Math. Proc. Camb. Phil. Soc. (1995), **117**, 345
Printed in Great Britain

Unavoidable systems of functions

By W. K. HAYMAN

Department of Mathematics, University of York, Heslington, York YO1 5DD, U.K.

and LEE A. RUBEL[1]

Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, Illinois 61801, *U.S.A.*

(*Received* 27 *April* 1994)

1. *Introduction*

Suppose that D is a plane domain and that $f(z)$, $g(z)$ are meromorphic in D and $f(z) \neq g(z)$ for all z in D. Then following Rubel and Yang [**11**], we say that $f(z)$ avoids $g(z)$ in D. A system of functions $g_1(z), \ldots, g_n(z)$ is said to be unavoidable if, whenever f is meromorphic in D, at least one of the equations $f(z) = g_\nu(z)$ has a root in D. Rubel and Yang [11] proved that if D is the open plane, then any two functions form an avoidable system, but three distinct polynomials a_1, a_2, a_3 such that $a_1 - a_2$ and $a_2 - a_3$ are not both constant form an unavoidable system.

In this paper we show that the above result extends to an arbitrary plane domain D.

Vincent Blondel has informed us that, in the case of the unit disc, our work is closely related to the 'simultaneous stabilization problem of linear systems' of control theory. It is also called the 'bistable (or stable) minimum phase problem.' (See Bertilsson and Blondel [**3**], and Blondel [**4**], theorem 5.4.)

THEOREM 1. *Let D be a domain in the open complex plane. Then given $g(z)$ and $h(z)$ meromorphic in D, there exists $f(z)$ meromorphic in D such that $f(z) \neq g(z)$ and $f(z) \neq h(z)$ for all z in D.*

On the other hand, there exists a function $\phi(z)$ holomorphic in D such that if $f(z)$ is meromorphic in D than at least one of the three equations $f(z) = \pm\phi(z)$, ∞ has infinitely many roots in D. Thus $\pm\phi(z)$, ∞ form an unavoidable system in D.

Roughly, we can choose for $\phi(z)$ any meromorphic function having only multiple zeros and many of them.

2. *Two functions are avoidable*

We require two classical interpolation results (see e.g. Leucking and Rubel [**10**]).

[1] Rubel's research was partially supported by a grant from the National Science Foundation. His research on this paper was partially conducted while he was Visiting Professor at the Hong Kong University of Science and Technology, which he thanks for its hospitality and support.

https://doi.org/10.1017/S0305004100073175 Published online by Cambridge University Press

LEMMA 1. *Suppose that (a_n) is a sequence of distinct points in D and that a_n tends to the boundary of D as $n \to \infty$. Then*

(i) *if (q_n) is a sequence of positive integers there exists $F(z)$ holomorphic in D and having zeros of multiplicity q_n at a_n and no other zeros;*

(ii) *if, further, $\{P_n(z)\}$ is an arbitrary sequence of polynomials, there exists $G(z)$ holomorphic in D such that*

$$G(z) - P_n(z) = o(z-a_n)^{q_n} \quad as \quad z \to a_n$$

for each fixed n.

Given (i) and (ii), the proof proceeds as in Rubel and Yang [11], but we give it for completeness. Suppose that $g(z)$ is meromorphic in D with poles at b_n with multiplicity p_n. We form a function $g_2(z)$ holomorphic in D and having zeros of multiplicity p_n at the points b_n. This is possible by (i). Thus $g_1(z) = g(z)g_2(z)$ is holomorphic in D and $g_1(b_n) \neq 0$. Hence $g(z) = g_1(z)/g_2(z)$ where g_1, g_2 are holomorphic in D, and have no common zeros. Similarly, if $h(z)$ is a second meromorphic function, we may write $h(z) = h_1(z)/h_2(z)$ where h_1, h_2 are holomorphic in D and have no common zeros.

