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Weak bisimulations for the Giry monad[†]

ERNST-ERICH DOBERKAT[‡]

Chair for Software Technology,
Technische Universität Dortmund
Email: ernst-erich.doberkat@udo.edu

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We study the existence of bisimulations for Kleisli morphisms associated with the Giry monad of subprobabilities over Polish spaces. We first investigate these morphisms and show that the problem can be reduced to the existence of bisimulations for objects in the base category of stochastic relations using simulation equivalent congruences. This leads us to a criterion for two objects to be bisimilar.

1. Introduction

The Giry monad (Giry 1981) provides a categorical approach to the measure-theoretic foundations of probability theory, and because it allows the representation of a variety of models in computer science, it has recently attracted some attention with respect to: the foundations of concurrency theory (like bisimulations); Kripke models for modal logics with applications to model checking; and even to modelling software architectures (see Doberkat (2007b; 2009) for overviews of recent developments.

E. Moggi has proposed (Moggi 1991) a monadic model of computation. Functor $\mathbf{T}$ acts as a type constructor, so that if A is a type, then $\mathbf{T}A$ is the object of computations of type A. Assuming that $\mathbf{T}$ is the functorial part of a monad, the Kleisli category for this monad is identified as the category of programs: morphisms in this category are the programs.

The current paper is about these morphisms in the Giry monad, which are usually referred to as *stochastic relations*. We study the problem of determining the conditions under which two stochastic relations are bisimilar in the Kleisli category for the Giry monad. Thus, given two stochastic relations $\mathscr{K}$ and $\mathscr{L}$, we ask under which conditions can we find a stochastic relation $\mathscr{M}$ and two weak morphisms $\Phi : \mathscr{M} \rightharpoonup \mathscr{K}$ and $\Psi : \mathscr{M} \rightharpoonup \mathscr{K}$, hence we want to know the conditions under which a span of morphisms in the Kleisli category for the Giry monad exists. The strong case, that is, the case where F and G are both replaced by Borel measurable maps and Kleisli composition is replaced by the ordinary composition of maps, has been studied extensively in different settings (Doberkat 2007b; 2009; Doberkat and Schubert 2009), and general criteria have been established. The case of weak morphisms, however, has not received sufficient attention yet.

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This problem is interesting for a variety of reasons. First, it is known that there is an intimate connection between bisimilarity and Kripke models for modal logics, and not only in the case when the logic is interpreted coalgebraically through a set-based functor (Schröder and Pattinson 2007) but also when the subprobability based functor is used (Doberkat 2009). It can be shown that the weak probabilistic case is by no means as clear cut as the strong case, and is thus rather more interesting. Since bisimilarity is a basic notion in the theory of concurrency, greater knowledge about an interesting special case is welcome. From a technical point of view, it is worth noting that weak morphisms are closely related to strong ones, which provides some insight into the inner workings of the Kleisli construction for the Giry monad.

The problem is tackled here in terms of congruences with an approach similar to the one proposed in Doberkat (2006). After we have recalled some familiar constructions for the Giry monad in Section 2, we introduce congruences in Section 3. In Section 4, we reduce the problem of finding a weak morphism to the strong case through a construction that resembles a tensor product, and we analyse congruences that occur in this case. We know that the existence of simulation equivalent congruences implies the existence of a bisimulation for the strong case, hence we study simulation equivalence in the context of the transformed problem. After considering some transformations involving the corresponding factor spaces, we show that two stochastic relations are weakly bisimilar if we can find simulation equivalent congruences on the base spaces that are compatible with the Kleisli composition. In the final section, Section 5, we present our conclusions and propose further work.

2. Some constructions associated with the Giry monad

This section recalls some basic facts about the Giry monad, and at the same time introduces some tools for later use. These tools come mainly from measure theory and from topology.

We will need the following statement on uniqueness – it hides a proof principle that is applied quite often in this area. The principle is akin to mathematical induction: we first show that a property holds for a $\cap$-closed generator $\mathscr{A}_0$ of a σ-algebra, and then in a second step show that the sets for which the property holds is closed under complements and countable disjoint unions. We may then conclude that all elements of $\sigma(\mathscr{A}_0)$ enjoy this property. Formally, it is based on the well-known π-λ-Theorem of measure theory (Kechris 1994, Theorem 10.1) – see Doberkat (2007b, Theorem 1.1) for a discussion:

Lemma 2.1. Let $(X, \mathscr{A})$ be a measurable space and $\mathscr{A}_0 \subseteq \mathscr{A}$ with $\mathscr{A} = \sigma(\mathscr{A}_0)$ such that $\mathscr{A}_0$ is closed under finite intersections. Let $\mathscr{D} \subseteq \mathscr{A}$ with $\mathscr{A}_0 \subseteq \mathscr{D}$ such that $\mathscr{D}$ is closed under complements and countable disjoint unions. Then $\mathscr{A} = \mathscr{D}$. In particular, if $\mu_1, \mu_2 \in \mathbf{S}(X, \mathscr{A})$ are subprobabilities such that $\mu_1(A) = \mu_2(A)$ for all $A \in \mathscr{A}_0$, then $\mu_1(A) = \mu_2(A)$ for all $A \in \mathscr{A}$.

Given a measurable space $(X, \mathscr{A})$, the space $\mathbf{S}(X, \mathscr{A})$ is the space of all subprobability measures on $(X, \mathscr{A})$. It is again a measurable space with the $*$-σ-algebra $\mathscr{A}^\bullet$. This is the smallest σ-algebra on $\mathbf{S}(X, \mathscr{A})$ that makes all the evaluations $\mu \mapsto \mu(Q)$ with $Q \in \mathscr{A}$

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measurable. Defining for another measurable space $(Y, \mathscr{B})$ and an $\mathscr{A}$-$\mathscr{B}$-measurable map $f : X \to Y$ the map $\mathbf{S}(f) \to \mathbf{S}(Y)$ through $\mathbf{S}(f)(\mu)(B) := \mu(f^{-1}[B])$ constitutes an $\mathscr{A}^\bullet$-$\mathscr{B}^\bullet$-measurable map. The subprobability functor $\mathbf{S}$ is an endofunctor on the category of measurable spaces. This functor is the functorial part of a monad, the *Giry monad* (Giry 1981). The *Kleisli construction* (Mac Lane 1997) for this monad is particularly interesting: if $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ are measurable spaces, then a stochastic relation $K : (X, \mathscr{A}) \rightsquigarrow (Y, \mathscr{B})$ is a map $K : X \to \mathbf{S}(Y, \mathscr{B})$ that is $\mathscr{A}$-$\mathscr{B}^\bullet$-measurable – stochastic relations are just the Kleisli morphisms for the Giry monad. A comparison with the Kleisli morphisms for the powerset monad may be instructive: they are exactly the relations (Mac Lane 1997, VI.5), and, in fact, there are many other interesting and non-obvious similarities – see, for example, Doberkat (2007b). In probability theory, stochastic relations are known as subprobability kernels – the algebraic context will usually not be considered there.

