) \quad \text{for } i = 1, \dots, d, \qquad Pa\mu \leqslant r \leqslant bP,$$

which equals

$$S(P) := \sum_{\substack{Pa\mu \leqslant r \leqslant bP \\ r \text{ prime}}} \prod_{i=1}^{d} ([-c_i r] - [-(c_i r + \delta)]).$$

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Let

$$T(P) := \sum_{Pa\mu \leqslant n \leqslant bP} \prod_{i=1}^{d} ([-c_i n] - [-(c_i n + \delta)])\Lambda(n). \tag{14}$$

We aim to show that

$$T(P) = \delta^d (b - a\mu) P (1 + o(1)) + O(N^{1 - d\gamma_{d,k} + \varepsilon/2}) \quad \text{if } P\mu \leqslant N. \tag{15}$$

As usual, from (15), it follows that

$$S(P) = \delta^d (b - a\mu) P (\log 2P)^{-1} (1 + o(1)) + O(N^{1 - d\gamma_{d,k} + \varepsilon/2}) \quad \text{if } P\mu \leqslant N,$$

which, together with (13), gives

$$\begin{aligned} N(P) = R(P)S(P) &= \delta^d (b - a\mu)(\mu - 1) P^2 (\log 2P)^{-2} (1 + o(1)) \\ &\quad + O(PN^{1 - d\gamma_{d,k} + \varepsilon/2}) \quad \text{if } P\mu \leqslant N. \end{aligned}$$

Combining this with (8)–(12), we obtain

$$\begin{aligned} \sum_{P \leqslant p < \mu P} \lambda(\mathcal{B}_p) \geqslant \frac{(b-a)^2}{4a} \cdot \frac{\min(2, c_1, \ldots, c_d)^{d-1}}{2^d} \\ \cdot (\mu P)^{-1-(d+1)(\gamma_{d,k} - \varepsilon)} P^2 (\log 2P)^{-2} (1 + o(1)) \\ + O(P^{-\gamma_{d,k}} N^{1 - d\gamma_{d,k} + 3\varepsilon/2}) \quad \text{if } P\mu \leqslant N. \end{aligned} \tag{16}$$

By splitting the interval $[1, N)$ into intervals of the form $[P, \mu P)$ and summing up, it now follows, from (7) and (16), that

$$\begin{aligned} \int_a^b F_N(\alpha)\, d\alpha &\geqslant \frac{(b-a)^2}{4a} \cdot \frac{\min(2, c_1, \ldots, c_d)^{d-1}}{2^d} \\ &\quad \cdot \left(\sum_{k=0}^{\infty} \left(\frac{N}{\mu^k} \right)^{-1-(d+1)(\gamma_{d,k} - \varepsilon)} \left(\frac{N}{\mu^{k+1}} \right)^2 (\log N)^{-2} \right) (1 + o(1)) \\ &= \frac{(b-a)^2}{4a} \cdot \frac{\min(2, c_1, \ldots, c_d)^{d-1}}{2^d} \\ &\quad \cdot \mu^{-2} \cdot \frac{1}{1 - \mu^{-(1-(d+1)(\gamma_{d,k} - \varepsilon))}} \\ &\quad \cdot N^{1-(d+1)(\gamma_{d,k} - \varepsilon)} (\log N)^{-2} (1 + o(1)). \end{aligned}$$

Further, since $\mu > 1$,

$$\begin{aligned} 1 - \mu^{-(1-(d+1)(\gamma_{d,k} - \varepsilon))} &\leqslant (1 - (d+1)(\gamma_{d,k} - \varepsilon))(\mu - 1) \\ &= (1 - (d+1)(\gamma_{d,k} - \varepsilon)) \cdot \frac{b-a}{2a}. \end{aligned}$$

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Hence, we deduce that

$$\begin{aligned}\int_a^b F_N(\alpha)\,d\alpha &\geqslant (b-a)\cdot\frac{2a^2}{(a+b)^2}\cdot\frac{1}{1-(d+1)(\gamma_{d,k}-\varepsilon)}\\ &\quad\cdot\frac{\min(c_1,\dots,c_d,2)^{d-1}}{2^d}\\ &\quad\times N^{1-(d+1)(\gamma_{d,k}-\varepsilon)}(\log N)^{-2}(1+o(1))\\ &\geqslant (b-a)\cdot\frac{A^2}{B^2}\cdot\frac{\min(c_1,\dots,c_d,2)^{d-1}}{2^{d+1}}\\ &\quad\cdot N^{1-(d+1)(\gamma_{d,k}-\varepsilon)}(\log N)^{-2}(1+o(1)),\end{aligned}\tag{17}$$

which establishes the claim of Theorem 2.2(i). It remains to prove (15).

