Math. Proc. Camb. Phil. Soc. (1978), **83**, 321
Printed in Great Britain

Waves created against a vertical cliff – a uniform asymptotic solution

By A. K. PRAMANIK

Centre of Advanced Study in Applied Mathematics, University College of Science, Calcutta

(*Received* 13 *June* 1977)

Summary. The initial-value problem of waves generated by a moving oscillatory surface pressure against a vertical cliff is solved and a uniform asymptotic analysis of the unsteady state is given. The same problem with no cliff is solved by Kaplan, and by Debnath and Rosenblat but their solutions are not uniform.

1. *Introduction.* The two-dimensional problem of gravity waves generated by moving oscillatory surface pressure distributions in a fluid which is unbounded in both horizontal directions has been studied by Kaplan(2) and by Debnath and Rosenblat(1) in infinite depth and in finite uniform depth respectively. The corresponding problem in a fluid bounded on one side by a vertical cliff appears not to have been attempted although there exist well-known solutions for free harmonic oscillation in a fluid which is horizontally bounded on one side by a vertical cliff(3).

In the present paper we consider the transient development of two-dimensional gravity waves generated by a moving oscillatory surface pressure distribution in a homogeneous incompressible fluid which is horizontally bounded by an infinite vertical cliff. With the help of an initial-value formulation, the integral representation of the surface waves is obtained through integral transform of the equations of motion. These integrals are then analysed asymptotically for large time and distance. We have here employed a method which yields an asymptotic expression that remains uniformly valid even when the stationary points of the integrals coincide with the poles. None of the asymptotic expressions of the previous works is uniformly valid. Again our method of solution here gives interesting information regarding the position and distribution of the resulting wave pattern. Regarding the effect of the presence of the vertical cliff it is found here that in the steady state it introduces a new wave to the system of waves found in the corresponding situation in a fluid unbounded in both horizontal directions. This new wave is interpreted as the reflected wave.

2. *Formulation and formal solution.* We take the x-axis on the undisturbed free surface and the y-axis vertically upwards. The system being initially at rest, waves are produced by the continued application of the pressure distribution $p(x,t)$ which at the same time moves with uniform velocity V. Let $\phi(x,y,t)$ be the velocity potential, $\eta(x,t)$ the surface elevation, ρ the density and g the acceleration due to gravity. Then in the moving coordinate system in which the origin initially coincides

https://doi.org/10.1017/S030500410005458X Published online by Cambridge University Press

with the cliff and then moves with velocity V, we have the following initial-value problem,

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \quad (0\leqslant x<\infty, y\geqslant 0, t\geqslant 0), \tag{1}$$

$$\left.\begin{aligned}\frac{p}{\rho}+\frac{\partial\phi}{\partial t}-V\frac{\partial\phi}{\partial x}+g\eta=0,\\ \frac{\partial\eta}{\partial t}-V\frac{\partial\eta}{\partial x}-\frac{\partial\phi}{\partial y}=0,\end{aligned}\right\} \quad \text{at} \quad y=0, \tag{2, 3}$$

$$\frac{\partial\phi}{\partial x}=0 \quad \text{on the cliff}, \tag{4}$$

$$\phi\to 0 \quad \text{at} \quad y=-\infty, \tag{5}$$

and

$$\phi(x,y,0)=0, \quad \eta(x,0)=0. \tag{6}$$

We again suppose that ϕ, η, p are defined in the generalized sense. To solve the problem, we write ϕ and p in the following integral forms,

$$\phi=\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty A(\kappa,t)\exp\{\kappa y\}\cos\kappa(x+Vt)\,d\kappa, \tag{7}$$

$$p=\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \bar{p}(\kappa,t)\cos\kappa(x+Vt)\,d\kappa. \tag{8}$$

Elimination of η between (2) and (3) gives the following condition for ϕ at $y=0$,

$$\frac{\partial^2\phi}{\partial t^2}-2V\frac{\partial^2\phi}{\partial x\,\partial t}+V^2\frac{\partial^2\phi}{\partial x^2}+g\frac{\partial\phi}{\partial y}+\frac{1}{\rho}\left(\frac{\partial p}{\partial t}-V\frac{\partial p}{\partial x}\right)=0.$$