The *Kleisli composition* of the stochastic relations $K : (X, \mathscr{A}) \rightsquigarrow (Y, \mathscr{B})$ and $L : (Y, \mathscr{B}) \rightsquigarrow (Z, \mathscr{C})$ is the stochastic relation $L * K : (X, \mathscr{A}) \rightsquigarrow (Z, \mathscr{C})$ defined by

$$\big(L * K\big)(x)(G) := \int_Y L(y)(G)\; K(x)(dy)$$

with $(x \in X, G \in \mathscr{C})$. In probability theory, this operation is usually referred to as the *convolution of the kernels K and L*.

The Change of Variables Theorem will be an important tool. It uses the basic fact that a measurable map between measurable spaces induces a measurable map between the corresponding subprobabilities.

Lemma 2.2. Let $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ be analytic spaces, and assume that $f : X \to Y$ is $\mathscr{A}$-$\mathscr{B}$-measurable. Then

$$\int_Y g(y)\; \mathbf{S}(f)(\mu)(dy) = \int_X \big(g \circ f\big)(x)\; \mu(dx)$$

whenever $g : Y \to \mathbb{R}$ is $\mathscr{B}$-measurable and bounded. In particular,

$$\int_B g(y)\; \mathbf{S}(f)(\mu)(dy) = \int_{f^{-1}[B]} \big(g \circ f\big)(x)\; \mu(dx) \tag{1}$$

whenever $B \subseteq Y$ is a Borel set.

Proof. The first part is discussed, for example, in Doberkat (2007b, Chapter 1) and the second part follows easily (here $1_B : y \mapsto$ `if` $y \in B$ `then 1 else 0 fi` denotes the indicator function of a set B):

$$\begin{aligned}
\int_B g(y)\; \mathbf{S}(f)(\mu)(dy) &= \int_Y (g \cdot 1_B)(y)\; \mathbf{S}(f)(\mu)(dy) \\
&= \int_X (g \cdot 1_B)(f(x))\; \mu(dx) \\
&= \int_X (g \circ f)(x) \cdot (1_B \circ f)(x)\; \mu(dx) \\
&= \int_{f^{-1}[B]} \big(g \circ f\big)(x)\; \mu(dx).
\end{aligned}$$

□

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We define the coproduct $\mathscr{K} + \mathscr{K}'$ of the stochastic relations $\mathscr{K} = ((X, \mathscr{A}), (Y, \mathscr{B}), K)$ and $\mathscr{K}' = ((X', \mathscr{A}'), (Y', \mathscr{B}'), K')$ by $((X + X', \mathscr{A} + \mathscr{A}'), (Y + Y', \mathscr{B} + \mathscr{B}'), K + K')$, where

$$\big(K + K'\big)(z) := \begin{cases} \big(\mathbf{S}(\mathrm{inj}_Y) \circ K\big)(x) & \text{if } z = \mathrm{inj}_X(x) \\ \big(\mathbf{S}(\mathrm{inj}_{Y'}) \circ K'\big)(x') & \text{if } z = \mathrm{inj}_{X'}(x'). \end{cases}$$

Here, for example, inj_Y is the injection $Y \to Y + Y'$ from Y into the sum $Y + Y'$. Thus we have, for example, for $x \in X$ and the measurable set $T \subseteq Y + Y'$, the identity

$$\big(K + K'\big)(\mathrm{inj}_X(x))(T) = K(x)(T \cap Y).$$

It is not difficult to see that this defines the coproduct in the category of stochastic relations, which anticipates the definition of strong morphisms in Definition 4.1.

It is instructive to compute the integral with respect to this sum, even though the result is not too surprising.

Lemma 2.3. Using the above notation, let $f : Y + Y' \to \mathbb{R}$ be Borel measurable and bounded. Then

$$\int_{Y+Y'} f(\bar{y})\ \big(K + K'\big)(\mathrm{inj}_X(x))(d\bar{y}) = \int_Y \big(f \circ \mathrm{inj}_Y\big)(y)\ K(x)(dy)$$

holds for $x \in X$, and similarly for $x' \in X'$.

Proof. Additivity of the integral and the Change of Variables Theorem (Lemma 2.2) yield

$$\begin{aligned} &\int_{Y+Y'} f(\bar{y})\ \big(K + K'\big)(\mathrm{inj}_X(x))(d\bar{y}) \\ &\quad = \int_{\mathrm{inj}_Y[Y]} f(\bar{y})\ \big(K + K'\big)(\mathrm{inj}_X(x))(d\bar{y}) + \int_{\mathrm{inj}_{Y'}[Y']} f(\bar{y})\ \big(K + K'\big)(\mathrm{inj}_X(x))(d\bar{y}) \\ &\quad = \int_{\mathrm{inj}_Y[Y]} f(\bar{y})\ \big(\mathbf{S}(\mathrm{inj}_Y) \circ K\big)(x)(d\bar{y}) + \int_{\mathrm{inj}_{Y'}[Y']} f(\bar{y})\ \big(\mathbf{S}(\mathrm{inj}_Y) \circ K\big)(x)(d\bar{y}) \\ &\quad = \int_Y \big(f \circ \mathrm{inj}_Y\big)(y)\ K(x)(dy) + \int_\varnothing \big(f \circ \mathrm{inj}_Y\big)(y)\ K(x)(dy) \\ &\quad = \int_Y \big(f \circ \mathrm{inj}_Y\big)(y)\ K(x)(dy), \end{aligned}$$

where we have used the fact that both $Y = \mathrm{inj}_Y^{-1}[\mathrm{inj}_Y[Y]]$ and $\varnothing = \mathrm{inj}_Y^{-1}[\mathrm{inj}_{Y'}[Y']]$ hold. □

This construction will enable us to reduce cospans of weak morphisms to weak morphisms later in the paper.

If X is a *Polish space*, that is, a Hausdorff space with a countable dense subset for which a complete metric exists, then X becomes a measurable space with the *Borel sets* $\mathscr{B}(X)$ as a σ-algebra. $\mathscr{B}(X)$ is the smallest σ-algebra that contains the open (or, equivalently, the closed) sets of X. We suppress $\mathscr{B}(X)$ usually from the notation of a Polish space X. Then $\mathbf{S}(X)$ is a Polish space as well (Parthasarathy 1967), taking the topology of weak

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convergence as a topology, and it is well known that the $*$-σ-algebra is just given by the Borel sets for this topology.

We again let $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ be measurable spaces, and use $\mathscr{A} \otimes \mathscr{B}$ to denote the product σ-algebra on $X \times Y$. For a stochastic relation $K : (X, \mathscr{A}) \rightsquigarrow (Y, \mathscr{B})$ and a subprobability $\mu \in \mathbf{S}(Y, \mathscr{B})$ on $(X \times Y, \mathscr{A} \otimes \mathscr{B})$, we define the measure

$$\big(\mu \otimes K\big)(H) := \int_X K(x)(H_x)\ \mu(dx)$$

(here $H_x := \{y \in Y \mid \langle x, y\rangle \in H\}$ is the vertical cut of H at $x \in X$). It is folklore in probability theory that this defines a measure on the product space. The following lemma will sometimes be helpful in expressing the Kleisli product in a more convenient manner.

Lemma 2.4. $x \mapsto K(x) \otimes L$ defines a stochastic relation $K(\cdot) \otimes L : (X, \mathscr{A}) \rightsquigarrow (Y \times Z, \mathscr{B} \otimes \mathscr{C})$ such that

$$\big(L * K\big)(x)(G) = \mathbf{S}\left(\pi_{Y\times Z,Z}\right)\big(K(x) \otimes L\big)(G)$$

for all $G \in \mathscr{C}$.