3.2. *Reduction to exponential sums.* For $x\in\mathbb{R}$, let

$$\psi(x):=x-[x]-\tfrac{1}{2}.$$

Then we may write $T(P)$ in the form

$$\begin{aligned}T(P)&=\sum_{Pa\mu\leqslant n\leqslant bP}\Lambda(n)\prod_{i=1}^{d}(\delta-(\psi(-c_in)-\psi(-(c_in+\delta))))\\ &=\sum_{\mathcal{A}\subseteq\{1,\dots,d\}}\delta^{d-|\mathcal{A}|}T_{\mathcal{A}}(P),\end{aligned}\tag{18}$$

where

$$T_{\mathcal{A}}(P):=\sum_{Pa\mu\leqslant n\leqslant bP}\Lambda(n)\prod_{i\in\mathcal{A}}(\psi(-(c_in+\delta))-\psi(-c_in)).$$

By the prime number theorem,

$$T_{\emptyset}(P)=\delta^d\sum_{Pa\mu\leqslant n\leqslant bP}\Lambda(n)\sim\delta^d(b-a\mu)P\quad\text{as } P\to\infty.$$

Hence, to establish (15), it suffices to prove that, for any fixed $\varepsilon>0$, a bound of the form

$$T_{\mathcal{A}}(P)=O(N^{1-d\gamma_{d,k}+\varepsilon/2})\quad\text{if } P\mu\leqslant N\tag{19}$$

holds for all non-empty subsets $\mathcal{A}$ of $\{1,\dots,d\}$. We reduce the left-hand side to exponential sums, using the following Fourier analytic tool developed by Vaaler [**19**].

LEMMA 3.1 (Vaaler). *For* $0<|t|<1$, *let*

$$W(t)=\pi t(1-|t|)\cot\pi t+|t|.$$

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Fix a positive integer J. For $x \in \mathbb{R}$, define

$$\psi^*(x) := -\sum_{1 \leqslant |j| \leqslant J} (2\pi \mathrm{i} j)^{-1} W\left(\frac{j}{J+1}\right) \mathrm{e}(jx)$$

and

$$\tau(x) := \frac{1}{2J+2} \sum_{|j| \leqslant J} \left(1 - \frac{|j|}{J+1}\right) \mathrm{e}(jx).$$

Then $\tau(x)$ is non-negative, and

$$|\psi^*(x) - \psi(x)| \leqslant \tau(x)$$

for all real numbers x.

Proof. This is [7, Theorem A6] and it has its origin in [19]. □

We set

$$\tau^*(x) := \psi(x) - \psi^*(x).$$

Then

$$\begin{aligned} T_{\mathcal{A}}(P) &= \sum_{\mathcal{B} \subseteq \mathcal{A}} \sum_{Pa\mu \leqslant n \leqslant bP} \Lambda(n) \left(\prod_{i \in \mathcal{B}} (\psi^*(-(c_i n + \delta)) - \psi^*(-c_i n)) \right) \\ &\quad \times \left(\prod_{j \in \mathcal{A} \setminus \mathcal{B}} (\tau^*(-(c_j n + \delta)) - \tau^*(-c_j n)) \right) \\ &= U_{\mathcal{A}}(P) + O(V_{\mathcal{A}}(P)), \end{aligned} \tag{20}$$

where

$$U_{\mathcal{A}}(P) := \sum_{Pa\mu \leqslant n \leqslant bP} \Lambda(n) \left(\prod_{i \in \mathcal{A}} (\psi^*(-(c_i n + \delta)) - \psi^*(-c_i n)) \right)$$

and

$$V_{\mathcal{A}}(P) := (\log 2P) \sum_{i \in \mathcal{A}} \sum_{Pa\mu \leqslant n \leqslant bP} (\tau(-(c_i n + \delta)) + \tau(-c_i n)),$$

where the O-term $V_{\mathcal{A}}(P)$ arrives by using $|\psi^*(x)| \leqslant 2$, $|\tau^*(x)| \leqslant \tau(x) \leqslant 1$, $\Lambda(n) \leqslant \log n$ and the triangle inequality.

The definition of the function $\tau(x)$ gives

$$V_{\mathcal{A}}(P) = \frac{\log 2P}{2J+2} \sum_{i \in \mathcal{A}} \sum_{|j| \leqslant J} \left(1 - \frac{|j|}{J+1}\right) (1 + \mathrm{e}(j\delta)) \sum_{Pa\mu \leqslant n \leqslant bP} \mathrm{e}(jc_i n),$$

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and the definition of $\psi^*(x)$ gives, after multiplying out and rearranging summations,

$$U_{\mathcal{A}}(P) = -\sum_{1\leqslant|j_1|\leqslant J}\cdots\sum_{1\leqslant|j_h|\leqslant J}\left(\prod_{k=1}^{h}\left((2\pi\mathrm{i}j_k)^{-1}W\left(\frac{j_k}{J+1}\right)(1+\mathrm{e}(j_k\delta))\right)\right)$$
$$\times\sum_{Pa\mu\leqslant n\leqslant bP}\Lambda(n)\mathrm{e}\left(n\sum_{k\in\mathcal{A}}j_kc_{l_k}\right),$$

