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ARTICLE

Optimal tree-decompositions with bags of bounded treewidth

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Abstract

We prove that several natural graph classes have tree-decompositions with minimum width such that each bag has bounded treewidth. For example, every planar graph has a tree-decomposition with minimum width such that each bag has treewidth at most 3. This treewidth bound is best possible. More generally, every graph of Euler genus g has a tree-decomposition with minimum width such that each bag has treewidth in $O(g)$. This treewidth bound is best possible. Most generally, every K_p-minor-free graph has a tree-decomposition with minimum width such that each bag has treewidth at most some polynomial function $f(p)$. In such results, the assumption of an excluded minor is justified, since we show that analogous results do not hold for the class of 1-planar graphs, which is one of the simplest non-minor-closed monotone classes. In fact, we show that 1-planar graphs do not have tree-decompositions with width within an additive constant of optimal and with bags of bounded treewidth. On the other hand, we show that 1-planar n-vertex graphs have tree-decompositions with width $O(\sqrt{n})$ (which is the asymptotically tight bound) and with bounded treewidth bags. Moreover, this result holds in the more general setting of bounded layered treewidth, where the union of a bounded number of bags has bounded treewidth.

Keywords: graph; minor; tree-decomposition; planar graph

2020 MSC Codes: Primary: 05C83; Secondary: 05C10

1. Introduction

Tree-decompositions were introduced by Robertson and Seymour [78], as a key ingredient in their Graph Minor Theory.[1] Indeed, the dichotomy between minor-closed classes with or without bounded treewidth is a central theme of their work. Tree-decompositions arise in several other results, such as the Erdős-Pósa theorem for planar minors [21, 79] and Reed's beautiful theorem on k-near bipartite graphs [76]. Tree-decompositions are also a key tool in algorithmic graph

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[1] We consider simple undirected graphs G with vertex set $V(G)$ and edge set $E(G)$. A graph H is a minor of a graph G if a graph isomorphic to H can be obtained from G by a sequence of edge deletions, vertex deletions, and edge contractions. A graph H is a topological minor of G if a subdivision of H is a subgraph of G. A graph class is a collection of graphs closed under isomorphism. A graph class $\mathcal{G}$ is minor-closed if for every graph $G \in \mathcal{G}$ every minor of G is in $\mathcal{G}$. A graph class $\mathcal{G}$ is monotone if for every graph $G \in \mathcal{G}$ every subgraph of G is in $\mathcal{G}$. A graph class $\mathcal{G}$ is proper if some graph is not in $\mathcal{G}$.

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theory, since many NP-complete problems are solvable in linear time on graphs with bounded treewidth [37].

For a non-null tree T, a T-decomposition of a graph G is a collection $(B_x : x \in V(T))$ of subsets of $V(G)$, henceforth called bags, such that:

- $B_x \subseteq V(G)$ for each $x \in V(T)$,
- for each edge $vw \in E(G)$, there exists a node $x \in V(T)$ with $v, w \in B_x$, and
- for each vertex $v \in V(G)$, the set $\{x \in V(T) : v \in B_x\}$ induces a non-empty (connected) subtree of T.

The width of such a T-decomposition is $\max\{|B_x| : x \in V(T)\} - 1$. A tree-decomposition is a T-decomposition for any tree T. The treewidth of a graph G, denoted $\mathrm{tw}(G)$, is the minimum width of a tree-decomposition of G. A tree-decomposition of a graph G with width $\mathrm{tw}(G)$ is said to be optimal. Treewidth is the standard measure of how similar a graph is to a tree. Indeed, a connected graph has treewidth at most 1 if and only if it is a tree. See [20, 56, 77] for surveys on treewidth.

In addition to studying treewidth and optimal tree-decompositions, much recent work has studied tree-decompositions where the subgraph induced by each bag is well structured in some sense. In this direction, Adler [11] introduced the following definition. For a graph parameter f and graph G, let tree-$f(G)$ be the minimum integer k such that G has a tree-decomposition $(B_x : x \in V(T))$ such that $f(G[B_x]) \leqslant k$ for each node $x \in V(T)$. Tree-chromatic number tree-χ was introduced by Seymour [81] and has since attracted substantial interest [18, 62, 63, 66]. Tree-diameter (under the name 'tree-length') was introduced by Dourisboure and Gavoille [48]; variants of this parameter have been widely studied in connection with coarse graph theory [19, 49, 50, 60, 75]. Tree-independence number tree-α was introduced by Yolov [86] and independently by Dallard, Milanič and Štorgel in the 'Treewidth versus clique-number' series [41–44]; it has since been widely studied [16, 23, 39, 40, 59, 66, 68, 72], including the 'Tree-independence number' series [5, 24, 26, 28, 29, 33], and the 'Induced subgraphs and tree decompositions' series [1–4, 6–10, 12–15, 25, 27, 30–32].

Liu, Norin, and Wood [69] studied tree-tw (and to a lesser extent tree-pw, tree-bw, and tree-td, where pw, bw, and td denote pathwidth, bandwidth, and treedepth, respectively). For example, Liu et al. [69] showed that for each integer p there is an integer c such that tree-$\mathrm{tw}(G) \leqslant c$ for every K_p-minor-free graph G; that is, G has a tree-decomposition such that each bag has treewidth[2] at most c.

In this paper we study optimal tree-decompositions, but with a focus on the structural properties of their bags. The primary contribution of this paper is to show that graphs in any proper minor-closed class admit optimal tree-decompositions whose bags have bounded treewidth.

First consider planar graphs G. Liu et al. [69] showed that G has a tree-decompostition with bags of treewidth 3 (that is, tree-$\mathrm{tw}(G) \leqslant 3$). The treewidth 3 bound here is best possible whenever G contains K_4, since in any tree-decomposition of G each clique of G is contained in a single bag, and $\mathrm{tw}(K_4) = 3$. The result of Liu et al. [69] gives no bound on the width of this tree-decomposition (and the method used does not relate to the optimal width). We prove the following analogous result with optimal width.

Theorem 1. *Every planar graph has an optimal tree-decomposition in which every bag has treewidth at most 3.*

Theorem 1 is proved in Section 4, where in fact we give a more precise description of the bags.

[2] For the sake of brevity, if S is a set of vertices in a graph G, we define the treewidth of S to be $\mathrm{tw}(G[S])$.

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Next consider graphs embeddable in a fixed surface.[3] Liu et al. [69] proved that tree-tw$(G) \in O(g)$ for graphs G with Euler genus g, and that this $O(g)$ bound is best possible. The next theorem strengthens this result and generalises Theorem 1 for graphs embeddable on any fixed surface.

Theorem 2. *Every graph with Euler genus g has an optimal tree-decomposition in which every bag has treewidth at most* $\max\{4g + 2, 3\}$.

Theorem 2 is proved in Section 5, where we give further properties of the tree-decomposition in Theorem 2.

More generally, our next contribution says that graphs in any proper minor-closed class have optimal tree-decompositions with bags of bounded treewidth.

Theorem 3. *For any integer $p \geqslant 1$ there exists an integer c such that every K_p-minor-free graph has an optimal tree-decomposition in which every bag has treewidth at most c.*

As mentioned above, Liu et al. [69] proved the weakening of Theorem 3 with no bound on the size of the bags.

The key idea underlying our first two main results (Theorems 1 and 2) is surprisingly simple. Consider a tree-decomposition $\mathcal{D} = (B_x : x \in V(T))$ and a separation (A_1, A_2) of a graph G, such that some bag B_x contains $A_1 \cap A_2$, and B_x intersects both $A_1 \setminus A_2$ and $A_2 \setminus A_1$. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be the induced tree-decompositions of $G[A_1]$ and $G[A_2]$, respectively. We can combine $\mathcal{D}_1$ and $\mathcal{D}_2$ to obtain a new tree-decomposition of G by adding an edge between the nodes corresponding to x in $\mathcal{D}_1$ and $\mathcal{D}_2$. By repeatedly applying this operation, we can obtain a tree-decomposition of G such that there is no separation (A_1, A_2) of G where a bag B_x contains $A_1 \cap A_2$, and B_x intersects both $A_1 \setminus A_2$ and $A_2 \setminus A_1$; we call such a bag 'unbreakable'. For graphs embedded on a surface, every cycle that bounds a disc separates the vertices drawn inside the disc from those drawn outside the disc. Thus, if an unbreakabale bag B contains a cycle bounding a disc D, then B must live entirely inside or entirely outside D. This allows us to bound the treewidth of B, for example by applying the Grid Minor Theorem [79] and observing that every large grid G embedded in the plane has a cycle C such that vertices of G are embedded both inside and outside the disc bounded by C. This approach also leads to a proof that every graph with maximum degree at most 3 has an optimal tree-decomposition such that every bag has treewidth at most 3 (Corollary 11).

For K_p-minor-free graphs, the situation is complicated by the existence of vertices whose neighbourhood have large treewidth. Indeed, if G is obtained from a planar graph by adding a vertex v adjacent to all other vertices, then G is K_6-minor-free and every set not containing v is unbreakable. We therefore need a more sophisticated method of modifying tree-decompositions in this setting. To this end we introduce the notion of 'irreducible' sets. In Section 3, we explain how to obtain optimal tree-decompositions with irreducible bags, and in Section 6 we bound the treewidth of subgraphs of K_p-minor-free graphs induced on irreducible sets, thus proving Theorem 3.