Proof.

(1) It is clear that

$$K(x) \otimes L \in \mathbf{S}(Y \times Z, \mathscr{B} \otimes \mathscr{C}).$$

So we have to show that $x \mapsto \big(L * K\big)(x)(H)$ is $\mathscr{A}$-measurable for every $H \in \mathscr{B} \otimes \mathscr{C}$. To do this, we let

$$\mathscr{D} := \{H \in \mathscr{B} \otimes \mathscr{C} \mid x \mapsto \big(L * K\big)(x)(H) \text{ is } \mathscr{A}\text{-measurable}\}$$

Then $\mathscr{D} \subseteq \mathscr{B} \otimes \mathscr{C}$ is closed under complementation and countable disjoint unions. Because

(a) $\big(K(x) \otimes L\big)(E \times F) = \int_E L(y)(F)\ K(x)(dy)$,

(b) $x \mapsto \int_E f\ dK(x)$ is measurable whenever f is a measurable map, and

(c) E is a measurable set,

we can infer that the set of all measurable rectangles

$$\mathscr{V} := \{E \times F \mid E \in \mathscr{B}, F \in \mathscr{C}\}$$

is a subset of $\mathscr{D}$. Because $\mathscr{V}$ is closed under finite intersections, and since we have $\sigma(\mathscr{V}) = \mathscr{B} \otimes \mathscr{C}$, we know that $\mathscr{D} = \mathscr{B} \otimes \mathscr{C}$ by the π-λ-Theorem (Lemma 2.1).

(2) Let $G \in \mathscr{C}$. Then

$$\begin{aligned}
\mathbf{S}\left(\pi_{Y\times Z,Z}\right)\big(K(x) \otimes L\big)(G) &= \big(K(x) \otimes L\big)(\pi^{-1}_{Y\times Z,Z}[G]) \\
&= \big(K(x) \otimes L\big)(Y \times G) \\
&= \int_Y L(y)\big((Y \times G)_y\big)K(x)(dy) \\
&= \int_Y L(y)(G)\ K(x)(dy) \\
&= \big(L * K\big)(x)(G).
\end{aligned}$$

□

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Now let X and Y be Polish spaces, and assume that a measure $\mu \in \mathbf{S}(X \times Y)$ is given. It is well known that μ can be represented through a measure on X and a stochastic relation $K : X \rightsquigarrow Y$, as given by the following theorem (see Parthasarathy (1967, Theorem V.8.1) or Doberkat (2007b, Section 1.5.3))

Theorem 2.5. Given $\mu \in \mathbf{S}(X \times Y)$, we have

$$\mu = \mathbf{S}\left(\pi_{X\times Y,X}\right)(\mu) \otimes K$$

for some stochastic relation $K : X \rightsquigarrow Y$, and this K is unique $\mathbf{S}(\pi_{X\times Y,X})(\mu)$-almost everywhere. K is said to be a *disintegration* of μ with respect to X, Y.

Thus, whenever $\mu \in \mathbf{S}(X \times Y)$ and $H \in \mathscr{B}(X \times Y)$, we can write

$$\mu(H) = \int_X K(x)(H_x)\, \mathbf{S}\left(\pi_{X\times Y,X}\right)(\mu)(dx).$$

Because a disintegration is only determined uniquely up to a set of measure zero, if K_1 and K_2 are both disintegrations of μ with respect to X, Y, then

$$\mathbf{S}\left(\pi_{X\times Y,X}\right)(\mu)\big(\{x \in X \mid K_1(x) \neq K_2(x)\}\big) = 0.$$

3. Smooth relations

If $(X, \mathscr{A})$ is a measurable space, then an equivalence relation ϱ on X is said to be *smooth* (or *countably generated*) if and only if there exists a sequence $(Q_n)_{n\in\mathbb{N}}$ of sets in $\mathscr{A}$ such that

$$x \,\varrho\, x' \text{ iff } \forall n \in \mathbb{N} : [x \in Q_n \Leftrightarrow x' \in Q_n].$$

The sequence $(Q_n)_{n\in\mathbb{N}}$ is said to *determine* ϱ. For example, when we construct an equivalence relation from the sets on which the formulas of a modal logic are valid (Doberkat 2006), these sets form a determining sequence. In this case two states are equivalent if and only if they cannot be separated by the logic.

Turning to the measurable structure of the factor space, it is well known that if X is Polish, the factor space X/ϱ is analytic (that is, homeomorphic to a continuous image of a Polish space) whenever ϱ is smooth, provided the factor carries the largest σ-algebra that makes the factor map $\eta_\varrho : x \mapsto [x]_\varrho$ $\mathscr{B}(X)$-measurable (Srivastava 1998, Exercise 5.1.14). As a rule, the factor spaces are analytic spaces, even if the base spaces are Polish – see Doberkat and Schubert (2009, Example 2.7) for a discussion.

We say a set $B \subseteq X$ is *ϱ-invariant* if and only if $B = \bigcup\{[x]_\varrho \mid x \in B\}$, so that $x \in B$ and $x \,\varrho\, x'$ implies $x' \in B$. The *ϱ-invariant $\mathscr{A}$-measurable sets*

$$\mathbf{\Sigma}(\mathscr{A}, \varrho) := \{C \in \mathscr{A} \mid C \text{ is } \varrho\text{-invariant}\}$$

form a σ-algebra for a measurable space $(X, \mathscr{A})$. The following lemma provides a collection of results for smooth equivalence relations that will be used silently throughout the rest of the paper.

Lemma 3.1. Let X be a Polish space and ϱ be a smooth equivalence relation with the Borel sets $(Q_n)_{n\in\mathbb{N}}$ as determining sequence. Then:

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(a) $\mathbf{\Sigma}(\mathscr{B}(X), \varrho) = \sigma(\{Q_n \mid n \in \mathbb{N}\})$.
(b) $C \in \mathscr{B}(X/\varrho)$ if and only if $\eta_\varrho^{-1}[C] \in \mathbf{\Sigma}(\mathscr{B}(X), \varrho)$ and $\eta_\varrho^{-1}[\eta_\varrho[A]] = A$ for each $A \in \mathbf{\Sigma}(\mathscr{B}(X), \varrho)$.
(c) $x \,\varrho\, x'$ if and only if $[x \in Q \Leftrightarrow x' \in Q]$ holds for all $Q \in \mathbf{\Sigma}(\mathscr{B}(X), \varrho)$.
(d) The equivalence classes $[x]_\varrho$ are exactly the atoms of the σ-algebra $\mathbf{\Sigma}(\mathscr{B}(X), \varrho)$.

Proof.