We now write

$$f(z) = \frac{\phi(z)h_1(z) - \psi(z)g_1(z)}{\phi(z)h_2(z) - \psi(z)g_2(z)}, \tag{1}$$

and show that for a suitable choice of ϕ, ψ, the function $f(z)$ avoids $g(z)$ and $h(z)$. We write

$$k(z) = h_1(z)g_2(z) - g_1(z)h_2(z). \tag{2}$$

Then $k(z) = 0$ if and only if $g(z) = h(z)$. If $k(z) \equiv 0$, then g(z) $\equiv h(z)$. In this case we solve our problem with $g(z)$, $g(z)+1$ instead of $g(z)$, $h(z)$ ($\equiv g(z)$). Thus we may assume that $k(z) \not\equiv 0$. Let a_n be the distinct zeros of $k(z)$ in D with multiplicities q_n. We write for each n, $k(z) = (z-a_n)^{q_n}K_n(z)$ with $K_n(a_n) \neq 0, \infty$. Let C be a constant such that for every n

$$g(a_n) = h(a_n) \neq C \quad \text{and} \quad C \neq 0, \infty. \tag{3}$$

We define

$$\phi_n(z) = \frac{g_2 C - g_1}{K_n(z)}, \quad \psi_n(z) = \frac{h_2 C - h_1}{K_n(z)}. \tag{4}$$

Since $g(a_n) \neq C$ and g_1, g_2 have no common zeros, $\phi_n(a_n) \neq 0$ and similarly $\psi_n(a_n) \neq 0$. We now construct functions Φ, Ψ such that Φ, Ψ are holomorphic in D and have at a_n the same power series developments up to $(z-a_n)^{q_n}$ as $\log \phi_n(z)$, $\log \psi_n(z)$, respectively, where we choose the principal values of the logarithms at a_n. This choice of Φ, Ψ is possible by Lemma 1 (ii). We then set

$$\phi = \exp \Phi, \quad \psi = \exp \Psi, \quad \text{so that} \quad \phi(z) \neq 0, \infty \quad \text{and} \quad \psi(z) \neq 0, \infty \quad \text{in} \quad D. \tag{5}$$

Now we define

$$f_1(z) = \frac{\phi h_1 - \psi g_1}{k}, \quad f_2 = \frac{\phi h_2 - \psi g_2}{k}, \tag{6}$$

so that $f = f_1/f_2$ is given by (1). We proceed to show that f avoids g and h.

It follows from our choice of ϕ, ψ that, as $z \to a_n$,

$$\phi(z) = \phi(z) + O(z-a_n)^{q_n+1}, \quad \psi(z) = \psi_n(z) + O(z-a_n)^{q_n+1},$$

https://doi.org/10.1017/S0305004100073175 Published online by Cambridge University Press

so that, near $z = a_n$, we have from (4) and (5)

$$f_1(z) = \frac{\phi_n h_1 - \psi_n g_1}{k} + O(z-a_n) = \frac{C}{K_n(a_n)} + O(z-a_n).$$

Thus $f_1(a_n) = C/K_n(a_n)$. Similarly $f_2(a_n) = 1/K_n(a_n)$. Thus $f(a_n) = C$ for all n, so that by (3), f avoids g, h at a_n. Suppose next that $z \neq a_n$. If $g(z) = \infty$, so that $g_2(z) = 0$, we have $g_1(z) \neq 0$. Also $h_2(z) \neq 0$ since otherwise $g(z) = h(z) = \infty$, which contradicts $z \neq a_n$. Now (1) yields, since $\phi\psi g_1(z) \neq 0$ by (5),

$$f(z) = \frac{\phi h_2 - \psi g_1}{\phi h_2} \neq \infty = g(z).$$

Next, if $g(z) \neq \infty$, we deduce from (1), (6) that

$$f(z) - g(z) = \frac{\phi k}{g_2(\phi h_2 - \psi g_2)} \neq 0,$$

so that again $f(z) \neq g(z)$. Thus f avoids g and similarly f avoids h. This proves the first part of Theorem 1.

3. *Construction of an unavoidable triplet*

We require Nevanlinna's second fundamental theorem (Hayman [**9**], p. 43). We also refer to [**9**] for the basic notation.