Substitution from (7) and (8) reduces this to

$$\frac{\partial^2 A}{\partial t^2}+\sigma^2 A+\frac{1}{\rho}\frac{\partial p}{\partial t}=0,$$

where

$$\sigma^2=g\kappa.$$

Solution to this equation after utilization of the initial condition (6) gives

$$A=-\frac{1}{\rho}\int_0^t \bar{p}(\kappa,\tau)\cos\sigma(t-\tau)\,d\tau. \tag{9}$$

The surface elevation η can be found from (2) with the help of (7), (8) and (9) in the following form

$$\eta=-\frac{1}{\sqrt{(2\pi)}\,g\rho}\int_0^\infty \cos\kappa(x+Vt)\,d\kappa\int_0^t \bar{p}(\kappa,\tau)\sin\sigma(t-\tau)\,d\tau. \tag{10}$$

Inversion of (8) gives

$$\bar{p}(\kappa,t)=\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty p(x,t)\cos\kappa(x+Vt)\,dx. \tag{11}$$

In the following we shall consider the pressure distribution given by

$$p(x,t)=f(x)\exp\{i\omega t\}. \tag{12}$$

https://doi.org/10.1017/S030500410005458X Published online by Cambridge University Press

Substituting from (11) into (10) we get the following integral representation for η,

$$4\pi\rho g\eta = \sum_{m=1}^{4} I_m = \sum_{m=1}^{4}\sum_{n=1}^{4} I_{mn}, \tag{13}$$

where

$$I_{11} = \int_0^\infty F_1(\kappa)\exp\{i\kappa x + i(2\kappa V+\omega)t\}\,d\kappa, \quad I_{12} = \int_0^\infty F_1(\kappa)\exp\{-i\kappa x + i\omega t\}\,d\kappa,$$

$$I_{13} = -\int_0^\infty F_1(\kappa)\exp\{i\kappa x + i(\kappa V+\sigma)t\}\,d\kappa, \quad I_{14} = -\int_0^\infty F_1(\kappa)\exp\{-i\kappa x + i(-\kappa V+\sigma)t\}\times d\kappa,$$

$$I_{21} = -\int_0^\infty F_2(\kappa)\exp\{i\kappa x + i(2\kappa V+\omega)t\}\times d\kappa, \quad I_{22} = -\int_0^\infty F_2(\kappa)\exp\{-i\kappa x + i\omega t\}\,d\kappa,$$

$$I_{23} = \int_0^\infty F_2(\kappa)\exp\{i\kappa x + i(\kappa V-\sigma)t\}\,d\kappa, \quad I_{24} = \int_0^\infty F_2(\kappa)\exp\{-i\kappa x - i(\kappa V+\sigma)t\}\,d\kappa,$$

$$I_{31} = -\int_0^\infty F_3(\kappa)\exp\{-i\kappa x + i(-2\kappa V+\omega)t\}\,d\kappa, \quad I_{32} = -\int_0^\infty F_3(\kappa)\exp\{i\kappa x + i\omega t\}\,d\kappa,$$

$$I_{33} = \int_0^\infty F_3(\kappa)\exp\{i\kappa x + i(\kappa V+\sigma)t\}\,d\kappa, \quad I_{34} = \int_0^\infty F_3(\kappa)\exp\{-i\kappa x + i(-\kappa V+\sigma)t\}\times d\kappa,$$

$$I_{41} = \int_0^\infty F_4(\kappa)\exp\{-i\kappa x + i(-2\kappa V+\omega)t\}\,d\kappa, \quad I_{42} = \int_0^\infty F_4(\kappa)\exp\{i\kappa x + i\omega t\}\,d\kappa,$$

$$I_{43} = -\int_0^\infty F_4(\kappa)\exp\{i\kappa x + i(\kappa V-\sigma)t\}\,d\kappa, \quad I_{44} = -\int_0^\infty F_4(\kappa)\exp\{-i\kappa x - i(\kappa V+\sigma)t\}\times d\kappa,$$

with

$$F_1(\kappa) = \frac{\sigma f_1(\kappa)}{\kappa V-\sigma+\omega}, \quad F_2(\kappa) = \frac{\sigma f_1(\kappa)}{\kappa V+\sigma+\omega},$$

$$F_3(\kappa) = \frac{\sigma f_2(\kappa)}{\kappa V+\sigma-\omega}, \quad F_4(\kappa) = \frac{\sigma f_2(\kappa)}{\kappa V-\sigma-\omega},$$

and

$$f_1(\kappa) = \int_0^\infty f(x)\exp\{i\kappa x\}\,dx, \quad f_2(\kappa) = \int_0^\infty f(x)\exp\{-i\kappa x\}\,dx.$$