(1) Properties (a) and (b) are proved in Srivastava (1998). Let $\eta_\varrho(a) \in \eta_\varrho[A]$ for some ϱ-invariant Borel set $A \subseteq X$. Hence, there exists $a' \in A$ with $\eta_\varrho(a) = \eta_\varrho(a')$. Since A is ϱ-invariant, we conclude from $a \,\varrho\, a'$ that $a \in A$. This proves $\eta_\varrho^{-1}[\eta_\varrho[A]] \subseteq A$; the other inclusion is trivial.
(2) Property (c) follows from Srivastava (1998, Lemma 3.1.6), and is used to establish property (d) as follows. Let $x \in X$ and assume that there exists $\varnothing \neq C \subset [x]_\varrho$. Hence, there exists $x' \in [x]_\varrho \setminus C$. Then $x' \,\varrho\, x''$ for each $x'' \in C$, which is a contradiction. Thus $[x]_\varrho$ is an atom of $\mathbf{\Sigma}(\mathscr{B}(X), \varrho)$. Conversely, if A is an atom, let $x \in A$. Assuming $x \,\varrho\, x'$ with $x' \notin A$, this then contradicts property (d) again. Thus $[x]_\varrho \subseteq A$. Since both are atoms, they must be equal. □

As a consequence of the characterisation of the atoms of $\mathbf{\Sigma}(\mathscr{B}(X), \varrho)$, we note that the $\mathbf{\Sigma}(\mathscr{B}(X), \varrho)$-measurable real functions are constant on the equivalence classes.

Corollary 3.2. Let X and ϱ be as in Lemma 3.1. If $f : X \to \mathbb{R}$ is $\mathbf{\Sigma}(\mathscr{B}(X), \varrho)$-measurable, then $\varrho \subseteq \mathsf{ker}(f)$.

Proof. Assume that there exist for some $\overline{x} \in X$ elements $x', x'' \in [\overline{x}]_\varrho$ such that $f(x') < q < f(x'')$. Then $\{x \in X \mid f(x) < q\} \in \mathbf{\Sigma}(\mathscr{B}(X), \varrho)$ is a proper subset of $[\overline{x}]_\varrho$, which contradicts the fact that $[\overline{x}]_\varrho$ is an atom of $\mathbf{\Sigma}(\mathscr{B}(X), \varrho)$. □

A pair (α, β) of smooth equivalence relations α on X and β on Y (with both X and Y Polish) is said to be a *congruence* for the stochastic relation $K : X \rightsquigarrow Y$ if and only if $K(x)(B) = K(x')(B)$ whenever $x \,\alpha\, x'$ and $B \subseteq Y$ is a β-invariant Borel set. Thus, if α cannot distinguish x from x', and if β cannot discern the elements of B, then $K(x)$ and $K(x')$ both assign the same probability to B. Given a congruence (α, β) for $K : X \rightsquigarrow Y$, we construct the factor relation $K_{(\alpha,\beta)} : X/\alpha \rightsquigarrow Y/\beta$ by setting

$$K_{(\alpha,\beta)}\big([x]_\alpha\big)(G) := K(x)\big(\eta_\beta^{-1}[G]\big) = \big(\mathbf{S}\,(\eta_\beta) \circ K\big)(x)(G)$$

for $x \in X, G \in \mathscr{B}(Y/\beta)$. The following proposition provides a characterisation of congruences in terms of the relation involved (Doberkat 2007a, Lemma 2.8).

Proposition 3.3. The following statements are equivalent for the Polish spaces X and Y

(a) (α, β) is a congruence for $K : X \rightsquigarrow Y$.
(b) $K : \mathbf{\Sigma}(\mathscr{B}(X), \alpha) \to \mathbf{\Sigma}(\mathscr{B}(Y), \beta)$ is a stochastic relation.

The notion of simulation equivalent congruences will be important in our investigation of bisimilarity. A necessary condition for the bisimilarity of stochastic relations is the existence of equivalent congruences on them. This condition is technically a bit involved because it requires the notion of the mutual generation of smooth equivalence relations;

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this concept is called *spawning*, and is discussed in detail in Doberkat (2006), as is the closely related concept of simulation equivalent congruences.

Lemma 3.4. Let α and β be smooth equivalence relations on the Polish spaces X and Y, respectively, and assume that $\mathfrak{p} : X/\alpha \to Y/\beta$ is a map between the equivalence classes. For $A \subseteq X$, put

$$\sqrt{\mathfrak{p}}(A) := \bigcup\{\mathfrak{p}([x]_\alpha) \mid x \in A\}.$$

Then

$$\sqrt{\mathfrak{p}} : \mathbf{\Sigma}(\mathscr{B}(X), \alpha) \to \mathbf{\Sigma}(\mathscr{B}(Y), \beta).$$

If $\mathfrak{p}$ is injective, $\sqrt{\mathfrak{p}}$ is a Boolean σ-morphism.

Proof.

(1) Because, by Lemma 3.1, the equivalence classes are the atoms of the invariant Borel sets, the Blackwell–Mackey Theorem (Doberkat 2007b, Theorem 1.54) shows that $\sqrt{\mathfrak{p}}(A) \in \mathbf{\Sigma}(\mathscr{B}(Y), \beta)$ whenever $A \in \mathbf{\Sigma}(\mathscr{B}(X), \alpha)$. This is because β is smooth, and thus $\mathbf{\Sigma}(\mathscr{B}(Y), \beta)$ is countably generated.

(2) It is clear that $\sqrt{\mathfrak{p}}$ is monotone. Let A_1 and A_2 be α-invariant. Then we have

$$\sqrt{\mathfrak{p}}(A_1 \cap A_2) = \sqrt{\mathfrak{p}}(A_1) \cap \sqrt{\mathfrak{p}}(A_2).$$

In fact, let $y \in \sqrt{\mathfrak{p}}(A_1) \cap \sqrt{\mathfrak{p}}(A_2)$ so that $[y]_\beta = \mathfrak{p}([x_i]_\alpha)$ for some $x_i \in A_i, i = 1, 2$. Since $\mathfrak{p}$ is injective, we know that $[x_1]_\alpha = [x_2]_\alpha$, so we may conclude that $x_2 \in A_1$ and $x_1 \in A_2$ from the invariance of A_1 and A_2. Thus we infer

$$\sqrt{\mathfrak{p}}(A_1 \cap A_2) \supseteq \sqrt{\mathfrak{p}}(A_1) \cap \sqrt{\mathfrak{p}}(A_2).$$

The other inclusion is trivial. The same argument applies for countable intersections. We see also from this argument that $\sqrt{\mathfrak{p}}(A_1)$ and $\sqrt{\mathfrak{p}}(A_2)$ are disjoint provided A_1 and A_2 are, and it is easy to see that countable unions are also preserved. Since for $A_1 \subseteq A_2$ we have

$$\sqrt{\mathfrak{p}}(A_1 \setminus A_2) \cup \sqrt{\mathfrak{p}}(A_2) = \sqrt{\mathfrak{p}}(A_1),$$

and since

$$\sqrt{\mathfrak{p}}(A_1 \setminus A_2) \cap \sqrt{\mathfrak{p}}(A_2) = \sqrt{\mathfrak{p}}\big((A_1 \setminus A_2) \cap A_2\big) = \varnothing,$$

we can conclude that

$$\sqrt{\mathfrak{p}}(A_1 \setminus A_2) = \sqrt{\mathfrak{p}}(A_1) \setminus \sqrt{\mathfrak{p}}(A_2).$$

□

The fact that $\sqrt{\mathfrak{p}}(A) = \eta_\beta^{-1}[\mathfrak{p}[\eta_\alpha[A]]]$ will be useful in the following.