where we suppose that J is a positive integer satisfying $J\leqslant P$ and

$$h := |\mathcal{A}| \quad\text{and}\quad \mathcal{A}=\{l_1,\dots,l_h\}.$$

We further estimate $V_{\mathcal{A}}(P)$ by

$$\begin{aligned}V_{\mathcal{A}}(P) &\ll \frac{P\log 2P}{J}+\frac{\log 2P}{J}\cdot\sum_{i=1}^{d}\sum_{1\leqslant j\leqslant J}\left|\sum_{Pa\mu\leqslant n\leqslant bP}\mathrm{e}(jc_in)\right|\\ &\ll \frac{P\log 2P}{J}+\frac{\log 2P}{J}\cdot\sum_{i=1}^{d}\sum_{1\leqslant j\leqslant J}\min(P,\|jc_i\|^{-1})\\ &\ll \frac{P\log 2P}{J}+(\log 2P)\sum_{i=1}^{d}\sum_{1\leqslant j\leqslant J}\min\left(\frac{P}{j},\|jc_i\|^{-1}\right)\\ &=: \frac{P\log 2P}{J}+(\log 2P)\tilde{V}_d(P),\end{aligned}\tag{21}$$

where the term $P(\log 2P)/J$ bounds the contribution of $j=0$, and we estimate $U_{\mathcal{A}}(P)$ by

$$\begin{aligned}U_{\mathcal{A}}(P) &\ll \tilde{U}_{\mathcal{A}}(P)\\ &:= \sum_{1\leqslant|j_1|\leqslant J}\cdots\sum_{1\leqslant|j_h|\leqslant J}\frac{1}{|j_1\cdots j_h|}\left|\sum_{Pa\mu\leqslant n\leqslant bP}\Lambda(n)\mathrm{e}\left(n\sum_{k\in\mathcal{A}}j_kc_{l_k}\right)\right|.\end{aligned}\tag{22}$$

It remains to estimate $\tilde{U}_{\mathcal{A}}(P)$ and $\tilde{V}_d(P)$. The estimation of $\tilde{V}_d(P)$ is clearly easier than that of $\tilde{U}_{\mathcal{A}}(P)$. We first deal with the term $\tilde{U}_{\mathcal{A}}(P)$.

3.3. *Application of Vaughan's identity.* We convert the inner sum involving the von Mangoldt function on the right-hand side of (22) into bilinear sums using Vaughan's identity.

LEMMA 3.2 (Vaughan). *Let $u\geqslant 1$, $v\geqslant 1$, $uv\leqslant x$. Then, for every arithmetic function $f:\mathbb{N}\to\mathbb{C}$, we have the estimate*

$$\sum_{u<n\leqslant x}f(n)\Lambda(n)\ll(\log 2x)T_1+T_2$$

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with

$$T_1 := \sum_{l \leqslant uv} \max_{w} \left| \sum_{w \leqslant m \leqslant x/l} f(ml) \right|$$

and

$$T_2 := \left| \sum_{u < m \leqslant x/v} \sum_{v < l \leqslant x/m} \Lambda(m) b(l) f(ml) \right|,$$

where $b(l)$ is an arithmetic function that only depends on v and which satisfies the inequality $b(l) \leqslant \tau(l)$, with $\tau(l)$ being the number of divisors of l.

Proof. This is [3, Satz 6.1.2] and it has its origin in [20]. □

We use Lemma 3.2 with parameters u and v satisfying $1 \leqslant u = v \leqslant (Pa\mu)^{1/2}$, to be fixed later, $x := bP$ and

$$f(n) := \begin{cases} \mathrm{e}(nc) & \text{if } Pa\mu \leqslant n \leqslant bP, \\ 0 & \text{if } n < Pa\mu, \end{cases}$$

with

$$c := \sum_{k \in \mathcal{A}} j_k c_{l_k},$$

to deduce that

$$\tilde{U}_{\mathcal{A}}(P) \ll (\log 2P) Z_1 + Z_2, \tag{23}$$

where

$$Z_1 := \sum_{1 \leqslant |j_1| \leqslant J} \cdots \sum_{1 \leqslant |j_h| \leqslant J} \frac{1}{|j_1 \cdots j_h|} \sum_{l \leqslant u^2} \max_{Pa\mu/l \leqslant w \leqslant bP/l} \times \left| \sum_{w \leqslant m \leqslant bP/l} \mathrm{e}\left(ml \sum_{k \in \mathcal{A}} j_k c_{l_k} \right) \right|$$

and

$$Z_2 := \sum_{1 \leqslant |j_1| \leqslant J} \cdots \sum_{1 \leqslant |j_h| \leqslant J} \frac{1}{|j_1 \cdots j_h|} \times \left| \sum_{u < m \leqslant bP/u} \sum_{\max(u, Pa\mu/m) \leqslant l \leqslant bP/m} \Lambda(m) b(l) \mathrm{e}\left(ml \sum_{k \in \mathcal{A}} j_k c_{l_k} \right) \right|.$$

Obviously,

$$Z_1 \ll \sum_{1 \leqslant |j_1| \leqslant J} \cdots \sum_{1 \leqslant |j_h| \leqslant J} \frac{1}{|j_1 \cdots j_h|} \sum_{l \leqslant u^2} \min\left(\frac{P}{l}, \left\| l \sum_{k \in \mathcal{A}} j_k c_{l_k} \right\|^{-1} \right). \tag{24}$$