LEMMA 2. *Suppose that $f(z)$ is meromorphic in $|z| < R_0 < \infty$, and*

$$\overline{\lim_{r \to R_0}} \frac{T(r,f)}{\log(1/R_0 - r)} = \infty. \tag{7}$$

Then if

$$\Theta(a) = 1 - \overline{\lim_{r \to R_0}} \frac{\bar{N}(r,a)}{T(r)}, \tag{8}$$

we have

$$\sum \Theta(a) \leqslant 2,$$

where the sum is taken over all a such that $\Theta(a) \neq 0$.

Suppose now that D is a domain in the closed plane whose complement contains at least one point. We may suppose that the complement of D contains a finite point, since otherwise this can be achieved by the map $w = 1/z$. We also assume that the origin lies in D since this can be achieved by a further translation. Let now $|z| < R_0$ be the largest disc centred at the origin which lies in D. Then there exists a point $z_0 = R_0 e^{i\Theta_0}$ which is not in D. We deduce the next result from Lemma 2.

THEOREM 2. *Suppose that $\phi(z)$ is meromorphic in $|z| < R_0$ and has only multiple zeros there, whose number $n(r)$ in $|z| < r$, counting multiplicity, satisfies*

$$\overline{\lim_{r \to R_0}} \frac{N(r)}{\log\{1/(R_0 - r)\}} = \infty \tag{9}$$

where

$$N(r) = \int_0^r (n(t) - n(0)) \frac{dt}{t}.$$

https://doi.org/10.1017/S0305004100073175 Published online by Cambridge University Press

Then if $f(z)$ is meromorphic in $|z| < R_0$, at least one of the equations $f(z) = \pm\phi(z), \infty$, has infinitely many roots in $|z| < R_0$.

Suppose that, contrary to this, the total number n_1 of the roots of the equations $f(z) = \pm\phi(z), \infty$, is finite. Consider

$$F(z) = f(z)/\phi(z).$$

Then $F(z) = \infty$ only when $f(z) = \infty$ or $\phi(z) = 0$.

Also since $f = \infty$ has only finitely many roots, all but a finite number of the poles of $F(z)$ are multiple, and coincide with the multiple zeros of ϕ. We have, by the first fundamental theorem (l.c. p.5),

$$T(r, F) \geqslant N(r) + O(1)$$

so that

$$\overline{\lim} \frac{T(r, F)}{\log\{1/(R_0 - r)\}} = \infty.$$

Thus, Lemma 2 is applicable. We have, since F has only finitely many simple poles,

$$\bar{N}(r, \infty, F) \leqslant \tfrac{1}{2}N(r, \infty, F) + O(1) \leqslant \tfrac{1}{2}T(r, F) + O(1)$$

as $r \to R_0$. Thus, $T(r, F)$ is unbounded, and being increasing, tends to ∞ as $r \to R_0$. Thus $\Theta(\infty, F) \geqslant \frac{1}{2}$.

Similarly, the equations $F(z) = \pm 1$ have only finitely many roots, so that $\Theta(1, F) = 1, \Theta(-1, F) = 1$. This contradicts Lemma 2 and so Theorem 2 is proved.

We may for instance, choose

$$\phi(z) = \left\{\sin\left(\frac{z_0}{z_0 - z}\right)^2\right\}^2, \quad z_0 = R_0 e^{i\theta_0}.$$

This function is holomorphic except at z_0 and so in D. It has only double zeros, given by

$$\left(\frac{z_0}{z_0 - z_n}\right)^2 = n\pi, \quad \text{i.e.} \quad z_n = z_0(1 - (n\pi)^{-\frac{1}{2}}).$$

Hence

$$n(r) \sim \frac{2}{\pi}\left(\frac{R_0}{R_0 - r}\right)^2$$

and (9) is satisfied. We deduce that $\pm\phi(z), \infty$ form an unavoidable triplet in $|z| < R_0$ and so in D, since ϕ is holomorphic in D.