Definition 3.5. Let α and β be smooth equivalence relations on the Polish spaces X and Y, respectively, and assume that $\mathfrak{p} : X/\alpha \to Y/\beta$ is an injective map between the equivalence classes. We say that α *spawns* β *via* $(\mathfrak{p}, \mathscr{A}_0)$ if and only if $\mathscr{A}_0$ is a countable generator $\mathbf{\Sigma}(\mathscr{B}(X), \alpha)$ such that $\{\sqrt{\mathfrak{p}}(A) | A \in \mathscr{A}_0\}$ is a generator of $\mathbf{\Sigma}(\mathscr{B}(Y), \beta)$.

Thus, if α spawns β, then the measurable structure induced by α on X is all we need for constructing the measurable structure induced by β on Y : the map $\mathfrak{p}$ can be made to carry over the generator $\mathscr{A}_0$ from $\mathbf{\Sigma}(\mathscr{B}(X), \alpha)$ to $\mathbf{\Sigma}(\mathscr{B}(Y), \beta)$ and to transport the atoms from

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one σ-algebra to the other. This is of particular interest since the atoms constitute the equivalence classes. We will show that the specific generator in question is unimportant because the map between the classes is injective, so any generator will do.

Lemma 3.6. Let X, Y, α, β and $\mathfrak{p} : X/\alpha \to Y/\beta$ be as in Definition 3.5, and assume that

$$\mathbf{\Sigma}(\mathscr{B}(Y), \beta) = \sigma\left(\left\{\sqrt{\mathfrak{p}}(A) \mid A \in \mathscr{A}_0\right\}\right)$$

for some countable generator $\mathscr{A}_0$ of $\mathbf{\Sigma}(\mathscr{B}(X), \alpha)$. If $\widetilde{\mathscr{A}_0}$ is another countable generator of $\mathbf{\Sigma}(\mathscr{B}(X), \alpha)$, then $\left\{\sqrt{\mathfrak{p}}(A) \mid A \in \widetilde{\mathscr{A}_0}\right\}$ generates the σ-algebra $\mathbf{\Sigma}(\mathscr{B}(Y), \beta)$.

Proof.

(1) Since each element A of $\widetilde{\mathscr{A}_0}$ is a union of equivalence classes, we can infer from the Blackwell–Mackey Theorem that $\sqrt{\mathfrak{p}}(A)$ is a union of equivalence classes, and thus is a member of $\mathbf{\Sigma}(\mathscr{B}(Y), \beta)$. Consequently,

$$\mathscr{D} := \sigma\left(\left\{\sqrt{\mathfrak{p}}(A) \mid A \in \widetilde{\mathscr{A}_0}\right\}\right)$$

is a σ-algebra that is countably generated and is contained in $\mathbf{\Sigma}(\mathscr{B}(Y), \beta)$.

(2) Let τ be the smooth equivalence relation associated with $\left\{\sqrt{\mathfrak{p}}(A) \mid A \in \widetilde{\mathscr{A}_0}\right\}$, and assume that $y \,\tau\, y'$. Either $[y]_\beta$ and $[y']_\beta$ are both in the image $\mathfrak{p}[X/\alpha]$ or neither of them is. In the latter case, we may infer $y \,\beta\, y'$, and in the former case, we know that $[y]_\beta = \mathfrak{p}([x]_\alpha)$ and $[y']_\beta = \mathfrak{p}\left(\left[x'\right]_\alpha\right)$ for some $x, x' \in X$. Thus we know for each α-invariant Borel set $A \subseteq X$ that

$$y \in \sqrt{\mathfrak{p}}(A) \iff y' \in \sqrt{\mathfrak{p}}(A).$$

Hence we may conclude, since each A is α-invariant and because $\mathfrak{p}$ is injective, that $[x]_\alpha = [x']_\alpha$, which means that $y \,\beta\, y'$, and thus that $\tau \subseteq \beta$. The converse inclusion is obvious, so $\beta = \tau$, and

$$\mathscr{D} = \mathbf{\Sigma}(\mathscr{B}(Y), \tau) = \mathbf{\Sigma}(\mathscr{B}(Y), \beta).$$ □

Definition 3.7. Let $K : X \rightsquigarrow Y$ and $K' : X' \rightsquigarrow Y'$ be stochastic relations over analytic spaces on which congruences (α, β) and $\left(\alpha', \beta'\right)$, respectively, are defined. Then:

(a) We say (α, β) is *proportional to* (α', β') (written $(\alpha, \beta) \propto (\alpha', \beta')$) if and only if α spawns α' via $(\mathfrak{p}, \mathscr{A}_0)$ and β spawns β' via $(\mathfrak{q}, \mathscr{B}_0)$ such that both $\mathscr{A}_0$ and $\mathscr{B}_0$ are closed under finite intersections and

$$\forall x \in X \forall x' \in \mathfrak{p}([x]_\alpha) \forall B \in \mathscr{B}_0 : K(x)(B) = K'(x')(\sqrt{\mathfrak{q}}(B)).$$

(b) We say these congruences are *simulation equivalent* if and only if both $(\alpha, \beta) \propto (\alpha', \beta')$ and $(\alpha', \beta') \propto (\alpha, \beta)$

We require the generators to be closed under finite intersections since this is, by the π-λ-Theorem, Lemma 2.1, a helpful condition for ensuring the uniqueness of a measure. Simulation equivalent congruences behave in identical fashion on the class structure induced by the respective congruences.

If α and α' are smooth equivalence relations on X and X', respectively, we define for $\bar{x}, \bar{x}' \in X + X'$ the equivalence relation $\alpha + \alpha'$ on the coproduct $X + X'$ through

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$[\mathrm{inj}_X(x)]_{\alpha+\alpha'} := [x]_\alpha$ whenever $x \in X$, and $[\mathrm{inj}_{X'}(x')]_{\alpha+\alpha'} := [x']_{\alpha'}$ whenever $x' \in X'$. Thus $(X + X')/(\alpha + \alpha')$ is isomorphic to $X/\alpha + X'/\alpha'$, so both spaces will be identified. Simulation equivalence is preserved through coproducts. The proof is fairly straighforward.

Lemma 3.8. Let $\mathscr{K}$ and $\mathscr{L}$ be stochastic relations with simulation equivalent congruences (χ, α) and (υ, β) and assume that $\mathscr{K}'$ and $\mathscr{L}'$ are stochastic relations with simulation equivalent congruences (χ', α') and (υ', β'). Then $(\chi + \chi', \alpha + \alpha')$ and $(\upsilon + \upsilon', \beta + \beta')$ are simulation equivalent congruences for $\mathscr{K} + \mathscr{K}'$ and $\mathscr{L} + \mathscr{L}'$.

4. Weak bisimilarity

We will now define weak and strong morphisms in order to be able to define bisimilarity. We will introduce this property through a span of morphisms, and thus take a coalgebraic point of view here. In the following, all spaces are Polish.

Definition 4.1. Let $\mathscr{K} = (X, Y, K)$ and $\mathscr{L} = (A, B, L)$ be stochastic relations. Then a pair $(f, g) : \mathscr{K} \to \mathscr{L}$ is said to be a *strong morphism* if and only if $f : X \to A$ and $g : Y \to B$ are surjective Borel maps such that $L \circ f = \mathbf{S}(g) \circ K$. Hence the following diagram commutes in the category of sets:

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & A \\ {\scriptstyle K}\downarrow & & \downarrow{\scriptstyle L} \\ \mathbf{S}(Y) & \xrightarrow[\mathbf{S}(g)]{} & \mathbf{S}(B) \end{array}$$

So strong morphisms are based on Borel maps, while weak morphisms are based on Kleisli morphisms.