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We shall boil down Z_2 to similar terms. Rearranging the summation gives

$$Z_2 = \sum_{1\leqslant|j_1|\leqslant J}\cdots\sum_{1\leqslant|j_h|\leqslant J}\frac{1}{|j_1\cdots j_h|} \times\Bigg|\sum_{a\mu u/b\leqslant l\leqslant bP/u} b(l)\sum_{M_1(l)\leqslant m\leqslant M_2(l)}\Lambda(m)\mathrm{e}\Bigg(ml\sum_{k\in\mathcal{A}}j_kc_{l_k}\Bigg)\Bigg|,$$

with

$$M_1(l) := \max([u]+1,\, Pa\mu/l) \quad\text{and}\quad M_2(l) := bP/l.$$

We now observe that

$$Z_2 \ll (\log 2P)\max_{u\leqslant L\leqslant bP/u} Z_2(L), \tag{25}$$

where

$$Z_2(L) := \sum_{1\leqslant|j_1|\leqslant J}\cdots\sum_{1\leqslant|j_h|\leqslant J}\frac{1}{|j_1\cdots j_h|}\sum_{L\leqslant l\leqslant 2L} b(l) \times\Bigg|\sum_{M_1(l)\leqslant m\leqslant M_2(l)}\Lambda(m)\mathrm{e}\Bigg(ml\sum_{k\in\mathcal{A}}j_kc_{l_k}\Bigg)\Bigg|.$$

Recall that $1\leqslant J\leqslant P$. Using the Cauchy–Schwarz inequality and the bound

$$\sum_{L\leqslant l\leqslant 2L}|b(l)|^2 \leqslant \sum_{L\leqslant l\leqslant 2L} d(l)^2 \ll L(\log 2L)^3,$$

with $d(l)$ being the divisor function, and by expanding the square, we obtain

$$\begin{aligned} Z_2(L)^2 &\ll \Bigg(\sum_{L\leqslant l\leqslant 2L}|b(l)|^2\Bigg)\Bigg(\sum_{1\leqslant|j_1|\leqslant J}\cdots\sum_{1\leqslant|j_h|\leqslant J}\frac{1}{|j_1\cdots j_h|}\Bigg) \\ &\quad\times\sum_{1\leqslant|j_1|\leqslant J}\cdots\sum_{1\leqslant|j_h|\leqslant J}\frac{1}{|j_1\cdots j_h|} \\ &\quad\times\sum_{L\leqslant l\leqslant 2L}\Bigg|\sum_{M_1(l)\leqslant m\leqslant M_2(l)}\Lambda(m)\mathrm{e}\Bigg(ml\sum_{k\in\mathcal{A}}j_kc_{l_k}\Bigg)\Bigg|^2 \\ &\ll \Bigg(\sum_{L\leqslant l\leqslant 2L}|b(l)|^2\Bigg)(\log 2P)^h\sum_{1\leqslant|j_1|\leqslant J}\cdots\sum_{1\leqslant|j_h|\leqslant J}\frac{1}{|j_1\cdots j_h|} \\ &\quad\times\sum_{L\leqslant l\leqslant 2L}\Bigg|\sum_{M_1(l)\leqslant m\leqslant M_2(l)}\Lambda(m)\mathrm{e}\Bigg(ml\sum_{k\in\mathcal{A}}j_kc_{l_k}\Bigg)\Bigg|^2. \end{aligned}$$

Expanding the square, and splitting the resulting expression into a diagonal and non-diagonal term, we estimate the above further by

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$$Z_2(L)^2 \ll L(\log 2P)^{h+3} \sum_{1\leqslant|j_1|\leqslant J} \cdots \sum_{1\leqslant|j_h|\leqslant J} \frac{1}{|j_1\cdots j_h|} \times \sum_{L\leqslant l\leqslant 2L} \sum_{M_1(l)\leqslant m\leqslant M_2(l)} \Lambda(m)^2 + L(\log 2P)^{h+3} \sum_{1\leqslant|j_1|\leqslant J} \cdots \sum_{1\leqslant|j_h|\leqslant J} \frac{1}{|j_1\cdots j_h|} \times \Bigg| \sum_{L\leqslant l\leqslant 2L} \sum_{M_1(l)\leqslant m_1<m_2\leqslant M_2(l)} \Lambda(m_1)\Lambda(m_2)\mathrm{e}\Bigg((m_2-m_1)l\sum_{k\in\mathcal{A}} j_k c_{l_k}\Bigg)\Bigg|.$$