4. *Extensions and generalizations*

We may consider the corresponding problem when D is a Riemann surface. If D is compact, then any non-constant function $\phi(z)$ is unavoidable in D. For suppose that $f(z)$ is meromorphic in D and $f(z) \neq \phi(z)$ for all z in D. Then $F(z) = \{f(z) - \phi(z)\}^{-1}$ remains finite everywhere. Thus by the maximum principle $F(z) =$ constant, so that $f(z) - \phi(z) =$ constant. Thus f and ϕ have the same poles and this gives a contradiction.

If D is a non-compact Riemann surface, then, as shown in the appendix, Lemma 1 extends to D and so, given any two meromorphic functions g, h on D, there exists

https://doi.org/10.1017/S0305004100073175 Published online by Cambridge University Press

f meromorphic in D and avoiding g, h. Finally, we can construct an unavoidable triplet in D. We consider a function $w = \psi(z)$ mapping the unit disk onto the universal cover over D. This function has a fundamental domain Δ in $|z| < 1$ which is non-compact in $|z| < 1$. We choose a sequence (z_n) of distinct points in Δ such that $|z_n| \leqslant 1 - n^{-\frac{1}{2}}$, with $|z_n| \to 1$ as $n \to \infty$. If w_n is the image of the point z_n in D then we construct a function $\Phi(w)$ holomorphic in D, having double zeros at the points w_n. This is possible by the Appendix. Then $\phi(z) = \Phi(\psi(z))$ is holomorphic in $|z| < 1$, $\phi(z)$ has only multiple zeros, and has zeros at all points equivalent to the z_n. Thus if $N(r)$ is the integrated counting function of these zeros, we have

$$N(r) > \frac{1}{2(1-r)},$$

for r near 1, so that (9) holds. Suppose now that $F(w)$ is holomorphic in D and that $F(w) \neq \pm\Phi(w)$. Then $f(z) = F(\psi(z))$ is holomorphic in $|z| < 1$ and $f(z) \neq \pm\phi(z)$, and we obtain a contradiction from Theorem 2.

We may also ask about unavoidable sets of functions which avoid each other. The following result is due to Bers and Royden [2, Proposition 3, p. 261].

THEOREM 3. *Suppose that ϕ_1, ϕ_2, ϕ_3, ϕ_4 are meromorphic in $|z| < 1$ and that $\phi_i(z) \neq \phi_j(z)$ for $|z| < 1$ and $i \neq j$. Then if $\lambda \neq \phi_i(0)$, $i = 1, \ldots, 4$, there exists $\phi_5(z)$ meromorphic in $|z| < 1$ and such that*

$$\phi_5(0) = \lambda$$
$$\phi_5(z) \neq \phi_i(z), \quad |z| < 1, \quad i = 1, \ldots, 4.$$

In fact the authors proved the above result with $\phi_1, \phi_2, \phi_3 = 0, 1, \infty$ and $\phi_4 = \mu$ where μ is a map of the unit disc onto the universal covering surface over the plane punctured at 0, 1, ∞. From this the general result follows at once. Suppose in fact that $\phi_1, \phi_2, \phi_3, \phi_4$, are as in Theorem 3. If $\phi(z)$ is meromorphic in $|z| < 1$, we define $\psi(z) = T(\phi)$ by

$$\psi(z) = T(\phi) = \frac{\phi - \phi_1}{\phi - \phi_3}\frac{\phi_2 - \phi_3}{\phi_2 - \phi_1}.$$

Then $\phi(z) = \phi_1(z)$ if and only if $\psi(z) = 0$; $\phi(z) = \phi_2(z)$ if and only if $\psi(z) = 1$; $\phi(z) = \phi_3(z)$ if and only if $\psi(z) = \infty$. Thus

$$\psi_4(z) = T(\phi_4) \neq 0, 1, \infty \quad \text{in} \quad |z| < 1.$$

It now follows that $\psi_4(z)$ is subordinate to $\mu(z)$ where μ maps $|z| < 1$ onto the universal cover of the complex plane punctured at 0, 1, ∞, and $\mu(0) = \psi_4(0)$. Thus $\psi_4(z) = \mu(\omega(z))$ where $\omega(z)$ satisfies the hypotheses of Schwarz's Lemma. Let $\psi(z)$ be a Bers-Royden function such that $\psi(z) \neq 0, 1, \infty, \mu(z)$ and $\psi(0) = T(\lambda)$. Then