3. *Uniform asymptotic analysis of the unsteady state.* In this section we wish to study the unsteady wave motion for large times and distances. This motion has to be described apparently in terms of asymptotics which are non-uniform across certain moving lines on the surface and the pressure segment. We shall, however, attempt to present here a uniform asymptotic analysis of the unsteady state. For this purpose it is convenient to refer to the fixed parallel coordinate system with the origin on the cliff. If x denotes the horizontal distance from this point then we shall use the same integral representation of η as given in (13), except that we shall write $x - Vt$ in place of x. Throughout the next portion of this paper x denotes the distance from the cliff.

https://doi.org/10.1017/S030500410005458X Published online by Cambridge University Press

We note that the integrands in I_1 have poles at $\kappa = \alpha_1$ and $\kappa = \alpha_2$ the integrands in I_3 have poles at $\kappa = \alpha_3$, while the integrands in I_4 have poles at $\kappa = \alpha_4$, where

$$\sqrt{\alpha_1} = \frac{1}{2V}(\sqrt{g} + \sqrt{(g - 4\omega V)}), \quad \sqrt{\alpha_2} = \frac{1}{2V}(\sqrt{g} - \sqrt{(g - 4\omega V)}) \quad (g > 4\omega V),$$

$$\sqrt{\alpha_3} = \frac{1}{2V}(-\sqrt{g} + \sqrt{(g + 4\omega V)}), \quad \sqrt{\alpha_4} = \frac{1}{2V}(\sqrt{g} + \sqrt{(g + 4\omega V)}). \tag{14}$$

We shall also use the following notations

$$\sigma_n = \sqrt{(g\alpha_n)} \quad (n = 1, 2, 3, 4). \tag{15}$$

We observe next that there are in all three integrals, one in each of I_1, I_3 and I_4, in each of which the integrand possesses a real saddle point together with one or two real poles within the range of integration. These integrals are

$$I_{14} = \left(-\frac{2}{g}\right) \int_0^\infty \frac{\sigma^2 f_1(\sigma^2/g)}{Vg^{-1}\sigma^2 - \sigma + \omega} \exp\{i(\sigma t - \sigma^2 g^{-1} x)\}\, d\sigma, \tag{16}$$

$$I_{34} = \left(\frac{2}{g}\right) \int_0^\infty \frac{\sigma^2 f_2(\sigma^2/g)}{Vg^{-1}\sigma^2 + \sigma - \omega} \exp\{i(\sigma t - \sigma^2 g^{-1} x)\}\, d\sigma, \tag{17}$$

$$I_{43} = \left(-\frac{2}{g}\right) \int_0^\infty \frac{\sigma^2 f_2(\sigma^2/g)}{Vg^{-1}\sigma^2 - \sigma - \omega} \exp\{-i(\sigma t - \sigma^2 g^{-1} x)\}\, d\sigma. \tag{18}$$

In each case the saddle point is $\sigma = gt/2x$. The complete set of poles is

$$\begin{aligned} &\sigma = (\sigma_1, \sigma_2) \quad \text{if} \quad g > 4\omega V, \quad \text{in (16)}, \\ &\sigma = (\sigma_3, -\sigma_4), \qquad\qquad\quad \text{in (17)}, \\ &\sigma = (\sigma_4, -\sigma_3), \qquad\qquad\quad \text{in (18)}. \end{aligned}$$

For such integrals it is usual to seek asymptotic values for large (gt/V) which shall remain uniformly valid even when $(gt/2x)$ is near to σ_j $(j = 1, 2, 3, 4)$. This objective is easily realized through the application of a method which is essentially due to B. L. Van der Waerden (4) whose procedure is outlined below in regard to I_{14}. When $g > 4\omega V$, we first replace (16) by

$$\begin{aligned} -\frac{V}{2} I_{14} = &\int_0^\infty f_1(\sigma^2/g) \exp\{i(\sigma t - \sigma^2 g^{-1} x)\}\, d\sigma + \frac{g}{V(\sigma_1 - \sigma_2)} \\ &\times \left\{(\sigma_1 - \omega) \int_0^\infty f_1(\sigma^2/g) \frac{\exp\{i(\sigma t - \sigma^2 x g^{-1})\}}{\sigma - \sigma_1}\, d\sigma \right. \\ &\left. - (\sigma_2 - \omega) \int_0^\infty f_1(\sigma^2/g) \frac{\exp\{i(\sigma t - \sigma^2 x g^{-1})\}}{\sigma - \sigma_2}\right\} d\sigma. \end{aligned} \tag{19}$$