Exchanging summation and estimating geometric sums, we deduce that

$$\begin{aligned} Z_2(L)^2 &\ll LP(\log 2P)^{2h+5} + L(\log 2P)^{h+3} \sum_{1\leqslant|j_1|\leqslant J} \cdots \sum_{1\leqslant|j_h|\leqslant J} \frac{1}{|j_1\cdots j_h|} \\ &\quad \times \sum_{u\leqslant m_1<m_2\leqslant bP/L} \Bigg| \sum_{\max(L,Pa\mu/m_1)\leqslant l\leqslant \min(2L,bP/m_2)} \\ &\qquad\qquad \times \mathrm{e}\Bigg((m_2-m_1)l\sum_{k\in\mathcal{A}} j_k c_{l_k}\Bigg)\Bigg| \\ &\ll LP(\log 2P)^{2h+5} + L(\log 2P)^{h+3} \sum_{1\leqslant|j_1|\leqslant J} \cdots \sum_{1\leqslant|j_h|\leqslant J} \frac{1}{|j_1\cdots j_h|} \\ &\quad \times \sum_{u\leqslant m_1<m_2\leqslant bP/L} \min\Bigg(L, \Bigg\|(m_2-m_1)\sum_{k\in\mathcal{A}} j_k c_{l_k}\Bigg\|^{-1}\Bigg) \\ &\ll LP(\log 2P)^{2h+5} + P(\log 2P)^{h+3} \sum_{1\leqslant|j_1|\leqslant J} \cdots \sum_{1\leqslant|j_h|\leqslant J} \frac{1}{|j_1\cdots j_h|} \\ &\quad \times \sum_{1\leqslant m\leqslant bP/L} \min\Bigg(\frac{P}{m}, \Bigg\|m\sum_{k\in\mathcal{A}} j_k c_{l_k}\Bigg\|^{-1}\Bigg). \end{aligned} \tag{26}$$

3.4. *Construction of a Dirichlet approximation.* The treatments of Z_1 and Z_2 lead to sums of the form

$$R_{\mathcal{A}}(M,x) := \sum_{1\leqslant|j_1|\leqslant J} \cdots \sum_{1\leqslant|j_h|\leqslant J} \frac{1}{|j_1\cdots j_h|} \sum_{1\leqslant m\leqslant M} \min\Bigg(\frac{x}{m}, \Bigg\|m\sum_{k\in\mathcal{A}} j_k c_{l_k}\Bigg\|^{-1}\Bigg), \tag{27}$$

which we estimate in the following. Here the Diophantine properties of the vector $(c_1,\dots,c_d)$ come into play.

Splitting each of the j_k-intervals into $O(\log P)$ dyadic intervals and summing up their contributions, we obtain

$$R_{\mathcal{A}}(M,x) \ll (\log 2P)^d \sup_{1\leqslant H_1,\dots,H_d\leqslant J} \frac{R_d(H_1,\dots,H_d,M,x)}{H_1\cdots H_d}, \tag{28}$$

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where

$$R_d(H_1,\dots,H_d,M,x) := \sum_{\substack{|j_1|\leqslant H_1,\dots,|j_d|\leqslant H_d\\ \mathbf{j}\neq\mathbf{0}}} \sum_{1\leqslant m\leqslant M} \min\left(\frac{x}{m}, \|m\mathbf{j}\cdot\mathbf{c}\|^{-1}\right), \quad (29)$$

with

$$\mathbf{j} := (j_1,\dots,j_d) \quad \text{and} \quad \mathbf{c} := (c_1,\dots,c_d).$$

The key point is now to approximate $\mathbf{j}\cdot\mathbf{c}$ by rational numbers, using Dirichlet's approximation theorem, where the denominators are uniformly bounded from below. Let $\mathbf{j}\in\mathbb{Z}^d\setminus\mathbf{0}$ and let X be a positive integer, to be fixed later. By Dirichlet's approximation theorem, there exist integers a,q such that $\gcd(a,q)=1$, $1\leqslant q\leqslant X$ and

$$\left|\mathbf{j}\cdot\mathbf{c}-\frac{a}{q}\right| \leqslant \frac{1}{qX}.$$

Hence,

$$\|q\mathbf{j}\cdot\mathbf{c}\| \leqslant \frac{1}{X}.$$

On the other hand, by condition (1) in Theorem 1.1,

$$\frac{C}{q^k\|\mathbf{j}\|_\infty^k} < \|q\mathbf{j}\cdot\mathbf{c}\|.$$

It follows that

$$q > \frac{(CX)^{1/k}}{\|\mathbf{j}\|_\infty}.$$

Now we apply the following well-known lemma.

LEMMA 3.3. *Let $L\geqslant 1$ and $x>1$. Suppose that $|c-a/q|\leqslant q^{-2}$ with $a\in\mathbb{Z}$, $q\in\mathbb{N}$ and $(a,q)=1$. Then*

$$\sum_{1\leqslant l\leqslant L} \min\left(\frac{x}{l}, \|lc\|^{-1}\right) \ll \left(\frac{x}{q}+L+q\right)(\log 2Lqx).$$

Proof. This is Lemma 6.4.4 in [3]. □

It follows that

$$\sum_{1\leqslant m\leqslant M} \min\left(\frac{x}{m}, \|m\mathbf{j}\cdot\mathbf{c}\|^{-1}\right) \ll \left(\frac{x\|\mathbf{j}\|_\infty}{X^{1/k}}+M+X\right)(\log 2MXx)$$

and hence

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$$R_d(H_1, \dots, H_d, M, x) \ll H_1 \cdots H_d \left(\frac{x \max(H_1, \dots, H_d)}{X^{1/k}} + M + X \right) (\log 2MXx). \tag{30}$$