$$\psi_5(z) = \psi(\omega(z)) \neq 0, 1, \infty, \mu(\omega(z))$$

since, if $\zeta = \omega(z)$, $\psi(\zeta) \neq \mu(\zeta)$. Also $\psi_5(0) = T(\lambda)$. If now

$$\phi_5(z) = T^{-1}\{\psi_5(z)\} = \frac{\psi_5\phi_3(\phi_2 - \phi_1) - \phi_1(\phi_2 - \phi_3)}{\psi_5(\phi_2 - \phi_1) - (\phi_2 - \phi_3)}$$

we see that $\phi_5 \neq \phi_1, \ldots, \phi_4$ so that the quadruplet $\phi_1, \ldots, \phi_4$ is avoidable. This proves

https://doi.org/10.1017/S0305004100073175 Published online by Cambridge University Press

Theorem 3. It has recently been proved for any n (See Slodkowski [12], theorem 1.4, p. 348) that a family of n functions that avoid each other can be enlarged to a family of $n+1$ with that property.

Appendix

We state here the analogue of Lemma 1 for non-compact Riemann surfaces. The proof of part (i) can be found in Forster [5], theorem 26·5, p. 203 ff. One proof of part (ii) could be patterned after the proof of theorem 26·7 on p. 204 ff, loc. cit., given the Mittag-Leffler theorem, theorem 26·3 on pages 202 ff, loc. cit., but we give a different proof based on functional analysis considerations.

THEOREM. (i) *Given an open Riemann surface R, a sequence of distinct points $z_1, z_2, \ldots$, of R that has no limits points in R, and positive integers $n_1, n_2, \ldots$, there exists a holomorphic function f on R that has a zero of order n_k at z_k for $k = 1, 2, \ldots$, and has no other zeros.*

(ii) *Given an open Riemann surface R, a sequence of distinct points $z_1, z_2, \ldots$, of R that has no limit points in R, given analytic univalent maps $C_n: \mathbb{D} \to R$ from the open disc $\mathbb{D} \ni 0$ into R such that $C_n(0) = z_n$ for $n = 1, 2, 3, \ldots$, and given a triangular array of complex numbers $w_{j,0}, w_{j,1}, \ldots, w_{j,N_j}$ for $j = 1, 2, 3, \ldots$, then there exists a holomorphic function $f: R \to \mathbb{C}$ such that in the local coordinate systems given by the C_n, $f(z_n) = w_{n,0}$, $f'(z_n) = w_{n,1}, \ldots, f^{(N_n)}(z_n) = w_{n,N_n}$ which means that $(f \circ C_n) = w_{n,0}$,*

$$\frac{d}{dz}(f \circ C_n)(0) = w_{n,1}, \ldots, \frac{d^{N_n}}{dz^{N_n}}(f \circ C_n)(0) = w_{n,N_n} \quad \text{for} \quad n = 1, 2, 3, \ldots.$$

That is, free multiple interpolation by holomorphic functions on R is possible.

Proof. First of all, if free interpolation is possible in one sequence $\{C_n\}$ of coordinate maps, then it is possible in any other sequence $\{C_n^*\}$ of coordinate maps. This is because the matrix M connecting

$$\left[\frac{d^j}{dz^j}(f \circ C_n)(0)\right]_{j=0}^{N_j} \quad \text{with} \quad \left[\frac{d^j}{dz^j}(f \circ C_n^*)(0)\right]_{j=0}^{N_j}$$

is lower triangular, with diagonal entries $\lambda^0, \lambda^1, \ldots, \lambda^{N_j}$ where $\lambda = dC_j^*/dC_j(0) \neq 0$, so that M is invertible (its determinant is $\lambda^{N(N+1)/2} \neq 0$.)