1.1 Monotone graph classes

Inspired by the above results for minor-closed classes, it is natural to ask for a given monotone graph class $\mathcal{G}$, does every graph in $\mathcal{G}$ have an optimal tree-decomposition such that every bag has bounded treewidth? We show that the answer is 'no' for a surprisingly simple graph class. A graph is 1-planar if it has a drawing in the plane with at most one crossing on each edge; see [65] for a survey on 1-planar graphs. Since every graph has a 1-planar subdivision, 1-planar graphs contain arbitrarily large complete graph minors [51, 58, 61]. On the other hand, we consider 1-planar graphs to be one of the simplest monotone classes that contains subdivisions of

[3] The Euler genus of an orientable surface with h handles is $2h$. The Euler genus of a non-orientable surface with c cross-caps is c. The Euler genus of a graph G is the minimum Euler genus of a surface in which G embeds (with no crossings).

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Table 1. Graphs that have tree-decompositions with bounded treewidth bags and with the given width bound

Graph class	Optimal Width $\mathrm{tw}(G)$	Near-optimal Width $\mathrm{tw}(G)+c$	Asymptotically tight Width $O(\sqrt{n})$
Proper minor-closed	Yes	Yes	Yes
1-planar	No	No	Yes
Bounded layered treewidth	No	No	Yes

any graph. We show that 1-planar graphs do not have optimal tree-decompositions with bags of bounded treewidth. In fact, the same result holds for tree-decompositions of width within an additive constant of optimal.

Theorem 4. *For any integers $c, w \geqslant 0$ there is a 1-planar graph G such that every tree-decomposition of G with width at most $\mathrm{tw}(G)+c$ has a bag with treewidth greater than w.*

This result is proved in Section 7, where we in fact prove a significant strengthening.

In contrast to Theorem 4, we show that 1-planar graphs with n vertices have tree-decompositions with width $O(\sqrt{n})$ such that each bag has bounded treewidth (Corollary 35). Here the $O(\sqrt{n})$ width bound is best possible, since the $\sqrt{n}\times\sqrt{n}$ grid graph is planar with treewidth $\sqrt{n}$. This shows a marked contrast between the settings of optimal width and $O(\sqrt{n})$ width. Moreover, the upper bound holds in the more general setting of graphs of bounded layered treewidth (Lemma 31), where we in fact show that the union of any bounded number of bags has bounded treewidth. This material is presented in Section 8.

We conclude the paper in Section 9 by discussing several open problems that arise from our work. Table 1 summarises all our results.

2. Normal, atomic, and refined tree-decompositions

This section introduces some elementary results about tree-decompositions that underpin the main proofs.

Let $(B_x : x \in V(T))$ be a tree-decomposition of a graph G. If $B_x \subseteq B_y$ for some edge $xy \in E(T)$, then let T' be the tree obtained from T by contracting xy into a new vertex z, and let $B_z := B_y$. Then $(B_x : x \in V(T'))$ is a tree-decomposition of G with width equal to the width of $(B_x : x \in V(T))$, and $|V(T')| < |V(T)|$. To normalise a tree-decomposition means to apply this operation until $B_x \not\subseteq B_y$ for each edge $xy \in E(T)$. This process terminates, since each contraction decreases $|V(T)|$ by exactly 1. A tree-decomposition $(B_x : x \in V(T))$ with $B_x \not\subseteq B_y$ for each $xy \in E(T)$ is said to be normal. Applying normalisation to an optimal tree-decomposition of a graph G gives a tree-decomposition of G that is optimal and normal.

Say $(B_x : x \in V(T))$ is a normal tree-decomposition of a graph G with $V(G) \neq \varnothing$. Root T at an arbitrary node $r \in V(T)$. For each edge $xy \in E(T)$ where y is the parent of x, let $f(xy)$ be any vertex in $B_x \setminus B_y$ (one exists since $B_x \not\subseteq B_y$). Thus f is an injection from $E(T)$ to $V(G - B_r)$. So $|V(G)| - 1 \geqslant |V(G)| - |B_r| \geqslant |E(T)| = |V(T)| - 1$, and $|V(T)| \leqslant |V(G)|$. That is, every normal tree-decomposition of a graph G has at most $|V(G)|$ bags, which is a well-known fact.

For a tree-decomposition $\mathcal{D} = (B_x : x \in V(T))$ of a graph G and a positive integer i, let $n_i(\mathcal{D})$ be the number of nodes $x \in V(T)$ such that $|B_x| = i$, and let

$$N(\mathcal{D}) := (n_{|V(G)|}(\mathcal{D}), n_{|V(G)|-1}(\mathcal{D}), \ldots, n_1(\mathcal{D})).$$

Sometimes $N(\mathcal{D})$ is called the fatness of $\mathcal{D}$. A tree-decomposition $\mathcal{D}$ is atomic if there is no tree-decomposition $\mathcal{D}'$ of G such that $N(\mathcal{D}')$ is lexicographically less than $N(\mathcal{D})$. This concept was

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implicitly introduced by Thomas [83] and has since been studied by various authors [3, 27, 36, 46, 47, 53, 74, 84].

We implicitly use the following simple lemma throughout the paper.

Lemma 5. *Every graph has an atomic tree-decomposition, and every atomic tree-decomposition is normal.*

Proof. Let G be a graph. Since every normal tree-decomposition of G has at most $|V(G)|$ bags, there are finitely many normal tree-decompositions of G. So there exists a normal tree-decomposition $\mathcal{D}$ of G that lexicographically minimises $N(\mathcal{D})$ among all normal tree-decompositions of G. Suppose for contradiction that there is a tree-decomposition $\mathcal{D}'$ of G such that $N(\mathcal{D}')$ is lexicographically less than $N(\mathcal{D})$. Let $\mathcal{D}''$ be obtained by normalising $\mathcal{D}'$. Then $N(\mathcal{D}'')$ is lexicographically less than $N(\mathcal{D})$, contradicting our choice of $\mathcal{D}$. This shows that $\mathcal{D}$ is an atomic tree-decomposition of G.

Now suppose that G has an atomic tree-decomposition $\mathcal{D}$ that is not normal, and let $\mathcal{D}'$ be the normalisation of $\mathcal{D}$. Since normalisation only removes bags from $\mathcal{D}$, $N(\mathcal{D}')$ is lexicographically less than $N(\mathcal{D})$. □

Many of the known properties of atomic tree-decompositions actually hold in the broader setting of refined tree-decompositions, which we now introduce. A tree-decomposition $\mathcal{D}'$ of a graph G is a refinement of a tree-decomposition $\mathcal{D}$ of G if each bag of $\mathcal{D}'$ is a subset of a bag of $\mathcal{D}$. If $\mathcal{D}'$ is a refinement of $\mathcal{D}$ and not vice versa, then $\mathcal{D}'$ is a proper refinement of $\mathcal{D}$. A tree-decomposition is refined if it is normal and has no proper refinement.

Lemma 6. *Every atomic tree-decomposition is refined.*

Proof. By Lemma 5, every atomic tree-decomposition is normal. Suppose for contradiction that $\mathcal{D}'$ is a proper refinement of an atomic tree-decomposition $\mathcal{D}$ of a graph G. Let $\mathcal{D}''$ be a tree-decomposition of G obtained by normalising $\mathcal{D}'$, and note that $\mathcal{D}''$ is also a proper refinement of $\mathcal{D}$. We will show that $N(\mathcal{D}'')$ is lexicographically less than $N(\mathcal{D})$, thus contradicting that $\mathcal{D}$ is atomic. To this end, let B_x be the largest bag of $\mathcal{D}$ that is not a subset of any bag of $\mathcal{D}''$. Now, for any bag B_u of $\mathcal{D}$ larger than B_x, there is some bag B''_v of $\mathcal{D}''$ and some bag B_w of $\mathcal{D}$ such that $B_u \subseteq B''_v \subseteq B_w$. The second part of Lemma 5 implies $u = w$, and thus $B_u = B''_v = B_w$.

Suppose for contradiction that there is a bag B''_u of $\mathcal{D}''$ of size at least $|B_x|$ that is not a bag of $\mathcal{D}$. Since $\mathcal{D}''$ is a refinement of $\mathcal{D}$, there is a bag B_v of $\mathcal{D}$ containing B''_u as a proper subset. By the above argument, B_v is also a bag of $\mathcal{D}''$, contradicting that $\mathcal{D}''$ is normal. Hence, every bag of $\mathcal{D}''$ of size at least $|B_x|$ is also a bag of $\mathcal{D}$. Since every bag of $\mathcal{D}''$ of size at least $|B_x|$ is also a bag of $\mathcal{D}$ but not vice versa and both are normal tree-decompositions, $N(\mathcal{D}'')$ is lexicographically less than $N(\mathcal{D})$, a contradiction. □

To the best of our knowledge, refined tree-decompositions have not been explicitly defined in the literature, although several known results about atomic tree-decompositions are in fact proved only using the properties of refined tree-decompositions.

Lemmas 5 and 6 imply that every graph has a refined tree-decomposition, which we henceforth use implicitly.

Lemma 7. *For every refined tree-decomposition $\mathcal{D} = (B_x : x \in V(T))$ of a graph G, for all nodes $x, y \in V(T)$ (not necessarily distinct), for any subset $S \subseteq B_y$, the bag B_x intersects exactly one component of $G - S$, unless $x = y$ and $S = B_y$.*

Proof. Since $\mathcal{D}$ is normal, B_x is not a subset of S unless $y = x$ and $S = B_y$. Suppose for contradiction that B_x intersects multiple components of $G - S$, and let H_1 be one such component.

Let T_1 and T_2 be disjoint copies of T, where u_i is the copy of u in T_i, for each $u \in V(T)$ and $i \in \{1, 2\}$. Let T' be obtained from $T_1 \cup T_2$ by adding the edge y_1y_2. Let $D_1 := V(H_1) \cup S$ and $D_2 := V(G) \setminus V(H_1)$. For each node $u_i \in V(T_i)$, let $B_{u_i} := B_u \cap V(D_i)$. So $\mathcal{D}' := (B_u : u \in V(T'))$

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is a tree-decomposition of G and is a proper refinement of $\mathcal{D}$ since B_x is not a subset of any bag of $\mathcal{D}'$. This contradiction completes the proof. □

The statement and proof of Lemma 7 is similar to a result of [74, Lemma 3.8] in the setting of atomic tree-decompositions.