The method of stationary phase gives for the first σ-integral the asymptotic value

$$\left(\frac{\pi g}{x}\right)^{\frac{1}{2}} f_1\left(\frac{gt^2}{4x^2}\right) \exp\left\{i\left(\frac{gt^2}{4x} - \tfrac{1}{4}\pi\right)\right\} \quad \text{for} \quad \frac{gt^2}{4x} \gg 1. \tag{20}$$

To use Van der Waerden's method in the second σ-integral we chose a new variable of integration

$$u_1 = \frac{\sigma}{\sigma_1} - \frac{gt}{2x\sigma_1}.$$

https://doi.org/10.1017/S030500410005458X Published online by Cambridge University Press

Let us suppose that the pole $\sigma = \sigma_1$ (i.e. $u_1 = u_{01} = 1 - (gt/2x\sigma_1)$ is sufficiently close to the saddle point $\sigma = gt/2x$ (or $u = 0$). The σ-integral in question is now

$$\int_{-gt/2x\sigma_1}^{\infty} (\sigma - \sigma_1)^{-1} f_1(\sigma^2/g) \frac{d\sigma}{du_1} \exp\{i(gt^2/4x - \sigma^2 x g^{-1} u_1^2)\} du_1. \tag{21}$$

Since $\sigma = \sigma_1$ (i.e. $u_1 = u_{01}$) is a simple pole of

$$(\sigma - \sigma_1)^{-1} f_1(\sigma^2/g) \frac{d\sigma}{du_1},$$

one may write

$$\frac{f_1(\sigma^2/g)\dfrac{d\sigma}{du_1}}{\sigma - \sigma_1} = \frac{A(\sigma_1)}{u_1 - u_{01}} + \text{a regular function } \sum_{n=0}^{\infty} \alpha_{n1} u_1^n.$$

It follows that

$$A(\sigma_1) = f_1(\sigma_1^2/g), \quad \alpha_{01} = \frac{\sigma_1\{f_1(gt^2/4x^2) - f_1(\sigma_1^2/g)\}}{(gt/2x) - \sigma_1}. \tag{22}$$

For $(\sigma_1^3 x/g) \to \infty$, Watson's lemma then reduces the integral in (21) to

$$A(\sigma_1) \exp\frac{igt^2}{4x} \int_{-\infty}^{\infty} (u_1 - u_{01})^{-1} \exp\{-i\sigma_1^2 x g^{-1} u_1^2\} du_1 + \alpha_{01} \exp\frac{igt^2}{4x} \int_{-\infty}^{\infty} \exp\{-i\sigma_1^2 x g^{-1} u_1^2\} du_1. \tag{23}$$

The extension of the range of integration is permissible as it contributes only a term $O(x^{-1})$ to (21). A little manipulation shows that the first integral in (23) is equal to

$$-\pi(1+i) \operatorname{sgn} u_{01} \exp\{-i\sigma_1^2 x g^{-1} u_{01}^2\} \operatorname{Cis}(\sigma_1^2 x g^{-1} u_{01}^2), \tag{24A}$$

where

$$\operatorname{Cis}(x) = C(x) + iS(x),$$

$$[C(x), S(x)] = (2\pi)^{-1} \int_0^x x^{-\frac{1}{2}} [\cos x, \sin x]\, dx.$$

The second integral in (23) is easily found to be

$$(1-i)\left(\frac{\pi g}{2\sigma_1^2 x}\right)^{\frac{1}{2}}. \tag{24B}$$

Combining the results from (21) to (24B), we obtain

$$\int_0^{\infty} \frac{f_1(\sigma^2/g)}{\sigma - \sigma_1} \exp\{i(\sigma t - \sigma^2 x g^{-1})\} d\sigma = -\pi(1+i) \operatorname{sgn} u_{01} f_1(\sigma_1^2/g) \operatorname{Cis}(\sigma_1^2 x g^{-1} u_{01}^2)$$
$$\times \exp\{igt^2/4x - i\sigma_1^2 x g^{-1} u_{01}^2\} + (1-i)\left(\frac{\pi g}{2\sigma_1^2 x}\right)^{\frac{1}{2}} \alpha_{01} \exp\{igt^2/4x\}. \tag{25}$$