Combining (28) and (30), we obtain

$$R_{\mathcal{A}}(M, x) \ll (\log 2P)^{d+1} \left(\frac{xJ}{X^{1/k}} + M + X \right).$$

Now, choosing

$$X := [(xJ)^{1-1/(k+1)}],$$

we deduce that

$$R_{\mathcal{A}}(M, x) \ll (\log 2P)^{d+1} (M + (xJ)^{1-1/(k+1)}). \tag{31}$$

3.5. *Completion of the proof.* Using (24), (26), (28) and (31), we get

$$Z_1 \ll (\log 2P)^{d+1} (u^2 + (PJ)^{1-1/(k+1)}) \tag{32}$$

and

$$Z_2(L) \ll (\log 2P)^{3d/2+5/2} (P^{1/2} L^{1/2} + PL^{-1/2} + P^{1-1/(2(k+1))} J^{1/2-1/(2(k+1))})$$

and hence, by (25),

$$Z_2 \ll (\log 2P)^{3d/2+7/2} (Pu^{-1/2} + P^{1-1/(2(k+1))} J^{1/2-1/(2(k+1))}). \tag{33}$$

Combining (22), (23), (32) and (33), we get

$$\tilde{U}_{\mathcal{A}}(P) \ll P^{\varepsilon} (u^2 + Pu^{-1/2} + (PJ)^{1-1/(k+1)} + P^{1-1/(2(k+1))} J^{1/2-1/(2(k+1))}).$$

Now, choosing $u := P^{2/5}$, it follows that

$$\tilde{U}_{\mathcal{A}}(P) \ll P^{\varepsilon} (P^{4/5} + (PJ)^{1-1/(k+1)} + P^{1-1/(2(k+1))} J^{1/2-1/(2(k+1))}). \tag{34}$$

We now turn to the term $\tilde{V}_d(P)$, defined in (21). By (29) and (30),

$$\tilde{V}_d(P) \ll R_d(1, \dots, 1, J, P) \ll \left(\frac{P}{X^{1/k}} + J + X \right) (\log 2JXP)$$

for any positive integer X. Choosing

$$X := [P^{1-1/(k+1)}]$$

gives

$$\tilde{V}_d(P) \ll \log(2P)(J + P^{1-1/(k+1)}).$$

Combining this with (21), we find that

$$V_{\mathcal{A}}(P) \ll (\log 2P)(PJ^{-1} + J + P^{1-1/(k+1)}). \tag{35}$$

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Now, putting (20), (34) and (35) together, we arrive at

$$T_{\mathcal{A}}(P) \ll P^{\varepsilon}(PJ^{-1}+J+P^{4/5}+(PJ)^{1-1/(k+1)}+P^{1-1/(2(k+1))}J^{1/2-1/(2(k+1))}).$$

Now, choosing

$$J := [P^{1/(3k+2)}], \tag{36}$$

we get

$$T_{\mathcal{A}}(P) \ll P^{1-1/(3k+2)+\varepsilon},$$

which proves (19), upon replacing ε by $\varepsilon/2$. This completes the proof of Theorem 2.2(i).

§4. *Proof of Theorem 2.2(ii).*

4.1. *Sieve theoretical approach.* We are broadly following the treatment in [**12**] with appropriate modifications because the linear case, considered here, requires a different treatment. In particular, as in the previous section, the Diophantine properties of the vector $(c_1, \dots, c_d)$ will come into play. Let $\{\cdot\}$ represent the fractional part, and put

$$\mu := N^{\varepsilon-\gamma_{d,k}}.$$

Write

$$\mathcal{A} = \mathcal{A}(\alpha) = \{n[n\alpha] : 1 \leqslant n \leqslant N, \{n\alpha\} < \mu, \{nc_i\alpha\} < \mu \text{ for } i = 1, \dots, d\}.$$

We desire to show that $\mathcal{A}$ does not contain too many products of two primes. To this end, we apply a two-dimensional upper bound sieve (see [**8**], Theorem 5.2). We therefore need to obtain an asymptotic formula for the number of solutions to

$$n[n\alpha] \equiv 0 \bmod q, \quad 1 \leqslant n \leqslant N,$$

with

$$\{n\alpha\} < \mu \quad \text{and} \quad \{nc_i\alpha\} < \mu \quad \text{for } i = 1, \dots, d, \tag{37}$$

where

$$q \leqslant Q := N^{\varepsilon}.$$

For this, it suffices to establish a formula for the number of solutions to

$$n \equiv 0 \bmod t_1, \qquad [n\alpha] \equiv 0 \bmod t_2,$$

subject to (37). We can combine (37) with the congruence conditions to require

$$1 \leqslant n \leqslant \frac{N}{t_1}, \qquad \left\{\frac{nt_1\alpha}{t_2}\right\} < \frac{\mu}{t_2}, \qquad \{nt_1c_i\alpha\} < \mu \quad \text{for } i = 1, \dots, d, \tag{38}$$

and count the number $S(\alpha; t_1, t_2)$ of solutions to (38), using Fourier analysis. To this end, we write