3. Breakability and reducibility

This section introduces the notions of breakable and reducible sets in a graph, which are key tools in the proofs of our main results.

A separation of a graph G is an ordered pair (A, B) such that $A, B \subseteq V(G)$ and $G = G[A] \cup G[B]$. The order of (A, B) is $|A \cap B|$. A separation (A, B) of a graph G breaks a set $S \subseteq V(G)$ if $S \setminus A$ and $S \setminus B$ are both nonempty, and $A \cap B \subseteq S$. A set $S \subseteq V(G)$ is breakable if there exists a separation (A, B) that breaks S, otherwise S is unbreakable.

Lemma 8. *Every bag of every refined tree-decomposition $(B_x : x \in V(T))$ of a graph G is unbreakable.*

Proof. Consider a separation (A, B) of G and a node $y \in V(T)$ such that $A \cap B \subseteq B_y$ and $B_y \setminus A \neq \varnothing$. By Lemma 7, there is a unique component C of $G - (A \cap B)$ that intersects B_y. Since $V(C) \cap B \supseteq B_y \setminus A \neq \varnothing$, we have $V(C) \cap A = \varnothing$, so $B_y \setminus B = \varnothing$. Thus every bag of $(B_x : x \in V(T))$ is unbreakable. □

Together with Lemma 6, this yields the following immediate corollary.

Corollary 9. *Every bag of every atomic tree-decomposition is unbreakable.*

As an aside, breakability leads to the following results about graphs of bounded maximum degree.

Proposition 10. *Given a graph G with maximum degree Δ, every bag of every refined tree-decomposition of G either has size at most $\Delta + 1$ or induces a subgraph of maximum degree at most $\Delta - 1$.*

Proof. Let $\mathcal{D}$ be a refined tree-decomposition of G with width $\mathrm{tw}(G)$. Consider a bag B of $\mathcal{D}$. Suppose for the sake of contradiction that $|B| \geqslant \Delta + 2$ and there is a vertex $v \in B$ with degree Δ in $G[B]$. Thus, $N_G[v] \subseteq B$ and some vertex in B is not adjacent to v. Hence $(N_G[v], V(G) \setminus \{v\})$ is a separation of G that breaks B, which contradicts Lemma 8. □

Since every graph with at most four vertices or with maximum degree at most 2 has treewidth at most 3, Proposition 10 implies:

Corollary 11. *Every graph G with maximum degree at most 3 has an optimal tree-decomposition $\mathcal{D}$ such that every bag of $\mathcal{D}$ has treewidth at most 3.*

A set S of vertices in a graph G is reducible if there is a separation (A, B) of G such that both $|(S \cup B) \cap A|$ and $|(S \cup A) \cap B|$ are strictly less than $|S|$, otherwise S is irreducible.

Note that if S is breakable, then it is reducible (since $(A \cap B) \setminus S = \varnothing$). That is, if S is irreducible, then S is unbreakable. Thus, the following lemma strengthens Corollary 9.

Lemma 12. *Every bag of an atomic tree-decomposition $(B_x : x \in V(T))$ of a graph G is irreducible.*

Proof. Suppose for contradiction that for some $y \in V(T)$, there is a separation (A_1, A_2) of G such that both $|(B_y \cup A_1) \cap A_2|$ and $|(B_y \cup A_2) \cap A_1|$ are strictly less than $|B_y|$. Select such a separation of minimum order. In particular, there is no set S' of size smaller than $|A_1 \cap A_2|$ that separates $A_1 \cap B_y$ from $A_1 \cap A_2$, or a smaller order separation can be obtained by taking S' and the vertices in components of $G - S'$ intersecting $A_1 \cap B_y$ on one side and taking S' and all remaining

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vertices on the other. Thus, by Menger's Theorem, there is a set $\mathcal{P}_1$ of $|A_1 \cap A_2|$ disjoint paths from $A_1 \cap A_2$ to $B_y \cap A_1$ in G. By construction each of these paths is internally disjoint from $A_1 \cap A_2$ and hence is contained in $G[A_1]$. By symmetry, there is a set $\mathcal{P}_2$ of $|A_1 \cap A_2|$ disjoint paths from $A_1 \cap A_2$ to $B_y \cap A_2$ in $G[A_2]$.

For each $a \in A_1 \cap A_2$, let w_a be the closest vertex to y in T such that $a \in B_{w_a}$ (meaning that $w_a = y$ if $a \in B_y$). Now, for each $w \in V(T)$, define

$$B'_w := B_w \cup \{a \in A_1 \cap A_2 : w \in V(yTw_a)\}.$$

Here yTw_a is the yw_a-path in T. Note that for every vertex $u \in V(G)$, the set $\{w \in V(T) : u \in B'_w\}$ is a superset of $\{w \in V(T) : u \in B_w\}$ and induces a subtree of T, so $(B'_w : w \in V(T))$ is a tree-decomposition of G.

Let T'' be the tree with vertex set $V(T) \times \{1, 2\}$ such that vertices (u, i) and (w, j) are adjacent if $i = j$ and $uw \in E(T)$, or $i \neq j$ and $u = w = y$. For each $(u, i) \in V(T'')$, define $B''_{(u,i)} := B'_u \cap V(A_i)$. For each $i \in \{1, 2\}$, let $T_i := T[V(T) \times \{i\}]$ and note that $(B''_x : x \in V(T_i))$ is a tree-decomposition of $G[A_i]$. Since $A_1 \cap A_2 \subseteq B''_{(y,1)} \cap B''_{(y,2)}$, it follows that $(B''_x : x \in V(T''))$ is a tree-decomposition of G.

Claim. *For each $(v, i) \in V(T'')$, we have $|B''_{(v,i)}| \leqslant |B_v|$. Furthermore, if equality holds, then $|B''_{(v,3-i)}| < |B_y|$.* □

Proof. Let $X_v := B'_v \setminus B_v$, and let $j := 3 - i$. By construction, $X_v \subseteq (A_1 \cap A_2) \setminus B_y$. For each $a \in X_v$, consider the path $P_{a,j}$ in $\mathcal{P}_j$ that has a as an endpoint. Since both B_{w_a} and B_y contain a vertex of $P_{a,j}$ and $v \in V(yTw_a)$, it follows that B_v contains a vertex of $P_{a,j}$. In particular, since $a \notin B_v$, we see that B_v contains a vertex in $V(P_{a,j}) \setminus (A_1 \cap A_2)$. Thus $B_v \setminus A_i$ contains at least $|X_v|$ vertices, and so $|B''_{(v,i)}| = |B_v \cap A_i| + |X_v| \leqslant |B_v|$. If equality holds, then $|B_v \setminus A_i| = |X_v| \leqslant |(A_1 \cap A_2) \setminus B_y|$. This means

$$|B''_{(v,j)}| \leqslant |B_v \setminus A_i| + |A_1 \cap A_2| \leqslant |(A_1 \cap A_2) \setminus B_y| + |A_1 \cap A_2|.$$

Recall that $|(B_y \cup A_1) \cap A_2| < |B_y|$, meaning $|(A_1 \cap A_2) \setminus B_y| < |B_y \setminus A_2|$. Thus

$$|B''_{(v,j)}| < |B_y \setminus A_2| + |A_1 \cap A_2| = |(B_y \cup A_2) \cap A_1| < |B_y|,$$

as required. □

Note that $|B''_{(y,1)}| = |(A_1 \cap A_2) \cup (B_y \setminus A_2)| < |B_y|$, and similarly $|B''_{(y,2)}| < |B_y|$. Let k be the maximum integer such that for some $v \in V(T)$ with $|B_v| = k$ we have $|B''_{v,1}| < k$ and $|B''_{(v,2)}| < k$ (so $k \geqslant |B_y|$).

By the above claim, the number of bags of $\mathcal{D}''$ of size k' equals the number of bags of $\mathcal{D}$ of size k' for all $k' > k$, and $\mathcal{D}''$ has strictly fewer bags of size k than $\mathcal{D}$. This contradicts the fact that $\mathcal{D}$ is atomic. □

Lemma 12 highlights the distinction between atomic and refined tree-decompositions, since its proof critically uses the atomic property.

4. Planar graphs

This section proves Theorem 1 showing that every planar graph has an optimal tree-decomposition with bags of treewidth 3.

We employ the following definitions introduced by Dehkordi and Farr [45]. A planar graph G is separable with respect to a plane embedding Π of G if there is a cycle C in G such that there is a vertex of $G - V(C)$ in the interior of C with respect to Π, and there is a vertex of $G - V(C)$ in the exterior of C with respect to Π. Otherwise, G is non-separable with respect to Π. That is, for any

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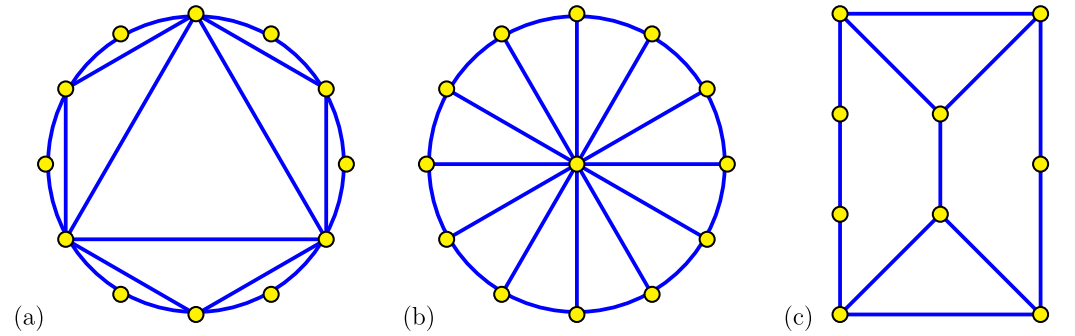


Figure 1. Non-separable planar graphs: (a) outerplanar, (b) wheel, (c) elongated triangular prism.

cycle C in G, all the vertices of $G - V(C)$ are in the interior of C, or all the vertices of $G - V(C)$ are in the exterior of C.