The asymptotic value of the third σ-integral in (19) is obtained in the same way and under the same conditions just on replacing σ_1 by σ_2, u_{02} and α_{02} denoting the quantities corresponding to u_{01} and α_{01}. Multiplying this result and the result (25) respectively by

$$g\{V(\sigma_1 - \sigma_2)\}^{-1}(\omega - \sigma_2, \sigma_1 - \omega),$$

and adding the total to (20), we obtain the defined asymptotic value for $(-\frac{1}{2} V I_{14})$.

When $g < 4\omega V$ the poles σ_1 and σ_2 are complex and the dominant contribution to I_{14} for large t arises from the stationary point of the phase function

$$I_{14} \simeq 2\left(\frac{\pi g}{x}\right)^{\frac{1}{2}} \frac{gt^2 f_1(gt^2/4x^2)}{Vgt^2 - 2gtx + 4\omega x^2} \exp\{i(gt^2/4x - \tfrac{1}{4}\pi)\},$$

https://doi.org/10.1017/S030500410005458X Published online by Cambridge University Press

The other two integrals I_{34} and I_{43} given in (17) and (18) may each be expressed as a sum of three integrals as in (19). One of these three which corresponds to a real positive pole may be evaluated as above by the use of Van der Waerden's method and other integrals by the method of stationary phase.

The asymptotic values of other integrals in (13) may be obtained by the method of generalized functions suggested by Debnath and Rosenblat (1). For example the asymptotic value of I_{12} is given for large $(x - Vt)$ as

$$I_{12} \simeq 2\pi i \operatorname{sgn}(x-Vt)\left(1-\frac{4\omega V}{g}\right)^{\frac{1}{2}} \{\alpha_1 f_1(\alpha_1)\exp\{i(\sigma_1 t-\alpha_1 x)\}-\alpha_2 f_1(\alpha_2)\exp\{i(\sigma_2 t-\alpha_2 x)\} \quad \text{for} \quad g > 4\omega V.$$

Collecting all the results so obtained we finally obtain the following uniformly valid asymptotic approximation for η. Here H denotes the Heaviside step function

$$\begin{aligned} -2\rho g\eta = {} & [H(g-4\omega V)\{P(\sigma_1)\exp\{i(\sigma_1 t-\alpha_1 x)\}-P(\sigma_2)\exp\{i(\sigma_2 t-\alpha_2 x)\}\} \\ & -Q(\sigma_3)\exp\{i(\sigma_3 t-\alpha_3 x)\}+R(\sigma_3)\exp\{i(\sigma_3 t+\alpha_3 x)\}-\{i\bar{Q}(\sigma_4) \\ & +R(\sigma_4)\}\exp\{i(\alpha_4 x-\sigma_4 t)\}] \\ & +\frac{1}{V}\left(\frac{g}{\pi x}\right)^{\frac{1}{2}}[\{H(g-4\omega V)\,T+H(4\omega V-g)\,S(x)+iU(\sigma_3,\sigma_4)\}\exp\{i(gt^2/4x-\tfrac{1}{4}\pi)\} \\ & +\{U(\sigma_4,\sigma_3)-iS(-x)\}\exp\{-i(gt^2/4x-\tfrac{1}{4}\pi)\}], \end{aligned} \tag{26}$$

where

$$P(\sigma_1) = \left(1-\frac{4\omega V}{g}\right)^{-\frac{1}{2}} i\alpha_1 f_1(\alpha_1)\left\{\operatorname{sgn}(x-Vt)-(1-i)\operatorname{sgn}\left(1-\frac{gt}{2x\sigma_1}\right)\right. \\ \left. \times \operatorname{Cis}(\alpha_1 x-\sigma_1 t+gt^2/4x)\right\},$$