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$$S(\alpha, t_1, t_2) = \sum_{1 \leqslant n \leqslant N/t_1} \left(\left[\frac{nt_1\alpha}{t_2} \right] - \left[\frac{nt_1\alpha}{t_2} - \frac{\mu}{t_2} \right] \right) \cdot \prod_{i=1}^{d} ([nt_1 c_i \alpha] - [nt_1 c_i \alpha - \mu])$$

and evaluate this term in a similar way as the term $T(P)$ defined in (14), using Vaaler's Lemma 3.1. By a chain of similar calculations, we arrive at the asymptotic estimate

$$S(\alpha; t_1, t_2) = \frac{N\mu^{d+1}}{t_1 t_2} + O\left(\frac{N\mu^d}{L} + E(\alpha; t_1, t_2) \right), \tag{39}$$

where we set

$$L := Q^3 \mu^{-1}$$

and

$$E(\alpha; t_1, t_2) := \frac{\mu^{d+1}}{t_2} \sum_{\substack{|m_0| \leqslant L, \dots, |m_d| \leqslant L \\ (m_0, \dots, m_d) \neq (0, \dots, 0)}} \left| \sum_{1 \leqslant n \leqslant N/t_1} \mathrm{e}\left(n\alpha t_1 \left(\frac{m_0}{t_2} + \sum_{i=1}^{d} c_i m_i \right) \right) \right|.$$

The above estimate (39) is analogous to the equation after (26) in [**12**].

Now, applying the upper bound sieve gives

$$F_N(\alpha) \leqslant \frac{C(\varepsilon) N \mu^d}{\log^2 N} + O(J_N(\alpha)), \tag{40}$$

where

$$\begin{aligned} J_N(\alpha) &:= \sum_{t_1 t_2 \leqslant Q} (t_1 t_2)^{\varepsilon} \left(\frac{N\mu^d}{L} + E(\alpha; t_1, t_2) \right) \\ &= \sum_{t_1 t_2 \leqslant Q} (t_1 t_2)^{\varepsilon} E(\alpha; t_1, t_2) + o\left(\frac{N\mu^{d+1}}{\log^2 N} \right) \quad \text{as } N \to \infty. \end{aligned}$$

Hence, to establish the claim in Theorem 2.2(i), it suffices to show that

$$\sum_{t_1 t_2 \leqslant Q} (t_1 t_2)^{\varepsilon} \int_A^B E(\alpha; t_1, t_2)\, d\alpha = o\left(\frac{N\mu^{d+1}}{\log^2 N} \right) \quad \text{as } N \to \infty, N \in \mathcal{S}. \tag{41}$$

4.2. *Average estimation for* $E(\alpha; t_1, t_2)$. To estimate the expression on the right-hand side of (41), we first observe that

$$\begin{aligned} &E(\alpha; t_1, t_2) \\ &\ll \frac{\mu^{d+1}}{t_2} \sum_{\substack{|m_0| \leqslant L, \dots, |m_d| \leqslant L \\ (m_0, \dots, m_d) \neq (0, \dots, 0)}} \min\left(\frac{N}{t_1}, \left\| \alpha t_1 \left(\frac{m_0}{t_2} + m_1 c_1 + \cdots + m_d c_d \right) \right\|^{-1} \right) \end{aligned} \tag{42}$$

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and note that if $(m_0, \dots, m_d) \neq (0, \dots, 0)$, then the term

$$t_1\left(\frac{m_0}{t_2} + m_1c_1 + \cdots + m_dc_d\right)$$

on the right-hand side of (42) is non-zero because $(c_1, \dots, c_d)$ is k-Diophantine. For an estimation of the integral in (41), we now use the following lemma.

LEMMA 4.1. *Suppose that* $0 < A < B$, $K \geqslant 2$ *and* $x \neq 0$. *Then*

$$\int_A^B \min(K, \|\alpha x\|^{-1})\, d\alpha = O_{A,B}(\min\{K, \max\{1, |x|^{-1}\}\} \log K). \tag{43}$$

Proof. We confine ourselves to the case when $x > 0$ since the case when $x < 0$ is similar. By change of variables $\beta = \alpha x$, we get

$$\int_A^B \min(K, \|\alpha x\|^{-1})\, d\alpha = \frac{1}{x}\int_{xA}^{xB} \min(K, \|\beta\|^{-1})\, d\beta. \tag{44}$$

By periodicity of the integrand, if $x(B - A) \geqslant 1$, then

$$\begin{aligned}\frac{1}{x}&\int_{xA}^{xB} \min(K, \|\beta\|^{-1})\, d\beta \\ &\leqslant \frac{[x(B - A) + 1]}{x}\int_0^1 \min(K, \|\beta\|^{-1})\, d\beta = O_{A,B}(\log K).\end{aligned} \tag{45}$$