Let Π be a plane embedding of a planar graph G. Suppose that $S \subseteq V(G)$ and $G[S]$ is separable with respect to the plane embedding of $G[S]$ induced by Π. So there is a cycle C in $G[S]$ with at least one vertex of S in the interior of C, and at least one vertex of S in the exterior of C. Let A be the set of all vertices of G in C or in the interior of C. Let B be the set of all vertices of G in C or in the exterior of C. By the Jordan Curve Theorem, (A, B) is a separation of G that breaks S. So if a set $S \subseteq V(G)$ is unbreakable, then S is non-separable with respect to Π. Lemma 8 thus implies:

Lemma 13. *Let Π be a plane embedding of a planar graph G. For every refined tree-decomposition $(B_y : y \in V(T))$ of G, for each $y \in V(T)$, $G[B_y]$ is non-separable with respect to the embedding of $G[B_y]$ induced by Π.*

A planar graph G is non-separable if G is non-separable with respect to some plane embedding of G. As illustrated in Figure 1, Dehkordi and Farr [45] showed that the class of non-separable planar graphs is minor-closed, and that a graph G is non-separable planar if and only if G does not contain $K_1 \cup K_4$ or $K_1 \cup K_{2,3}$ or $K_{1,1,3}$ as a minor. In fact, Dehkordi and Farrb [45] provided the following precise structural characterisation: any non-separable planar graph is either outerplanar, or a subgraph of a wheel, or a subgraph of an elongated triangular prism (a graph obtained from the triangular prism $K_3 \square K_2$ by subdividing edges that are not in triangles any number of times).

Lemma 14. *For every refined tree-decomposition $(B_y : y \in V(T))$ of a planar graph G, for each $y \in V(T)$, $G[B_y]$ is outerplanar, is a subgraph of a wheel, or is a subgraph of an elongated triangular prism.*

Each graph listed in Lemma 14 has treewidth at most 3. So Lemma 14 implies the following result, which implies Theorem 1.

Corollary 15. *Every planar graph G has an optimal tree-decomposition $(B_x : x \in V(T))$ such that for each $x \in V(T)$, the graph $G[B_x]$ is outerplanar, a subgraph of a wheel, or a subgraph of an elongated triangular prism. In particular, $G[B_x]$ has treewidth at most* 3.

A forthcoming companion paper [57] gives a more precise structural characterisation than Corollary 15, where we drop the requirement of outerplanarity.

5. Graphs on surfaces

This section proves Theorem 2 showing that every graph G with Euler genus g has an optimal tree-decomposition with bags of treewdith $O(g)$. For a graph G embedded in a surface Σ, for

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distinct vertices $a, b \in V(G)$, two ab-paths P_1 and P_2 in G are homotopic if $P_1 \cup P_2$ bounds a disc in Σ. We use the following lemma of Malnič and Mohar [71] (see Proposition 4.2.7 in [73] and the discussion that follows).

Lemma 16 [71, 73]. *For any graph G embedded in a surface with Euler genus $g \geqslant 1$, for any distinct vertices $a, b \in V(G)$, if $\mathcal{P}$ is a set of pairwise internally disjoint ab-paths in G and $|\mathcal{P}| \geqslant 2g+1$, then $\mathcal{P}$ contains a pair of homotopic paths.*

Lemma 17. *For any integer $c \geqslant 2$, for any graph G embedded in a surface with Euler genus $g \geqslant 1$, for any distinct vertices $a, b \in V(G)$, if $\mathcal{P}$ is a set of pairwise internally disjoint ab-paths in G and $|\mathcal{P}| \geqslant (c-1)(2g+1)$, then $\mathcal{P}$ contains a set of c pairwise homotopic paths.*

Proof. Homotopy between paths in $\mathcal{P}$ is an equivalence relation. If some equivalence class has c paths then we are done. Assume each equivalence class at most $c-1$ paths. The number of equivalence classes is at least $2g+1$. Taking one path from each equivalence class gives $2g+1$ pairwise non-homotopic ab-paths, contradicting Lemma 16. □

Lemma 18. *Every embedding of $K_{2,4g+2}$ in a surface of Euler genus $g \geqslant 1$ has a cycle bounding a disc and separating two vertices.*

Proof. Let $n := 4g+2$. Let a, b be the two vertices of degree n in $K_{2,n}$. There are n internally disjoint ab-paths $P_1, \dots, P_n$ in $K_{2,n}$. By Lemma 17, three of $P_1, \dots, P_n$ are pairwise homotopic. Without loss of generality, P_1, P_2, P_3 are pairwise homotopic, and there is a disc D in Σ bounded by $P_1 \cup P_2$ with the internal vertex of P_3 in the interior of D. Choose D to be minimal with this property. Then no other vertex is in the interior of D. Since $n \geqslant 4$, $P_1 \cup P_2$ is separating. □

Lemma 19. *For every graph G of Euler genus $g \geqslant 1$, for any $S \subseteq V(G)$, if $K_{2,4g+2}$ is a minor of $G[S]$, then S is breakable.*

Proof. Fix an embedding of G in a surface Σ of Euler genus g. Let H be obtained from $G[S]$ by contracting each branch set of the $K_{2,4g+2}$ minor. We obtain an embedding of H in Σ. By Lemma 18, there is a cycle C_0 in H bounding a disc and separating two vertices in H. The branch sets corresponding to C_0 thus contain a cycle C in $G[S]$ bounding a disc D and separating two vertices in S. Let A be the set of all vertices of G embedded in D (including C). Let B be the set of all vertices of G embedded in $\Sigma - D$ plus $V(C)$. So (A, B) is a separation of G that breaks S. □

We use the following lemma by Malnič and Mohar [70] (improving on a previous bound by Leaf and Seymour [67]). A graph H is an apex-forest if $H - v$ is a forest for some $v \in V(H)$.

Lemma 20 [70]. *For any apex-forest H with $|V(H)| \geqslant 2$, every H-minor-free graph has treewidth at most $|V(H)| - 2$.*

Lemma 21. *For any integer $g \geqslant 1$, every graph G with Euler genus at most g, every refined tree-decomposition $(B_x : x \in V(T))$ of G and every $x \in V(T)$:*

- *$G[B_x]$ has no $K_{2,4g+2}$ minor, and*
- *$\mathrm{tw}(G[B_x]) \leqslant 4g+2$.*

Proof. Lemmas 8 and 19 imply that $G[B_x]$ is $K_{2,4g+2}$-minor-free for each $x \in V(T)$. Since $K_{2,4g+2}$ is an apex-forest, by Lemma 20, every $K_{2,4g+2}$-minor-free graph has treewidth at most $4g+2$. The result follows. □

Lemma 21 implies Theorem 2 (using Theorem 1 in the $g = 0$ case). Moreover, the $O(g)$ bound on $\mathrm{tw}(G[B_x])$ in Theorem 2 is best possible, as we now explain. Liu et al. [69, Theorem 4.37] showed that for sufficiently large n, there is a 256-regular n-vertex graph G with tree-tw$(G) \geqslant \frac{n}{4}$. Let g be the Euler genus of G. So $g \leqslant 2|E(G)| \leqslant 256n$. Hence tree-tw$(G) \geqslant \frac{n}{4} \geqslant \frac{g}{1024}$. In short, there are graphs G with Euler genus g and tree-tw$(G) \in \Omega(g)$. That is, in any tree-decomposition

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of G (regardless of the size of the bags), some bag induces a subgraph with treewidth $\Omega(g)$. Nevertheless, grid minors in bags can be limited using the following lemma by Geelen, Richter, and Salazar [54].

Lemma 22 [54, Lemma 4]. *Let t, k, n be positive integers such that $n \geqslant t(k+1)$. Let G be an $n \times n$ grid graph. If G is embedded in a surface Σ of Euler genus at most $t^2 - 1$, then some $k \times k$ subgrid of G is embedded in a closed disc D in Σ such that the boundary cycle of the $k \times k$ grid is the boundary of D.*

Lemma 23. *For any graph G of Euler genus at most g, if $S \subseteq V(G)$ and the $4\lceil\sqrt{g+1}\,\rceil \times 4\lceil\sqrt{g+1}\,\rceil$ grid is a minor of $G[S]$, then S is breakable.*

Proof. Let $t := \lceil\sqrt{g+1}\,\rceil$. So $g \leqslant t^2 - 1$, and G embeds in a surface Σ of Euler genus at most $t^2 - 1$. Let $k := 3$ and $n := 4t$. By assumption, the $n \times n$ grid is a minor of $G[S]$. Let G' be the $n \times n$ grid, embedded in Σ, obtained from $G[S]$ by appropriate contractions and deletions. By Lemma 22, some 3×3 subgrid G'' of G' is embedded in a closed disc D_0 in Σ such that the boundary cycle C_0 of the subgrid is the boundary of D_0. Note that $G' - V(G'')$ is connected, and that at least 3 or the four corner vertices of G'' have neighbours in $G' - V(G'')$. Thus $G' - V(G'')$ is embedded in $\Sigma - D_0$. The branch sets corresponding to C_0 contain a cycle C in $G[S]$ bounding a disc D, where there is at least one vertex of S in the interior of D (corresponding to the internal vertex of the 3×3 grid), and there is at least one vertex of S in $\Sigma - D$. Let A be the set of all vertices of G embedded in D (including the boundary). Let B be the set of all vertices of G embedded in $\Sigma - D$ plus the vertices on the boundary of D. So (A, B) is a separation of G that breaks S. □

Lemma 23 implies that in addition to the properties in Lemma 21, in every refined tree-decomposition of a graph with Euler genus g, the subgraph induced by each bag has no $4\lceil\sqrt{g+1}\,\rceil \times 4\lceil\sqrt{g+1}\,\rceil$ grid minor. In particular:

Theorem 24. *For any integer $g \geqslant 1$, every graph G with Euler genus at most g has an optimal tree-decomposition $(B_x : x \in V(T))$, such that for each $x \in V(T)$:*

- *$G[B_x]$ has no $K_{2,4g+2}$ minor,*
- *$\mathrm{tw}(G[B_x]) \leqslant 4g + 2$,*
- *$G[B_x]$ has no $4\lceil\sqrt{g+1}\,\rceil \times 4\lceil\sqrt{g+1}\,\rceil$ grid minor.*

6. K_p-minor-free graphs

This section proves Theorem 3, which shows that for fixed p, every K_p-minor-free graph has an optimal tree-decomposition with bags of bounded treewidth.