$$Q(\sigma_3) = \left(1+\frac{4\omega V}{g}\right)^{-\frac{1}{2}} \alpha_3 f_2(\alpha_3)\left\{1-(1-i)\operatorname{sgn}\left(1-\frac{gt}{2x\sigma_3}\right)\operatorname{Cis}(\alpha_3 x-\sigma_3 t+gt^2/4x)\right\},$$

$$R(\sigma_3) = \left(1+\frac{4\omega V}{g}\right)^{-\frac{1}{2}} i\alpha_3 f_2(\alpha_3)\{\operatorname{sgn}(x-Vt)-1\},$$

$$S(x) = \frac{Vgt^2 f_1(gt^2/4x^2)}{Vgt^2-2gtx+4\omega x^2},$$

$$T = f_1\left(\frac{gt^2}{4x^2}\right)+\frac{V}{g}\left(1-\frac{4\omega V}{g}\right)^{-\frac{1}{2}}(\sigma_1\alpha_{01}-\sigma_2\alpha_{02})$$

$$U(\sigma_3,\sigma_4) = f_2\left(\frac{gt^2}{4x^2}\right)+\frac{V}{g}\left(1+\frac{4\omega V}{g}\right)^{-\frac{1}{2}}\left\{\sigma_3\alpha_{02}-\frac{\sigma_4^2 f_2(gt^2/4x^2)}{gt/2x+\sigma_4}\right\}.$$

Evidently the wave motion depicted by (26) consists of a system of decaying dispersive waves superposed on a system of non-decaying travelling waves which are represented by the terms in the first square bracket on the right of (26):

$$\eta = \eta_{\mathrm{tr}} + \eta_{\mathrm{dis}}.$$

The waves η_{tr} have slowly varying amplitudes. They move respectively in the order in which they occur in (26) with velocities

$$g(\sigma_1^{-1}, \sigma_2^{-1}, \sigma_3^{-1}, -\sigma_3^{-1}, \sigma_4^{-1})$$

https://doi.org/10.1017/S030500410005458X Published online by Cambridge University Press

away from the cliff and may be, for brevity, referred to as σ_1-wave, σ_2-wave, σ_3'-wave, σ_3-wave and σ_4-wave. In the important limiting case $t \gg 1$ this progressive wave pattern takes a simpler form everywhere (within our region of approximation) excluding a sufficiently small neighbourhood of the four moving lines given by

$$x = x_j \equiv \frac{gt}{2\sigma_j},$$

and the pressure strip $x = Vt$. To discern this distribution, one has to note the asymptotic approximation

$$\mathrm{Cis} \sim 2^{-\frac{1}{2}} e^{i\frac{1}{4}\pi} + O(x^{-\frac{1}{2}}) \quad \text{for} \quad x \gg 1,$$

and the following inequalities:

$$\left.\begin{aligned} &\sigma_3 < \sigma_2 < \sigma_1 < \sigma_4, \\ x_4 < x_1 < Vt < x_2 < x_3 \quad &\text{when} \quad 4\omega V < g, \\ x_4 < Vt < x_3 \quad &\text{when} \quad g < 4\omega V < 3g \\ x_4 < x_3 < Vt \quad &\text{when} \quad 3g < 4\omega V. \end{aligned}\right\} \tag{27}$$

Then we have the following three cases.

Case I. $4\omega V < g$. For

$$x < x_4, \quad \eta_{\mathrm{tr}} = \left(1+\frac{4\omega V}{g}\right)^{-\frac{1}{2}} (\rho g)^{-1} \alpha_3 f_2(\alpha_3) \left[i \exp\{i(\alpha_3 x + \sigma_3 t)\} + \exp\{i(\sigma_3 t - \alpha_3 x)\}\right]$$

$$\equiv \eta_4^- \quad \text{(say)},$$

$$x_4 < x < x_1, \quad \eta_{\mathrm{tr}} = \eta_4^- - i\left(1+\frac{4\omega V}{g}\right)^{-\frac{1}{2}} (\rho g)^{-1} \alpha_4 f_2(\alpha_4) \exp\{i(\alpha_4 x - \sigma_4 t)\}$$

$$\equiv \eta_4^+ \quad \text{(say)},$$

$$x_1 < x < Vt, \quad \eta_{\mathrm{tr}} = \eta_4^+ + i\left(1-\frac{4\omega V}{g}\right)^{-\frac{1}{2}} (\rho g)^{-1} \alpha_1 f_1(\alpha_1) \exp\{i(\sigma_1 t - \alpha_1 x)\}$$