If $1/K \leqslant x(B - A) < 1$, then

$$\begin{aligned}\frac{1}{x}&\int_{xA}^{xB} \min(K, \|\beta\|^{-1})\, d\beta \\ &\leqslant \frac{1}{x}\int_{-x(B-A)/2}^{x(B-A)/2} \min(K, \|\beta\|^{-1})\, d\beta = O_{A,B}\left(\frac{\log K}{x}\right).\end{aligned} \tag{46}$$

If $0 < x(B - A) < 1/K$, then, trivially,

$$\frac{1}{x}\int_{xA}^{xB} \min(K, \|\beta\|^{-1})\, d\beta = O_{A,B}(K). \tag{47}$$

Combining (44)–(47), we deduce the claim when $x > 0$, which completes the proof. □

Now, employing (42) and Lemma 4.1, we get

$$\begin{aligned}\int_A^B E(\alpha; t_1, t_2)\, d\alpha \ll \frac{\mu^{d+1}}{t_2} &\sum_{\substack{|m_0|\leqslant L, \dots, |m_d|\leqslant L \\ (m_0,\dots,m_d)\neq(0,\dots,0)}} \\ &\times \min\left\{\frac{N}{t_1}, \max\left\{1, \left|t_1\left(\frac{m_0}{t_2} + m_1c_1 + \cdots + m_dc_d\right)\right|^{-1}\right\}\log N\right\}\end{aligned}$$

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$$\ll \mu^{d+1}(\log N) \sum_{\substack{|m_0|\leqslant L,\dots,|m_d|\leqslant L\\(m_0,\dots,m_d)\neq(0,\dots,0)}} \times \min\left\{\frac{N}{t_1t_2}, \max\left\{\frac{1}{t_2}, |t_1m_0 + t_1t_2(m_1c_1 + \cdots + m_dc_d)|^{-1}\right\}\right\}.$$

The contribution of $(m_1, \dots, m_d) = (0, \dots, 0)$ to the last line is bounded by $O(\mu^{d+1}L \log N)$, and the contribution of $(m_1, \dots, m_d) \neq 0$ is bounded by

$$\begin{aligned}
&\ll \mu^{d+1}(\log N) \sum_{\substack{|m_1|\leqslant L,\dots,|m_d|\leqslant L\\(m_1,\dots,m_d)\neq(0,\dots,0)}} \sum_{|m_0|\leqslant L} \\
&\quad \times \min\left\{\frac{N}{t_1t_2}, \max\left\{\frac{1}{t_2}, |t_1m_0 + t_1t_2(m_1c_1 + \cdots + m_dc_d)|^{-1}\right\}\right\} \\
&\ll \mu^{d+1}(\log N) \sum_{\substack{|m_1|\leqslant L,\dots,|m_d|\leqslant L\\(m_1,\dots,m_d)\neq(0,\dots,0)}} \\
&\quad \times \min\left\{\frac{N}{t_1t_2}, \|t_1t_2(m_1c_1 + \cdots + m_dc_d)\|^{-1}\right\} + \mu^{d+1}L^{d+1}\log N.
\end{aligned}$$

So, altogether,

$$\begin{aligned}
\int_A^B E(\alpha; t_1, t_2)\, d\alpha &\ll \mu^{d+1}L^{d+1}\log N \\
&+ \mu^{d+1}(\log N) \sum_{\substack{|m_1|\leqslant L,\dots,|m_d|\leqslant L\\(m_1,\dots,m_d)\neq(0,\dots,0)}} \min\left\{\frac{N}{t_1t_2}, \|t_1t_2(m_1c_1 + \cdots + m_dc_d)\|^{-1}\right\}.
\end{aligned}$$

Summing up over t_1 and t_2, writing $q = t_1t_2$ and using $d(q) = O(q^\varepsilon)$, with $d(q)$ being the divisor function, we obtain

$$\begin{aligned}
&\sum_{t_1t_2\leqslant Q} (t_1t_2)^\varepsilon \int_A^B E(\alpha; t_1, t_2)\, d\alpha \\
&\quad \ll N^\varepsilon QL^{d+1}\mu^{d+1} + N^\varepsilon\mu^{d+1}R_d(L, \dots, L, Q, N),
\end{aligned}$$

where $R_d(L, \dots, L, Q, N)$ is defined as in (29). Combining this with (30), we get

$$\sum_{t_1t_2\leqslant Q} (t_1t_2)^\varepsilon \int_A^B E(\alpha; t_1, t_2)\, d\alpha \ll N^\varepsilon L^d\mu^{d+1}\left(L + \frac{LN}{X^{1/k}} + Q + X\right)\log(2QNX)$$

for any positive integer X. Choosing $X := [(LN)^{1-1/(k+1)}]$, we deduce that

$$\sum_{t_1t_2\leqslant Q} (t_1t_2)^\varepsilon \int_A^B E(\alpha; t_1, t_2)\, d\alpha \ll N^\varepsilon L^d\mu^{d+1}(L + Q + (LN)^{1-1/(k+1)}),$$

which implies (41), upon recalling that $\mu := N^{\varepsilon-\gamma_{d,k}}$, $Q := N^\varepsilon$, $L := Q^3\mu^{-1}$ and (2).

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