We begin by defining walls. For positive integers m and n, the $m \times n$-grid is the graph with vertex set $\{1, \dots, m\} \times \{1, \dots, n\}$ such that vertices (i, j) and (i', j') are adjacent if and only if $|i - i'| + |j - j'| = 1$. The elementary h-wall W_h is the graph obtained from the $2h \times h$-grid by deleting every edge of the form $(i, j)(i+1, j)$ with $i + j$ even, and then deleting the two vertices of degree 1 in the resulting graph. For each $i \in [h]$, the path of W_h induced by the vertices of the form $(2i - 1, j)$ together with the vertices of the form $(2i, j)$ is a column of W_h, and the path induced by the vertices of the form (j, i) is a row of W_h. An h-wall is a graph that is isomorphic to a subdivision of the elementary h-wall. We always implicitly fix such an isomorphism. We use the following variant of the Grid Minor Theorem of Robertson and Seymour [79].

Theorem 25 [79]. *There is a function f such that for any integer $k \geqslant 1$, every graph of treewidth at least $f(k)$ contains a k-wall as a subgraph.*

We use the following precursor to the Flat Wall Theorem of Robertson and Seymour [80], which depends on the following definitions. The rows and columns of an h-wall are the paths

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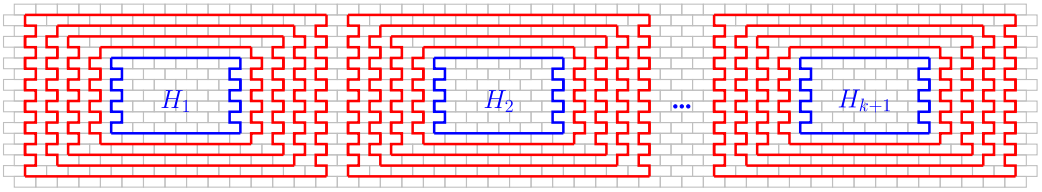


Figure 2. Subwalls $H_1, \ldots, H_{k+1}$, and cycle collections $\mathcal{C}_1, \ldots, \mathcal{C}_{k+1}$.

corresponding to the rows and columns of W_h, respectively. Any graph that is a k-wall for some k is a wall, and we say its height is k. A subwall of W_h is a subgraph $H \subseteq W_h$ such that H is a wall, each column of H is contained in a column of W_h and each row of H is contained in a row of W_h. The subwalls of a wall W of height h are the corresponding subgraphs of W. The perimeter of W_h is the unique shortest cycle containing all degree 2 vertices, and the perimeter of an h-wall W is the corresponding cycle of W. Given a wall W in a graph G, a subwall H of W is dividing if every path in G from H to the perimeter of W contains a vertex in the perimeter of H. For a wall W, define a distance function $\hat{d}_W$, where $\hat{d}_W(v, v) = 0$ for all $v \in V(W)$, and for distinct $v, w \in V(W)$, $\hat{d}_W(v, w)$ is the minimum integer k such that for some planar embedding of W there is curve in the plane from v to w that intersects only k points of the drawing (including v and w).

Theorem 26 [80, (9.3)]. *For any integer $p \geqslant 1$ there exists $k, r \geqslant 0$ such that given a wall W in a K_p-minor-free graph G and subwalls $H_1, H_2, \ldots, H_t$ with $\hat{d}_W(H_i, H_j) \geqslant r$ for all distinct $i, j \in \{1, \ldots, t\}$, there is a subset $X \subseteq V(G)$ of size at most $\binom{p}{2}$ and a subset $I \subseteq \{1, \ldots, t\}$ of size at least $t - k$ such that for all $i \in I$, H_i is dividing in $(G - X) \cup W$.*

Lemma 27. *For any integer $p \geqslant 1$ there exists an integer c_p such that for every K_p-minor-free graph G, every irreducible set $S \subseteq V(G)$ satisfies* $\mathrm{tw}(G[S]) \leqslant c_p$.

Proof. Let $r, k \geqslant 0$ be integers as in Theorem 26, let $p^* = \max\{\binom{p}{2}, 2r(k+1) + (k+1)(p+2)\}$, and let $c_p = f(p^*)$, where f is from Theorem 25. Assume that $\mathrm{tw}(G[S]) \geqslant c_p$. Our goal is to show that S is reducible. By Theorem 25, $G[S]$ contains a p^*-wall W as a subgraph. As illustrated in Figure 2, in W there are:

- $k+1$ pairwise disjoint subwalls $H_1, \ldots, H_{k+1}$, each of height $p+2$ and disjoint from the perimeter of W, and
- for each $i \in \{1, \ldots, k+1\}$ there is a set $\mathcal{C}_i$ of r pairwise disjoint cycles in W, each disjoint from $H_1 \cup \ldots \cup H_{k+1}$, such that each $C \in \mathcal{C}_i$ separates H_i and $\bigcup\{H_j : j \neq i\}$.

Since W is a subdivision of a 3-connected planar graph, by a theorem of Whitney [85], W has a unique embedding in the plane (up to the choice of the outerface). In this embedding, any curve in the plane from H_i to H_j (for $i \neq j$) must intersect each cycle $C \in \mathcal{C}_i$. Thus $\hat{d}_W(H_i, H_j) \geqslant r$, and so by Theorem 26 there is some $X \subseteq V(G)$ of size at most $\binom{p}{2}$ and some $i \in \{1, \ldots, k+1\}$ such that H_i is dividing in $(G - X) \cup W$. Let S' be the union of X and the vertex set $\mathrm{per}(H_i)$ of the perimeter of H_i. Let A be the union of S' and the vertex sets of all components of $G - S'$ that intersect the perimeter of W. Let $B := V(G) \setminus (A \setminus S')$. So (A, B) is a separation of G with $A \cap B = S'$ and $V(H_i) \subseteq B$. Let X_W be the vertex set of the perimeter of W, and note that $|(S \cup A) \cap B| \leqslant |S| - |X_W| + |X|$, which is less than $|S|$ since $|X_W| > p^* \geqslant \binom{p}{2} \geqslant |X|$. Likewise $|(S \cup B) \cap A| \leqslant |S| - |V(H_i)| + |S'|$, which is less than $|S|$ since there are at least p^2 vertices of H_i that are not in the perimeter of H_i, there are at most $\binom{p}{2}$ vertices of S' that are not in the perimeter of H_i, and all vertices in the perimeter of H_i are in $S \cap S'$. Thus S is reducible, as required. □

Lemmas 12 and 27 imply the next result, which implies Theorem 3.

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Theorem 28. *For any integer $p \geqslant 1$ there exists an integer c_p such that for every K_p-minor-free graph G, for every bag B of any atomic tree-decomposition of G, we have* $\mathrm{tw}(G[B]) \leqslant c_p$.

Much work has gone into optimising the bounds in the Grid Minor Theorem [22, 34, 35] and the Flat Wall Theorem [34, 55, 64]. These results imply polynomial bounds for the functions in Theorems 25 and 26, and thus for the implicit function in Lemma 27 and Theorem 28. See [55] for an in-depth discussion of these results.

7. 1-planar graphs

This section proves our negative results for 1-planar graphs introduced in Section 1. The following notation is helpful. For a tree T and edge $xy \in E(T)$, let $T_{x:y}$ and $T_{y:x}$ be the subtrees of $T - xy$ respectively containing x and y. For a tree-decomposition $(B_x : x \in V(T))$ of a graph G and for each edge $xy \in E(T)$, let $G_{x:y} := G[\bigcup\{B_z \setminus B_y : z \in V(T_{x:y})\}]$.

Lemma 29. *Let c be a positive integer, G be a graph and S be a set of vertices in G such that for any collection $\{\{a_1, b_1\}, \dots, \{a_z, b_z\}\}$ of disjoint pairs of vertices in S, there is a collection $\{P_{i,j} : i \in \{1, \dots, z\}, j \in \{1, \dots, 2c+4\}\}$ of internally disjoint paths, each internally disjoint from S, such that for each $i \in \{1, \dots, z\}$ and $j \in \{1, \dots, 2c+4\}$ the endvertices of $P_{i,j}$ are a_i and b_i. For any tree-decomposition of G of width less than $|S| + c$, some bag contains S.*

Proof. Consider any tree-decomposition $\mathcal{D} = (B_x : x \in V(T))$ of G in which no bag contains S. Our goal is to show that $\mathcal{D}$ has width at least $|S| + c$. For every edge $xy \in E(T)$, orient xy towards x if $G_{x:y}$ has more vertices of S than $G_{y:x}$, orient xy towards y if $G_{y:x}$ has more vertices of S than $G_{x:y}$, and otherwise orient xy arbitrarily. Let t_0 be a sink of the resulting oriented tree, and let $t_1, \dots, t_d$ be the neighbours of t_0. For each $i \in \{1, \dots, d\}$, let $S_i := S \cap V(G_{t_i:t_0})$, and let $S_0 := S \setminus \bigcup\{S_1, \dots, S_d\}$. Without loss of generality, $|S_1| = \max\{|S_1|, \dots, |S_d|\}$.

