Student Problems

Students up to the age of 19 are invited to send solutions to either of the following problems to Stan Dolan, 126A Harpenden Road, St. Albans, Herts AL3 6BZ.

Two prizes will be awarded – a first prize of £25, and a second prize of £20 – to the senders of the most impressive solutions for either problem. It is, therefore, not necessary to submit solutions to both. Solutions should arrive by May 20th 2016. Please give your School year, the name and address of your School or College, and the name of a teacher through whom the award may be made. Please print your own name clearly! The names of all successful solvers will be published in the July 2016 edition.

Problem 2016.1 (Chris Starr)

The four rods, AB, BC, CD and DE, are of equal length and are hinged at B, C and D. They lie on a flat surface in such a way that A touches CD and E touches AB.

Given that triangle ADE is isosceles, find $\angle ADE$.

Problem 2016.2 (Paul Stephenson)

Consider this truncated pyramid of numbers. The k th row starts with k 1s. Each subsequent number is then the sum of the k previous numbers. The second row is therefore the first five Fibonacci numbers.

			1	1	1			
		1	1	2	3	5		
	1	1	1	3	5	9	17	
1	1	1	1	4	7	13	25	49
.	.	.	.	.	.	.	.	.

(a) What do you notice about the differences between vertical entries? Explain your conclusion.

(b) Find a formula for the i th number in the j th row.

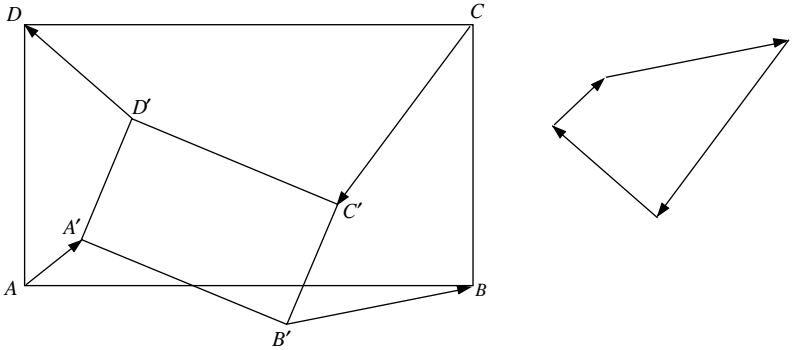
https://doi.org/10.1017/mag.2016.32 Published online by Cambridge University Press

Solutions to 2015.5 and 2015.6

Problem 2015.5 was solved by James Chong (Charterhouse) and Thomas Gessey-Jones (Tamworth Sixth Form). Problem 2015.6 was solved by Ed Turnbull (Bishop Wordsworth's School). Both problems were solved by Andrew Ng (Dulwich College) and Victor Wong (Charterhouse).

Problem 2015.5 (Dao Thanh Oai)

Draw any two similar rectangles $ABCD$ and $A'B'C'D'$ with vertices corresponding in the natural way.



Prove that, as shown above, the vectors $\overrightarrow{AA'}$, $\overrightarrow{B'B}$, $\overrightarrow{CC'}$ and $\overrightarrow{D'D}$ always form a closed quadrilateral. What can you determine about the diagonals of this quadrilateral?

Solution

The diagonals are perpendicular. It is also true (but not spotted by respondents) that the lengths of the diagonals are in proportion to the lengths of the sides of the original rectangle.

James, Thomas and Victor all obtained a closed quadrilateral by the same neat method:

$$\overrightarrow{AA'} + \overrightarrow{A'B'} + \overrightarrow{B'B} + \overrightarrow{BC} + \overrightarrow{CC'} + \overrightarrow{C'D'} + \overrightarrow{D'D} + \overrightarrow{DA} = \mathbf{0} \quad \text{(Closed loop)}$$

$\overrightarrow{BC} + \overrightarrow{DA} = \mathbf{0}$ and $\overrightarrow{A'B'} + \overrightarrow{C'D'} = \mathbf{0}$ (Opposite sides of a rectangle)

Therefore $\overrightarrow{AA'} + \overrightarrow{B'B} + \overrightarrow{CC'} + \overrightarrow{D'D} = \mathbf{0}$, as required.

Thomas then completed the proof using the scalar product and the fact that $ABCD$ can be transformed into $A'B'C'D'$ by an enlargement of scale factor K, a rotation of angle θ and a translation.

The diagonals are $\overrightarrow{AB} - \overrightarrow{A'B'}$ and $\overrightarrow{A'D'} - \overrightarrow{AD}$. The scalar product of these vectors then simplifies to

$$\overrightarrow{AB}.\overrightarrow{A'D'} + \overrightarrow{AD}.\overrightarrow{A'B'} = k|AB||AD|\left(\cos\left(\theta + \frac{\pi}{2}\right) + \cos\left(\theta - \frac{\pi}{2}\right)\right) = 0.$$

Beautiful!

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Problem 2015.6 (Paul Stephenson)

Consider powers of 10 written as binary numbers, for example:

$$10^1 = 1\mathbf{0}10_2, \qquad 10^2 = 11\mathbf{00}100_2, \qquad 10^3 = 1111\mathbf{0}1000_2.$$

For 10^n, the number of zero digits to the right of the final 1 is just n. Can you find and prove a formula for the number of zeros between the penultimate and final 1s? These zeros are shown emboldened above.

Solution

Victor, Ed and Andrew started the proof in similar fashion. What follows is based upon Ed's very clearly expressed solution.

First note that $10^n = 2^n \times 5^n$. The factor of 2^n accounts for the n trailing zeroes and can be ignored, leaving powers of 5 to be considered as binary numbers.

If n is odd, then $5^n - 1 = (5 - 1)(1 + 5 + 5^2 + \ldots + 5^{n-1})$ is 4 times an odd number.

If n is even let $n = 2^k m$, where m is odd. Then

$$5^n - 1 = (5^m - 1)(5^m + 1)(5^{2m} + 1)\ldots(5^{2^{k-1}m} + 1).$$

Each factor $5^i + 1 \equiv 2 \pmod 4$ and so, for both odd and even n, $5^n - 1$ is 2^{k+2} times an odd number.

The number of emboldened zeroes is therefore 1 more than the power of 2 which divides n.

The first prize of £25 is awarded to Thomas Gessey-Jones. The second prize of £20 is awarded to Ed Turnbull.

doi:10.1017/mag.2016.32 STAN DOLAN

https://doi.org/10.1017/mag.2016.32 Published online by Cambridge University Press