Definition 4.2. Let $\mathscr{K} = (X, Y, K)$ and $\mathscr{L} = (A, B, L)$ be stochastic relations. Then a pair $(F, G) : \mathscr{K} \rightsquigarrow \mathscr{L}$ is said to be a *weak morphism* if and only if $F : X \rightsquigarrow A$ and $G : Y \rightsquigarrow B$ are Kleisli morphisms with $L * F = G * K$. Consequently, the following diagram commutes in the Kleisli category of the Giry monad:

$$\begin{array}{ccc} X & \overset{F}{\cdots\!\!\succ} & A \\ {\scriptstyle K}\,\vdots & & \vdots\,{\scriptstyle L} \\ Y & \underset{G}{\cdots\!\!\succ} & B \end{array}$$

The condition $L * F = G * K$ entails that, for each $x \in X$ and each Borel set $P \in \mathscr{B}(B)$,

$$\begin{aligned} \big(L * F\big)(x)(P) &= \int_A L(a)(P)\ F(x)(da) \\ &= \int_Y G(y)(P)\ K(x)(dy) \\ &= \big(G * K\big)(x)(P). \end{aligned}$$

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If $(f,g) : \mathscr{K} \to \mathscr{L}$ is a strong morphism, then $L(f(x))(P) = K(x)(g^{-1}[P])$. Clearly, strong morphisms are special cases of weak ones. This is because $\delta_f : x \mapsto \delta_{f(x)}$ is a stochastic relation $X \rightsquigarrow Y$ whenever $f : X \to Y$ is a Borel map, and because $\int_Y h(y)\ \delta_f(x)(dy) = h(f(x))$ whenever $f : Y \to \mathbb{R}$ is a measurable map (note also that $\delta_f * N = \mathbf{S}(f) \circ N$ provided $N : A \rightsquigarrow X$, and $M * \delta_f = M \circ f$ whenever $M : Y \rightsquigarrow Z$). Some properties of weak morphisms and their relation to modal logics are discussed in Doberkat (2009, Chapter 3).

The following technical construction involving the coproduct construction allows us to reduce cospans of weak morphisms to weak morphisms proper.

Lemma 4.3. Let

$$\begin{array}{ccccc} X & \overset{F}{\cdots\!\!>} & A & \overset{F'}{\cdots\!\!<} & X' \\ {\scriptstyle K}\downarrow & & {\scriptstyle N}\downarrow & & \downarrow{\scriptstyle K'} \\ Y & \underset{G}{\cdots\!\!>} & B & \underset{G'}{\cdots\!\!<} & Y' \end{array}$$

be a span of weak morphisms $\mathscr{K} \rightsquigarrow \mathscr{N} \rightsquigarrow \mathscr{K}'$. Then

$$(F + F', G + G') : \mathscr{K} + \mathscr{K}' \rightsquigarrow \mathscr{N} + \mathscr{N}$$

is a weak morphism.

Weak morphisms yield cospans of strong morphisms using the construction discussed in Lemma 2.4.

Proposition 4.4. The following conditions are equivalent for the stochastic relations $\mathscr{K} = (X, Y, K)$ and $\mathscr{L} = (A, B, L)$:

(a) $(F, G) : \mathscr{K} \rightsquigarrow \mathscr{L}$ is a weak morphism.

(b) There exists a stochastic relation $\mathscr{R} = (X, B, R)$ such that

$$\begin{array}{ccc} & & (X, Y \times B, K(\cdot) \otimes G) \\ & & \downarrow{\scriptstyle (id_X, \pi_{Y\times B,B})} \\ (X, A \times B, F(\cdot) \otimes L) & \xrightarrow[(id_X, \pi_{A\times B,B})]{} & \mathscr{R} \end{array} \qquad (2)$$

is a cospan of strong morphisms.

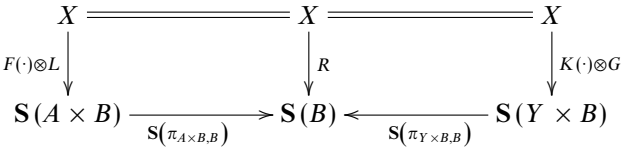
Proof.

(a)⇒(b): Set $R := L * F : X \rightsquigarrow B$, so that, by Lemma 2.4, for $x \in X$, we have

$$\begin{aligned} \mathbf{S}\left(\pi_{A\times B,B}\right) \circ \left(F(x) \otimes L\right) &= (L * F)(x) \\ &= R(x) \\ &= (G * K)(x) \\ &= \mathbf{S}\left(\pi_{Y\times B,B}\right) \circ \left(K(x) \otimes G\right). \end{aligned}$$

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(b)⇒(a): The following diagram commutes by assumption:



Consequently,

$$\begin{aligned}(L * F)(x) &= \mathbf{S}\left(\pi_{A\times B,B}\right)\left(F(x)\otimes L\right)\\ &= R(x)\\ &= \mathbf{S}\left(\pi_{Y\times B,B}\right)\left(K(x)\otimes G\right)\\ &= (G * K)(x).\end{aligned}$$

□

As an aside, we will now take a quick look at disintegrations. It turns out that weak morphisms may be characterised through disintegrations, yielding a partial converse to Lemma 2.4.

Proposition 4.5. A weak morphism $(F,G):(X,Y,K)\curvearrowright(A,B,L)$ is characterised through the stochastic relations $M_1 : X \rightsquigarrow A\times B$ and $M_2 : X \rightsquigarrow Y\times B$ with the following properties:

(a) $\mathbf{S}(\pi_{A\times B,B})\circ M_1 = \mathbf{S}(\pi_{Y\times B,B})\circ M_2$.
(b) The disintegration of $M_1(x)$ with respect to A,B is independent of $x\in X$ and coincides with L.
(c) $K = \mathbf{S}(\pi_{Y\times B,Y})\circ M_2$.
(d) The disintegration of $M_2(x)$ with respect to Y,B is independent of $x\in X$.

Then G is the disintegration of $M_2(x)$ with respect to Y,B, and $F = \mathbf{S}(\pi_{A\times B,A})\circ M_1$.

Proof.

(1) Given the weak morphism $(F,G):(X,Y,K)\curvearrowright(A,B,L)$, we put $M_1(x) := F(x)\otimes L$ and $M_2(x) := K(x)\otimes G$. It is then easy to see that M_1 and M_2 satisfy the properties (a)–(d).
(2) Conversely, let M_1 and M_2 be given. An easy calculation using Proposition 4.4 then shows that the definitions of F and G do indeed yield a weak morphism $(X,Y,K)\curvearrowright(A,B,L)$. □

For completeness, we will now give the definitions of strong and weak bisimilarity.

Definition 4.6. Let $\mathscr{K}=(X,Y,K)$ and $\mathscr{L}=(A,B,L)$ be stochastic relations.