First consider the case where $2|S_1| \geqslant |S \setminus S_0|$. Since t_0 is a sink, $|S_1| \leqslant |S \setminus B_{t_1}|$. Thus, there are disjoint pairs $\{a_1, b_1\}, \dots, \{a_{|S_1|}, b_{|S_1|}\}$, where each a_i is in S_1 and each b_i is in $S \setminus B_{t_1}$. By assumption, there is a set of internally disjoint paths $\{P_{i,j} : i \in \{1, \dots, |S_1|\}, j \in \{1, \dots, 2c+4\}\}$, each internally disjoint from S, such that for each $i \in \{1, \dots, |S_1|\}$ and $j \in \{1, \dots, 2c+4\}$ the endpoints of $P_{i,j}$ are a_i and b_i. By the definition of a tree-decomposition, $B_{t_0} \cap B_{t_1}$ separates S_1 from $S \setminus B_{t_1}$ in G, and so $B_{t_0} \cap B_{t_1}$ contains a vertex of each of these paths. Since $\{a_1, \dots, a_{|S_1|}\} \cap B_{t_0} = \varnothing$ and $\{b_1, \dots, b_{|S_1|}\} \cap B_{t_1} = \varnothing$ and each path is internally disjoint from S, we have $|(B_{t_0} \cap B_{t_1}) \setminus S| \geqslant (2c+4)|S_1|$. Thus, $|B_{t_0}| \geqslant |S_0| + (2c+4)|S_1| \geqslant |S_0| + (c+2)|S \setminus S_0|$. By assumption S is not a subset of B_{t_0}, so $|S \setminus S_0| \geqslant 1$ and $|B_{t_0}| \geqslant |S| + c + 1$.

Now consider the case where $2|S_1| \leqslant |S \setminus S_0| - 1$. Let $\mathcal{X} = \{\{a_1, b_1\}, \dots, \{a_z, b_z\}\}$ be a maximum sized collection of disjoint pairs of vertices such that each set $\{a_i, b_i\}$ intersects exactly two sets in $\{S_1, \dots, S_d\}$. Suppose for contradiction that $2z \leqslant |\bigcup\{S_1, \dots, S_d\}| - 2$. Then for some $i \in \{1, \dots, d\}$ there are at least two vertices v and w in S_i that are not in a pair in $\mathcal{X}$, and all vertices in $S \setminus (S_0 \cup S_i)$ are in a pair in $\mathcal{X}$. Since $|S_i| \leqslant |S_1|$ and $2|S_1| < |S \setminus S_0|$, we have $|S_i| < |S \setminus (S_0 \cup S_i)|$, there is a pair $\{a_j, b_j\}$ in $\mathcal{X}$ that is disjoint from S_i. Replacing $\{a_j, b_j\}$ by $\{v, a_j\}$ and $\{b_j, w\}$ increases the size of $\mathcal{X}$, yielding the desired contradiction. Thus, $2z \geqslant |S_1 \cup \ldots \cup S_d| - 1 = |S \setminus S_0| - 1$. By construction, there is a set of internally disjoint paths $\{P_{i,j} : i \in \{1, \dots, z\}, j \in \{1, \dots, 2c+4\}\}$, each internally disjoint from S, such that for each $i \in \{1, \dots, |S_1|\}$ and $j \in \{1, \dots, 2c+4\}$ the endpoints of $P_{i,j}$ are a_i and b_i. Note that for distinct $i, j \in \{1, \dots, d\}$, the set B_{t_0} separates S_i from S_j in G. Thus, B_{t_0} contains an internal vertex of each of these paths, and so $|B_{t_0}| \geqslant |S_0| + (2c+4)z \geqslant |S_0| + (c+2)(|S \setminus S_0| - 1)$. By assumption $S \not\subseteq B_{t_0}$, so $|S_1| \geqslant 1$ and $|S \setminus S_0| \geqslant 2|S_1| + 1 \geqslant 3$. Thus, $|B_{t_0}| \geqslant (|S| - 1) + 2(c+1) > |S| + c + 1$.

In both cases, $\mathcal{D}$ has width at least $|B_{t_0}| - 1 \geqslant |S| + c$. Thus, every tree-decomposition of G with width less than $|S| + c$ has a bag containing S. □

The following result implies and strengthens Theorem 4 by taking G_0 to be any graph with sufficiently large treewidth (such as a large complete graph or large grid).

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Theorem 30. *For any integer $c \geqslant 0$ and graph G_0 there is a 1-planar graph G such that every tree-decomposition of G with width less than $\mathrm{tw}(G) + c$ has a bag B such that G_0 is a topological minor of $G[B]$.*

Proof. Draw G_0 in the plane (allowing crossings), and let H be the planarisation of this drawing (that is, introduce a new vertex in H at each crossing). We introduce H simply to make it easier to talk about the drawing, it has not no essential role in the construction. Let T^* be a rooted spanning tree of the dual of H. Let $f_0, f_1, \ldots, f_s$ be the faces of H (equivalently, vertices of T^*) ordered so that f_s is the root of T^*, and for each edge f_if_j in T^*, if $i < j$ then f_j is the parent of f_i; we denote this edge f_if_j by e_i.

We will construct a 1-planar graph G and a set $S \subseteq V(G)$ such that $G[S]$ is a subdivision of G_0. We will then show that $\mathrm{tw}(G) \leqslant |S|$ and that S satisfies the conditions of Lemma 29. Let V_0 be the set of vertices of G_0 incident to f_0. For each $i \in \{1, \ldots, s\}$, let V_i be the set of vertices incident with f_i and not incident with any face in $\{f_1, \ldots, f_{i-1}\}$. Let E_0 be the set of edges of H incident to f_0 and not dual to edges in T^*, and for each $i \in \{1, \ldots, s\}$, let E_i be the set of edges of $E(H) \setminus (E_0 \cup E_1 \cup \cdots \cup E_{i-1})$ that are incident with f_i and not dual to the edge f_if_j of T^* with $j > i$. Thus the sets E_i partition $E(H)$, and for each $i \in \{1, \ldots, s-1\}$ the edge of H dual to e_i is in E_j where $e_i = f_if_j$ with $i < j$.

Iteratively construct G from G_0 as follows. First, for each $e \in E_0$, subdivide the edge-segment of G_0 corresponding to e once. Let X_0 be the set of all subdivision vertices introduced in this step, together with V_0. Now, for each $v \in X_0$, and each $j \in \{0, 1, \ldots, s\}$, create $2c+4$ new vertices $a_{v,j,1}, a_{v,j,2}, \ldots, a_{v,j,2c+4}$, and draw these vertices in f_j. Draw an edge from v to each of these new vertices that does not cross any edge in H except those dual to the edges of the path from f_0 to f_j in T^*. For each pair of distinct vertices $v, w \in X_0$ and each $y \in \{1, \ldots, 2c+4\}$, draw an edge from v to $a_{w,0,y}$ in f_0. At the final step, we will freely subdivide all edges incident to these newly created vertices, so the number of crossings on these edges is unimportant.

For $i = 1, 2, \ldots, s$ apply the following step. First, for each $e = pq \in E_i$, subdivide the edge-segment of G_0 corresponding to e once between p and the first crossing on e starting at p, once between q and the first crossing on e starting at q, and once between each pair of consecutive crossings on e. No future edges will cross e. Let X_i be the the set of all subdivision vertices added to edge-segments corresponding to edges in E_i, together with all vertices in V_i. For each $v \in X_i$, each $j \in \{i, i+1, \ldots, s\}$ and each $y \in \{1, \ldots, 2c+4\}$, create a new vertex $a_{v,j,y}$ and draw this vertex in f_j. Draw an edge from v to each of these new vertices that does not cross any edge in H except those dual to edges of the path from f_i to f_j in T^*. For each pair of distinct vertices $v \in X_i$ and $w \in X_0 \cup \ldots \cup X_i$ and each $y \in \{1, \ldots, 2c+4\}$, draw an edge from v to $a_{w,i,y}$ in f_i. These steps may produce arbitrarily many crossings between edges which are not drawn in the image of H, but we will freely subdivide all such edges in the final step of the construction.

Let G' be the graph constructed so far. Let S be the set of all vertices in $V(G_0)$ together with all subdivision vertices introduced so far (that is, all vertices not of the form $a_{v,j,y}$). By construction, the subgraph of G' induced by S is a subdivision of G_0, and every edge in this subgraph is crossed at most once in the entire graph. Subdivide the remaining edges (those incident to vertices of the form $a_{v,j,y}$) until the resulting graph is 1-planar, and call the final graph G.

We first construct a tree-decomposition of G'. Let T be the star with central vertex r and leaf-set $V(G') \setminus S$. Set $B'_r := S$ and $B'_w := N_{G'}[w]$ for each $w \in V(G') \setminus S$. Since $V(G') \setminus S$ is an independent set in G', $(B'_x : x \in V(T))$ is a tree-decomposition of G' with width $|S|$ (since $N_{G'}(w) \subseteq S$ and $|B'_w| = |N_{G'}[w]| \leqslant |S| + 1$ for each $w \in V(G') \setminus S$). Since G is a subdivision of G', $\mathrm{tw}(G) = \mathrm{tw}(G') \leqslant |S|$.