Let us denote the sequence of linear functions $f \mapsto d^j/dz^j(f \circ C_n)(0)$ by (L_n) in some convenient ordering. In the language of Gauthier and Rubel [11], we have to prove that (L_n) is an interpolating sequence, and by Theorem 1 of that paper, this is equivalent to (L_n) being totally linearly independent. This means, first of all, that (L_n) is linearly independent, and second, if V_n is the linear span of $L_1, L_2, \ldots, L_n$, then if $T_n \in V_n$, $n = 1, 2, 3, \ldots$, and $T_n \to 0$ weak-* as $n \to \infty$, there exists an N such that $T_n \in V_N$ for $n = 1, 2, 3, \ldots$.

Now we recall the Cauchy kernel of Behnke-Stein (see Gauthier and Rubel [7] for a complete description). This is a function Φ meromorphic on $R \times R$ such that $\Phi(\zeta, z) - (\zeta - z)^{-1}$ is locally holomorphic, where the local variable is written as the pair (ζ, z). This means the following (see [Gauthier-Rubel [7] again). R is spread over the finite z-plane by a mapping ρ whose existence is guaranteed by a theorem of Gunning and Narasimhan [8]. Hence $\rho \times \rho$ spreads $R \times R$ over $\mathbb{C} \times \mathbb{C}$ and we write the local variable as (ζ, z). We choose the maps C_n to be the inverse of ρ, locally, after a

https://doi.org/10.1017/S0305004100073175 Published online by Cambridge University Press

translation. Look at $L_n(\Phi(\zeta, z))$, L_n operates in the z variable. This will have a pole at some z_j and different L_n will either have poles at different z_j, or, if they have poles at the same z_j, the orders of the poles will be different. Hence, if a finite linear combination of the L_n is zero then the coefficients must all be zero, and the L_n are consequently linearly independent.

Suppose now that $T_n \in V_n$ and $T_n \to 0$ weak-*. Via the final proof of Gauthier–Rubel [6], the $L_n(\Phi(\zeta, z))$, as functions of z, must all be holomorphic outside some compact set K in R. Therefore the poles of the T_n must all lie inside K. Let $z_1, z_2, \ldots, z_N$ be the totality of the z_j that lie in K. Then $T_n \in V_N$ for $n = 1, 2, 3, \ldots$. This establishes the total linear independence of (L_n) and completes the proof.

REFERENCES

[1] H. Behnke and K. Stein. *Entwicklungen analytischer Funktionen auf Riemannschen Flächen*, *Math. Ann* **120** (1949), 430–461.

[2] L. Bers and H. L. Royden. Holomorphic families of injections, *Acta. Math.* **157** (1986), 259–286.

[3] D. Bertilsson and V. Blondel. Transcendence in simultaneous stabilization, preprint.

[4] V. Blondel *et al.* Simultaneous stabilization of three or more plants: conditions on the positive real axis do not suffice, *SIAM J. of Control and Optimization*, to appear.

[5] O. Forster *Lectures on Riemann Surfaces* (Springer-Verlag, 1981).

[6] P. M. Gauthier and L. A. Rubel. Interpolation in separable Fréchet spaces with applications to spaces of analytic functions, *Canadian J. Math* **27** (1975), 1110–1113.

[7] P. M. Gauthier and L. A. Rubel. Holomorphic functionals on open Riemann Surfaces, *Canadian J. Math* **28** (1976), 885–888.

[8] R. C. Gunning and R. Narasimhan. Immersion of open Riemann Surfaces, *Math. Ann.* **174** (1967), 103–108.

[9] W. K. Hayman. *Meromorphic functions* (Oxford University Press, 1964).

[10] D. H. Luecking and L. A. Rubel. *Complex analysis: a functional analysis approach* (Springer Verlag, 1984).

[11] L. A. Rubel and C. C. Yang. Interpolation and unavoidable families of meromorphic functions, *Michigan Math J.* **20** (1973), 289–296.

[12] Z. Slodkowski. Holomorphic motions and polynomial hulls, *Proc. Amer. Math. Soc.* **111** (1991), 347–355.

https://doi.org/10.1017/S0305004100073175 Published online by Cambridge University Press