$$Vt < x < x_2, \quad \eta_{\mathrm{tr}} = (\rho g)^{-1} \left[i\left(1-\frac{4\omega V}{g}\right)^{-\frac{1}{2}} \alpha_2 f_1(\alpha_2) \exp\{i(\sigma_2 t - \alpha_2 x)\} \right.$$

$$\left. + \left(1+\frac{4\omega V}{g}\right)^{-\frac{1}{2}} \alpha_3 f_2(\alpha_3) \exp\{i(\sigma_3 t - \alpha_3 x)\} \right],$$

$$x_2 < x < x_3, \quad \eta_{\mathrm{tr}} = (\rho g)^{-1} \left(1+\frac{4\omega V}{g}\right)^{-\frac{1}{2}} \alpha_3 f_2(\alpha_3) \exp\{i(\sigma_3 t - \alpha_3 x)\},$$

$$x_3 < x, \quad \eta_{\mathrm{tr}} = 0.$$

Case II. $g < 4\omega V < 3g$. For

$$x < x_4, \quad \eta_{\mathrm{tr}} = \eta_4^-$$

$$x_4 < x < Vt, \quad \eta_{\mathrm{tr}} = \eta_4^+$$

$$Vt < x < x_3, \quad \eta_{\mathrm{tr}} = (\rho g)^{-1} \left(1+\frac{4\omega V}{g}\right)^{-\frac{1}{2}} \alpha_3 f_2(\alpha_3) \exp\{i(\sigma_3 t - \alpha_3 x)\},$$

$$x_3 < x, \quad \eta_{\mathrm{tr}} = 0.$$

https://doi.org/10.1017/S030500410005458X Published online by Cambridge University Press

Case III. $3g < 4\omega V$. For

$$x < x_4, \quad \eta_{\text{tr}} = \eta_4^-,$$

$$x_4 < x < x_3, \quad \eta_{\text{tr}} = \eta_4^+,$$

$$x_3 < x < Vt, \quad \eta_{\text{tr}} = i(\rho g)^{-1}\left(1+\frac{4\omega V}{g}\right)^{-\frac{1}{2}}[\alpha_4 f_2(\alpha_4)\exp\{i(\alpha_4 x-\sigma_4 t)\} - \alpha_3 f_2(\alpha_3)\exp\{i(\alpha_3 x+\sigma_3 t)\}],$$

$$Vt < x, \quad \eta_{\text{tr}} = 0.$$

The waves denoted by σ_1, σ_2, σ_3 and σ_4 are exactly the same waves as found in the fluid unbounded in both horizontal directions. But our method gives more information regarding their positions. It is seen that the waves denoted by σ_1, σ_3 and σ_4 exist behind the pressure strip and the σ_2-wave exists ahead of the pressure strip. The three waves denoted by σ_1, σ_2 and σ_4 always lie in the vicinity of the pressure strip, while the σ_3-wave goes far remote from the pressure strip. It is to be noted that the σ_3-wave has the greatest velocity of all the waves. The σ_3'-wave is the addition to the system due to the presence of the cliff. This wave has the velocity and amplitude similar to the σ_3-wave and moves away from the cliff. This leads us to conclude that it is the reflected σ_3-wave. The reason why only σ_3-wave is reflected is clear, for it is the only wave that being generated from the pressure strip moves towards the cliff.

The author is grateful to Professor A. R. Sen of Visva-Bharati University for his help and guidance during the preparation of this paper.

REFERENCES

(1) DEBNATH, L. and ROSENBLAT, S. The ultimate approach to the steady state in the generation of waves on a running stream. *Quart. J. Mech. Appl. Math.* **22** (1969), 221.

(2) KAPLAN, P. The waves generated by the forward motion of oscillatory pressure distribution. *Proc. 5th Midwest Conf. Fluid Mech.* (1957), 316.

(3) STOKER, J. J. *Water waves* (New York, Interscience, 1957).

(4) VAN DER WAERDEN, B. L. *Appl. Sci. Res.* **2** (1951), 353.

https://doi.org/10.1017/S030500410005458X Published online by Cambridge University Press