Now consider an arbitrary collection $\{\{a_1, b_1\}, \ldots, \{a_z, b_z\}\}$ of disjoint pairs of vertices in S. By construction, for each $i \in \{1, \ldots, z\}$, there are at least $2c+4$ common neighbours $a_{v,j,y}$ of a_i and b_i in G' with $v \in \{a_i, b_i\}$. Thus there is a collection $\{P_{i,j} : i \in \{1, \ldots, z\}, j \in \{1, \ldots, 2c+4\}\}$ of internally disjoint paths in G, each internally disjoint from S, such that for each $i \in \{1, \ldots, z\}$ and $j \in \{1, \ldots, 2c+4\}$ the endvertices of $P_{i,j}$ are a_i and b_i. By Lemma 29, every tree-decomposition of

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G of width less than $|S| + c$ (and thus every tree-decomposition of G of width less than $\mathrm{tw}(G) + c$) has a bag containing S. □

8. Width $O(\sqrt{n})$

This section constructs tree-decompositions with bags of bounded treewidth in more general graph classes than those studied above, at the expense that the optimal width condition is relaxed to the asymptotically tight bound of $O(\sqrt{n})$ for n-vertex graphs. In fact, we show that the union of any bounded number of bags induces a subgraph with bounded treewidth.

The following definitions are key ingredients to the proofs. A layering of a graph G is an ordered partition $(V_1, \dots, V_n)$ of $V(G)$ into (possibly empty) sets such that for each edge $vw \in E(G)$ there exists $i \in \{1, 2, \dots, n-1\}$ such that $\{v, w\} \subseteq V_i \cup V_{i+1}$. The layered treewidth of a graph G, denoted by $\mathrm{ltw}(G)$, is the minimum nonnegative integer ℓ such that G has a tree-decomposition $\mathcal{X} = (X_x : x \in V(T))$ and a layering $(V_1, \dots, V_n)$, such that $|X_x \cap V_i| \leqslant \ell$ for each bag X_x and layer V_i. This implies that the subgraph induced by each layer has bounded treewidth, and moreover, a single tree-decomposition of G has bounded treewidth when restricted to each layer. In fact, these properties hold when considering a bounded sequence of consecutive layers. Layered treewidth was independently introduced by Dujmović, Morin, and Wood [52] and Shahrokhi [82].

The next lemma extends a result of Sergey Norin who proved the $2\sqrt{cn}$ treewidth bound [52]. Here we choose to present a relatively simple proof rather than optimising the bound on the treewidth of the union of bags.

Lemma 31. *Let G be a graph with n vertices and layered treewidth $c \geqslant 1$. Then G has a tree-decomposition $\mathcal{D}$ with width at most $2\sqrt{cn}$, such that the subgraph of G induced by the union of any k bags of $\mathcal{D}$ has treewidth at most $(3k+1)c - 1$.*

Proof. We may assume $c < n$, or else the lemma is satisfied by the trivial tree-decomposition with a single node. Let $(V_0, V_1, \dots, V_m)$ be a layering of G, and let $\mathcal{D} := (B_x : x \in V(T))$ be a tree-decomposition of G, such that $|B_x \cap V_i| \leqslant c$ for each $x \in V(T)$ and $i \in \{0, 1, \dots, m\}$. Let $p := \lceil \sqrt{n/c} \rceil$, which is at least 2 since $c < n$. For $i \in \{0, 1, \dots, p-1\}$ let $\widehat{V}_i := \bigcup\{V_j : j \equiv i \pmod p\}$. So $\widehat{V}_0, \widehat{V}_1, \dots, \widehat{V}_{p-1}$ is a partition of $V(G)$, and $|\widehat{V}_i| \leqslant \frac{n}{p}$ for some $i \in \{0, 1, \dots, p-1\}$. For each component C of $G - V_i$, let $\mathcal{D}_C$ be the tree-decomposition of C induced by $\mathcal{D}$ (so that each of bag in the intersection of some B_x with $V(C)$. Each component of $G - \widehat{V}_i$ is contained within $p-1$ layers, so $\mathcal{D}_C$ has width at most $c(p-1) - 1$. These tree-decompositions can be combined to obtain a tree-decomposition $\mathcal{D}'$ of $G - V_i$ of width at most $c(p-1) - 1$. By adding V_i to every bag of $\mathcal{D}'$, we obtain a tree-decomposition $\mathcal{D}''$ of G of width at most $\frac{n}{p} + c(p-1) - 1 \leqslant 2\sqrt{cn}$.

Consider arbitrary bags $C_1, \dots, C_k$ of $\mathcal{D}''$. By construction, there is a set $S \subseteq V(T)$ with $|S| \leqslant k$ such that $C_1 \cup \ldots \cup C_k \subseteq X := \widehat{V}_i \cup \bigcup\{B_x : x \in S\}$. We now construct a tree-decomposition of $G[X]$ with width at most $(3k+1)c - 1$, which implies that $\mathrm{tw}(G[C_1 \cup \ldots \cup C_k]) \leqslant (3k+1)c - 1$ as desired.

Fix a vertex r of T. Let T_j be a copy of T for each $j \equiv i \pmod p$, where these copies are pairwise disjoint. For each $x \in V(T)$, let x_j be the copy of x in T_j. Let $P := (y_0, y_1, \dots, y_m)$ be a path, where y_j is identified with the vertex r_j whenever $j \equiv i \pmod p$. We obtain a tree $U := P \cup \bigcup\{T_j : j \equiv i \pmod p\}$. For each node $x_j \in V(U)$, let

$$A_{x_j} := (B_x \cap V_j) \cup \bigcup\{(V_{j-1} \cup V_j \cup V_{j+1}) \cap B_z : z \in S\}.$$

For each node $y_j \in V(U)$ with $j \not\equiv i \pmod p$, let

$$A_{y_j} := \bigcup\{(V_{j-1} \cup V_j \cup V_{j+1}) \cap B_z : z \in S\}.$$

We now prove that $(A_u : u \in V(U))$ is a tree-decomposition of $G[X]$.

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Consider a vertex $v \in X$. Say $v \in V_j$. If $v \in \bigcup\{B_z : z \in S\}$ and $j \bmod p$ is in $\{i-1, i, i+1\}$ then $v \in A_{x_j}$ for every $x \in V(T)$, and $v \in A_{y_{j-1}} \cup A_{y_{j+1}}$, and v is in no other bag A_u with $u \in V(U)$. If $v \in \bigcup\{B_z : z \in S\}$ and $j \bmod p$ is not in $\{i-1, i, i+1\}$ then $v \in A_{y_{j-1}} \cup A_{y_j} \cup A_{y_{j+1}}$, and v is in no other bag A_u with $u \in V(U)$. Otherwise, $v \notin \bigcup\{B_z : z \in S\}$. Since $v \in X$, we have $j \equiv i \pmod p$, implying $v \in A_{x_j}$ whenever $v \in B_x$ with $x \in V(T)$, and v is in no other bag A_u with $u \in V(U)$. In each case, the set of bags A_u that contain v correspond to a connected subtree of U. Hence $(A_u : u \in V(U))$ satisfies the vertex-property of tree-decompositions.

Consider an edge vw of $G[X]$. So $v, w \in B_x$ for some node $x \in V(T)$. Say $v \in V_j$ and $w \in V_\ell$. If $j, \ell \equiv i \pmod p$, then since $p \geqslant 2$, we have $\ell = j$ and $v, w \in A_{x_j}$. If $j \equiv i \pmod p$ and $\ell \not\equiv i \pmod p$, then $|j - \ell| = 1$ and $w \in \cup\{B_z : z \in S\}$, implying that $v, w \in A_{x_j}$. If $j \not\equiv i \pmod p$ and $\ell \not\equiv i \pmod p$, then $|j - \ell| \leqslant 1$ and $v, w \in \cup\{B_z : z \in S\}$, implying that $v, w \in A_{y_j}$. Hence $(A_u : u \in V(U))$ satisfies the edge-property of tree-decompositions.

Therefore, $(A_u : u \in V(U))$ is a tree-decomposition of $G[X]$. Observe that $|A_{x_j}| \leqslant (3k+1)c$ and $|A_{y_j}| \leqslant 3ck$. Thus, the width of $(A_u : u \in V(U))$ is at most $(3k+1)c - 1$. Hence, $G[X]$ and $G[C_1 \cup \ldots \cup C_k]$ have treewidth at most $(3k+1)c - 1$. □

The next lemma shows the versatility of layered treewidth.