(1) $\mathscr{K}$ and $\mathscr{L}$ are said to be *strongly bisimilar* if and only if there exists a span $\mathscr{K}\leftarrow\mathscr{M}\rightarrow\mathscr{L}$ of strong morphisms.
(2) $\mathscr{K}$ and $\mathscr{L}$ are said to be *weakly bisimilar* if and only if there exists a span $\mathscr{K}\curvearrowleft\mathscr{M}\curvearrowright\mathscr{L}$ of weak morphisms.

In each case, the relation $\mathscr{M}$ is said to be *mediating*.

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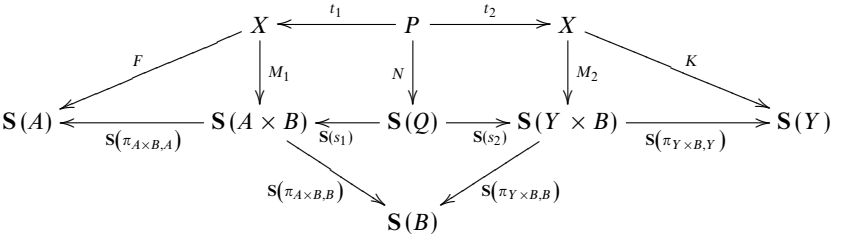


Fig. 1. Derived diagram for the proof of Proposition 4.7

Consequently, in order to establish, for example, weak bisimilarity for the relations $\mathscr{K}$ and $\mathscr{L}$, we need to find three objects:

— a mediating relation $\mathscr{M}$,
— a weak morphism $(F, G) : \mathscr{M} \curvearrowright \mathscr{K}$, and
— a weak morphism $(F', G') : \mathscr{M} \curvearrowright \mathscr{L}$.

Using the construction from Proposition 4.5, we can now derive a first criterion for the weak bisimilarity of two stochastic relations.

Proposition 4.7. Construct $M_1 : X \rightsquigarrow A \times B$ and $M_2 : X \rightsquigarrow Y \times B$ from the weak morphism $(F, G) : (X, Y, K) \curvearrowright (A, B, L)$. If M_1 and M_2 are strongly bisimilar, then K and L are weakly bisimilar.

Proof.

(1) The construction yields the diagram of commuting maps shown in Figure 1 with a suitable stochastic relation $N : P \rightsquigarrow Q$. Now define

$$\begin{aligned} S &:= F * \delta_{t_1} (= F \circ t_1) \\ T &:= \mathbf{S}\left(\pi_{A\times B,B}\right) \circ \delta_{s_1} \\ U &:= \delta_{t_2} \\ V &:= \mathbf{S}\left(\pi_{Y\times B,B}\right) \circ \delta_{s_2}. \end{aligned}$$

Hence

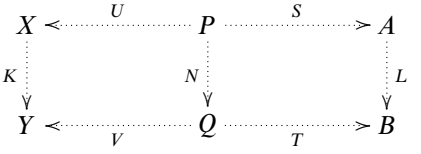
$$K * U = K \circ t_2 = \mathbf{S}\left(\pi_{Y\times B,B}\right) \circ \mathbf{S}(s_2) \circ N = V * N,$$

and, given $p \in P$ with $R \in \mathscr{B}(B)$,

$$\begin{aligned} \big(L * (F * \delta_{t_1})\big)(p)(R) &= \int_A L(a)(R)\ F(t_1(p))(da) \\ &= M_1(t_1(p))(A \times R) \\ &= \mathbf{S}\left(\pi_{A\times B,B}\right)\big(M_1(t_1(p))\big)(R) \\ &= \big(\mathbf{S}\left(\pi_{A\times B,B}\right) \circ M_1 \circ t_1\big)(p)(R) \\ &= \big((\mathbf{S}\left(\pi_{A\times B,B}\right) \circ \mathbf{S}(s_1)) \circ N\big)(p)(R) \\ &= \big((\mathbf{S}\left(\pi_{A\times B,B}\right) \circ \delta_{s_1}) * N\big)(p)(R). \end{aligned}$$

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Consequently, the diagrams



commute in the Kleisli category. □

This reduction result encourages us to look for conditions that ensure that the derived relations M_1 and M_2 are strongly bisimilar. We fix smooth equivalence relations α on A, β on B, χ on X and υ on Y, and a weak morphism $(F, G) : (X, Y, K) \rightsquigarrow (A, B, L)$ such that (χ, υ) is a congruence for K, (α, β) is a congruence for L, (χ, α) is a congruence for F and, finally, (υ, β) is a congruence for G. We assume that (χ, α) and (χ, υ) are simulation equivalent such that $\mathfrak{p} : A/\alpha \to Y/\upsilon$ is a bijection with inverse $\mathfrak{q}$. The maps $\mathfrak{p}$ and $\mathfrak{q}$ are used to transport the structures back and forth in the sense of Definition 3.5 with $\mathscr{A}_0$ and $\mathscr{Y}_0$ as the respective $\cap$-stable generators for $\mathbf{\Sigma}(\mathscr{B}(A), \alpha)$ and $\mathbf{\Sigma}(\mathscr{B}(Y), \upsilon)$. The identity on X/χ maps the factor structure onto itself so that

$$K(x)(W_0) = F(x')\big(\sqrt{\mathfrak{q}}(W_0)\big) \tag{3}$$

holds for every $W_0 \in \mathscr{Y}_0$ whenever $x \chi x'$. We define the product $\alpha \times \beta$ of the equivalence relations α and β through

$$\langle a, b\rangle \, \alpha \times \beta \, \langle a', b'\rangle \Leftrightarrow a \, \alpha \, a' \wedge b \, \beta \, b'.$$

So $\alpha \times \beta$ and, similarly, $\upsilon \times \beta$ are smooth equivalence relations on $A \times B$ and $Y \times B$, respectively, such that, for example (see Doberkat (2007b, Lemma 5.13)),

$$\mathbf{\Sigma}(\mathscr{B}(A \times B), \alpha \times \beta) = \mathbf{\Sigma}(\mathscr{B}(A), \alpha) \otimes \mathbf{\Sigma}(\mathscr{B}(B), \beta).$$

Let $\mathscr{B}_0$ be a $\cap$-stable countable generator of $\mathbf{\Sigma}(\mathscr{B}(B), \beta)$. Then

$$\{A_0 \times B_0 \mid A_0 \in \mathscr{A}_0, B_0 \in \mathscr{B}_0\}$$

is a countable and $\cap$-stable generator of

$$\mathbf{\Sigma}(\mathscr{B}(A \times B), \alpha \times \beta),$$

and similarly for

$$\mathbf{\Sigma}(\mathscr{B}(Y \times B), \upsilon \times \beta).$$

We claim that $(\chi, \alpha \times \beta)$ is simulation equivalent to $(\chi, \upsilon \times \beta)$. Hence it will be sufficient to prove that

$$M_2(x)(\sqrt{\mathfrak{p}}(A_0) \times B_0) = M_1(x')(A_0 \times B_0) \tag{4}$$

$$M_1(x)(\sqrt{\mathfrak{q}}(Y_0) \times B_0) = M_2(x')(Y_0 \times B_0) \tag{5}$$