Lemma 32. *Every graph G with layered treewidth $c \geqslant 1$ has a set S of at most $\sqrt{cn}$ vertices, such that $\mathrm{tw}(G[S]) \leqslant c - 1$ and $G - S$ has a tree-decomposition with width at most $\sqrt{cn}$ in which the union of any $k \geqslant 1$ bags has pathwidth at most $2ck - 1$.*

Proof. Let $(V_0, V_1, \ldots, V_m)$ be a layering of G, and let $(B_x : x \in V(T))$ be a tree-decomposition of G, such that $|B_x \cap V_i| \leqslant c$ for each $x \in V(T)$ and $i \in \{0, 1, \ldots, m\}$. Let $p := \lceil \sqrt{n/c} \rceil$. Since $n > c$, we have $p \geqslant 2$. For $i \in \{0, 1, \ldots, p-1\}$, let $\widehat{V}_i := \bigcup\{V_j : j \equiv i \pmod p\}$. So $\widehat{V}_0, \widehat{V}_1, \ldots, \widehat{V}_{p-1}$ is a partition of $V(G)$, and $|\widehat{V}_i| \leqslant \frac{n}{p}$ for some $i \in \{0, 1, \ldots, p-1\}$. Let $S := \widehat{V}_i$. Each component C of $G - S$ is contained within $p - 1$ layers, and thus $\mathcal{D}_C := (B_x \cap V(C) : x \in V(T))$ is a tree-decomposition of C with width at most $c(p-1) - 1$. Hence $G - S$ has a tree-decomposition $\mathcal{D}'$ of width at most $c(p-1) - 1 \leqslant \sqrt{cn}$, obtained by combining these tree-decompositions. Let X be the union of k bags from $\mathcal{D}'$. Then $X \subseteq \widehat{X} := B_{x_1} \cup \ldots \cup B_{x_k}$ for some $x_1, \ldots, x_k \in V(T)$. Observe that $(\widehat{X} \cap (V_0 \cup V_1), \widehat{X} \cap (V_1 \cup V_2), \ldots, \widehat{X} \cap (V_{m-1} \cup V_m))$ is a path-decomposition of $G[\widehat{X}]$ with width at most $2ck - 1$. The result follows. □

Dujmović et al. [52] showed that every planar graph has layered treewidth at most 3. Thus Lemmas 31 and 32 imply:

Corollary 33. *Every planar graph with n vertices has:*

- *(a) a tree-decomposition $\mathcal{D}$ with width at most $2\sqrt{3n}$, such that the subgraph of G induced by the union of any k bags of $\mathcal{D}$ has treewidth at most $9k + 2$, and*
- *(b) a set S of at most $\sqrt{3n}$ vertices, such that $\mathrm{tw}(G[S]) \leqslant 2$ and $G - S$ has a tree-decomposition with width at most $\sqrt{3n}$ in which the union of any $k \geqslant 1$ bags has pathwidth at most $6k - 1$.*

Dujmović et al. [52] showed that every graph of Euler genus at most g has layered treewidth at most $2g + 3$. Thus Lemmas 31 and 32 implies:

Corollary 34. *Every graph of Euler genus g with n vertices has:*

- *(a) a tree-decomposition $\mathcal{D}$ with width at most $2\sqrt{(2g+3)n}$, such that the subgraph of G induced by the union of any k bags of $\mathcal{D}$ has treewidth at most $(3k+1)(2g+3) - 1$, and*
- *(b) a set S of at most $\sqrt{(2g+3)n}$ vertices, such that $\mathrm{tw}(G[S]) \leqslant 2g + 2$ and $G - S$ has a tree-decomposition with width at most $\sqrt{(2g+3)n}$ in which the union of any $k \geqslant 1$ bags has pathwidth at most $2(2g+3)k - 1$.*

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Layered treewidth is of interest beyond minor-closed classes, since there are several natural graph classes that have bounded layered treewidth but contain arbitrarily large complete graph minors [51, 58, 61]. Here is one example. A graph G is (g, ℓ)-planar if G has a drawing in a surface of Euler genus at most g with at most ℓ crossings on each edge. Dujmović et al. [51] showed that every (g, ℓ)-planar graph has layered treewidth at most $(4g + 6)(\ell + 1)$. Thus Lemma 31 implies the following result:

Corollary 35. *Every (g, ℓ)-planar graph with n vertices has:*

- *(a) a tree-decomposition $\mathcal{D}$ with width at most $2\sqrt{(2g + 3)(\ell + 1)n}$, such that the subgraph of G induced by the union of any k bags of $\mathcal{D}$ has treewidth at most $(2k + 1)(4g + 6)(\ell + 1) - 1$, and*
- *(b) a set S of at most $\sqrt{(4g + 6)(\ell + 1)n}$ vertices, such that $\operatorname{tw}(G[S]) \leqslant (4g + 6)(\ell + 1) - 1$ and $G - S$ has a tree-decomposition with width at most $\sqrt{(4g + 6)(\ell + 1)n}$ in which the union of any $k \geqslant 1$ bags has pathwidth at most $2(4g + 6)(\ell + 1)k - 1$.*

Note that the $g = 0$ and $\ell = k = 1$ case of Corollary 35 says that every 1-planar graph with n vertices has a tree-decomposition with width at most $2\sqrt{6n}$, such that each bag has treewidth at most 35, as promised in Section 1. This treewidth bound can be improved by optimising Lemma 31 in the $k = 1$ case.

Now consider K_p-minor-free graphs where p is a fixed positive integer. Alon, Seymour, and Thomas [17] showed that every K_p-minor-free graph G with n vertices has treewidth $O(\sqrt{n})$. Liu et al. [69] showed that every K_p-minor-free graph G has a tree-decomposition such that the subgraph of G induced by any bag has bounded treewidth. The following theorem combines and generalises these two results.

Theorem 36. *For any integer $p \geqslant 1$ there exists c such that every K_p-minor-free graph G with n vertices has a tree-decomposition $\mathcal{D}$ with width at most $c\sqrt{n}$, such that the subgraph of G induced by the union of any $k \geqslant 1$ bags of $\mathcal{D}$ has treewidth at most ck.*

The proof of Theorem 36 employs the following version of the graph minor structure theorem in terms of layered treewidth, due to Dujmović et al. [52]. If $(B_x : x \in V(T))$ is a tree-decomposition of a graph G, then for each node $x \in V(T)$, the torso at x is the graph $G\langle B_x\rangle$ obtained from $G[B_x]$ by adding edges so that $B_x \cap B_y$ is a clique for each edge $xy \in E(T)$.

Theorem 37 [52]. *For any integer $p \geqslant 1$ there exists c such that every K_p-minor-free graph G has a tree-decomposition $(B_x : x \in V(T))$ such that for each $x \in V(T)$ there exists $A_x \subseteq B_x$ with $|A_x| \leqslant c$ and $\operatorname{ltw}(G\langle B_x\rangle - A_x) \leqslant c$.*

Proof of Theorem 36. Let $(B_x : x \in V(T))$ be the tree-decomposition of G from Theorem 37. For each $x \in V(T)$ there exists $A_x \subseteq B_x$ such that $|A_x| \leqslant c$ and $\operatorname{ltw}(G\langle B_x\rangle - A_x) \leqslant c$. By Lemma 31, $G\langle B_x\rangle - A_x$ has a tree-decomposition $\mathcal{D}_x$ with width at most $2\sqrt{cn}$, such that the subgraph of G induced by the union of any k bags of $\mathcal{D}_x$ has treewidth at most $(3k + 1)c - 1$. Add A_x to every bag of $\mathcal{D}_x$. Now $\mathcal{D}_x$ is a tree-decomposition of $G\langle B_x\rangle$ with width at most $2\sqrt{cn} + c$, such that the subgraph of $G\langle B_x\rangle$ induced by the union of any $k \geqslant 1$ bags of $\mathcal{D}_x$ has treewidth at most $(3k + 2)c - 1$. Let $\mathcal{D}$ be the tree-decomposition of G obtained as follows. For each edge $xy \in E(T)$, let $I_{xy} := B_x \cap B_y$. Since I_{xy} is a clique in $G\langle B_x\rangle$, there is a bag R_x in $\mathcal{D}_x$ and there is a bag R_y in $\mathcal{D}_y$ such that $I_{xy} \subseteq R_x \cap R_y$. Add an edge between (the nodes corresponding to the bags) R_x and R_y, to obtain a tree-decomposition of G with width at most $2\sqrt{cn} + c \leqslant c'\sqrt{n}$.

Let $\mathcal{C}$ be a set of k bags in this tree-decomposition. Let $X := \bigcup \mathcal{C}$. For each $x \in V(T)$ let $\mathcal{C}_x := \mathcal{C} \cap \mathcal{D}_x$. As shown above, for each $x \in V(T)$, the subgraph of $G\langle B_x\rangle$ induced by $\bigcup \mathcal{C}_x$ has a tree-decomposition $\mathcal{T}_x$ with width at most $(3|\mathcal{C}_x| + 2)c - 1 \leqslant (3k + 2)c - 1$. For each edge $xy \in E(T)$ let $I_{xy} := (B_x \cap \bigcup \mathcal{C}_x) \cap (B_y \cap \bigcup \mathcal{C}_y)$, which is a clique in $G\langle B_x\rangle$ and in $G\langle B_y\rangle$ (by the definition of

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torso). So there is a bag R_x in $\mathcal{T}_x$ and there is a bag R_y in $\mathcal{T}_y$ such that $I_{xy} \subseteq R_x \cap R_y$. Add an edge between (the nodes corresponding to the bags) R_x and R_y, to obtain a tree-decomposition of $\bigcup \mathcal{C}$ with width at most $(3k+2)c-1$. □

9. Open problems

We conclude by mentioning three open problems that arise from this work:

1. Recall that Corollary 11 says that every graph with maximum degree at most 3 has an optimal tree-decomposition such that every bag has treewidth at most 3. The following natural question arises: Is there a constant c such that every graph with maximum degree at most 4 has an optimal tree-decomposition such that every bag has treewidth at most c? For general tree-decompositions, Liu et al. [69] showed that tree-tw$(G) \leqslant 15$ for every graph G with maximum degree at most 4, and there is a constant c such that tree-tw$(G) \leqslant c$ for every graph G with maximum degree at most 5. This type of result does not hold for much larger maximum degree. In particular, Liu et al. [69] showed that the class of 146-regular n-vertex graphs (for even n) has tree-tw $\Omega(n)$.
2. Are there constants $c, k \geqslant 1$ such that every 1-planar graph G has a tree-decomposition of width at most $c\,\mathrm{tw}(G)$ such that each bag has treewidth at most k?
3. Can tree-decompositions with bags of small treewidth be used to speed up algorithms? Numerous problems can be solved on n-vertex graphs G with time complexity $2^{O(\mathrm{tw}(G))}n$ via dynamic programming on a tree-decomposition (see [38]). Can such results be improved using that each bag has bounded treewidth?

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