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both hold whenever $A_0 \in \mathscr{A}_0, B_0 \in \mathscr{B}_0, Y_0 \in \mathscr{Y}_0$ and $x, x' \in X$ with $x \chi x'$ are given. Looking at equation (4):

$$M_2(x)(\sqrt{\mathfrak{p}}(A_0) \times B_0) = \int_{\sqrt{\mathfrak{p}}(A_0)} G(y)(B_0)\ K(x)(dy) \tag{6}$$

$$= \int_{\sqrt{\mathfrak{p}}(A_0)} \big(G_{(\upsilon,\beta)} \circ \eta_\upsilon(y)\big)(\eta_\beta\,[B_0])\ K(x)(dy) \tag{7}$$

$$= \int_{\eta_\upsilon[\sqrt{\mathfrak{p}}(A_0)]} G_{(\upsilon,\beta)}(t)(\eta_\beta\,[B_0])\ \mathbf{S}(\eta_\upsilon)(K(x))(dt) \tag{8}$$

$$= \int_{\eta_\upsilon[\sqrt{\mathfrak{p}}(A_0)]} G_{(\upsilon,\beta)}(t)(\eta_\beta\,[B_0])\ K_{(\chi,\upsilon)}([x]_\chi)(dt). \tag{9}$$

Because (υ, β) is a congruence for G, we know from Proposition3.3 that $y \mapsto G(y)(B_0)$ is $\mathbf{\Sigma}(\mathscr{B}(Y), \upsilon)$-measurable since $B_0 \in \mathbf{\Sigma}(\mathscr{B}(B), \beta)$. From Corollary 3.2, we can infer that this map is constant on the atoms of $\mathbf{\Sigma}(\mathscr{B}(Y), \upsilon)$, which are just the equivalence classes, so we may conclude that

$$G(y)(B_0) = \big(G_{(\upsilon,\beta)} \circ \eta_\upsilon\big)(y)(\eta_\upsilon\,[B_0]),$$

which explains the equality of (6) and (7). Equation (8) comes from the Change of Variables Formula, applying the factor map η_υ, and from the observation that for a Borel set $T \in \mathscr{B}(Y/\upsilon)$, the equality $\eta_\upsilon^{-1}[T] = \sqrt{\mathfrak{p}}(A_0)$ is equivalent to $T = \eta_\upsilon[\sqrt{\mathfrak{p}}(A_0)]$ because $\sqrt{\mathfrak{p}}(A_0) \in \mathbf{\Sigma}(\mathscr{B}(Y), \upsilon)$. From the construction, it is evident that the measures $\mathbf{S}(\eta_\upsilon)(K(x))$ and $K_{(\chi,\upsilon)}([x]_\chi)$ coincide on $\mathscr{B}(Y/\upsilon)$, yielding equation (9).

Because (α, β) and (υ, β) are simulation equivalent, we may conclude that

$$G_{(\upsilon,\beta)}(t)(\eta_\beta\,[B_0]) = L_{(\alpha,\beta)}(\mathfrak{q}(t))(\eta_\beta\,[B_0]).$$

Thus we can continue from equation (9) as follows:

$$M_2(x)(\sqrt{\mathfrak{p}}(A_0) \times B_0) = \int_{\eta_\upsilon[\sqrt{\mathfrak{p}}(A_0)]} L_{(\alpha,\beta)}(\mathfrak{q}(t))(\eta_\beta\,[B_0])\ K_{(\chi,\upsilon)}([x]_\chi)(dt) \tag{10}$$

$$= \int_{\mathfrak{q}[\eta_\upsilon[\sqrt{\mathfrak{p}}(A_0)]]} L_{(\alpha,\beta)}(s)(\eta_\beta\,[B_0])\ \mathbf{S}(\mathfrak{q})\big(K_{(\chi,\upsilon)}([x]_\chi)\big)(ds) \tag{11}$$

$$= \int_{\mathfrak{q}[\eta_\upsilon[\sqrt{\mathfrak{p}}(A_0)]]} L_{(\alpha,\beta)}(s)(\eta_\beta\,[B_0])\ \mathbf{S}(\eta_\alpha)\big(F(x')\big)(ds) \tag{12}$$

$$= \int_{\eta_\alpha^{-1}[\mathfrak{q}[\eta_\upsilon[\sqrt{\mathfrak{p}}(A_0)]]]} L_{(\alpha,\beta)}(\eta_\alpha(a))(\eta_\beta\,[B_0])\ F(x')(da) \tag{13}$$

$$= \int_{A_0} L(a)(B_0)\ F(x')(da) \tag{14}$$

$$= M_1(x')(A_0 \times B_0). \tag{15}$$

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Equation (11) is inferred from (10) using the Change of Variables Formula and from $\mathfrak{q}^{-1}[\mathfrak{q}[\eta_\upsilon[\sqrt{\mathfrak{p}}(A_0)]]] = \eta_\upsilon[\sqrt{\mathfrak{p}}(A_0)]$, because $\mathfrak{q}$ is injective. Now let $G \in \mathscr{B}(A/\alpha)$. Then

$$\begin{aligned}\mathbf{S}(\mathfrak{q})\big(K_{(\chi,\upsilon)}([x]_\chi)\big)(G) &= K_{(\chi,\upsilon)}([x]_\chi)(\mathfrak{q}^{-1}[G]\\ &= K(x)\big((\eta_\upsilon^{-1}\circ\mathfrak{q}^{-1})[G]\big)\\ &\overset{(3)}{=} F(x')\big(\sqrt{\mathfrak{q}}(\eta_\upsilon^{-1}\circ\mathfrak{q}^{-1})[G]\big)\big)\\ &\overset{(\dagger)}{=} F(x')(\eta_\alpha^{-1}[G])\\ &= \mathbf{S}(\eta_\alpha)\big(F(x')\big)(G).\end{aligned}$$

This is derived from the definition of the factor relation, from the assumption that $x\,\chi\,x'$, which is used through equation (3), and from the assumption that $\mathfrak{q}$ is injective, since

$$\sqrt{\mathfrak{q}}\circ\eta_\upsilon^{-1}\circ\mathfrak{q}^{-1} = \big(\eta_\alpha^{-1}\circ\mathfrak{q}\circ\eta_\upsilon\big)\circ\eta_\upsilon^{-1}\circ\mathfrak{q}^{-1} = \eta_\alpha^{-1},$$

which gives equation (†). This justifies equation (12). Equation (13) is again an application of the Change of Variables Formula, and equation (14) uses $\eta_\alpha^{-1}\circ\mathfrak{q}\circ\eta_\upsilon\circ\eta_\upsilon^{-1}\circ\mathfrak{p}\circ\eta_\alpha = id$. Equation (5) is established in the same way, noting that the maps $\mathfrak{p}$ and $\mathfrak{q}$ act strictly symmetrically to each other.

Geometrically speaking, the equation $L * F = G * K$ that defines the composition defines the product measures $F(\cdot)\otimes L$ and $K(\cdot)\otimes G$ only on the respective *strips* given through Borel sets $A\times B_0$ and $Y\times B_0$ with $B_0\in\mathscr{B}(B)$. This is not enough to determine the values of these measures on proper rectangles $A_0\times B_0$ and $Y_0\times B_0$ with $A_0\in\mathscr{B}(A)$ and $Y_0\in\mathscr{B}(Y)$, respectively. As an aside, note that the proof does not use the assumption that (F,G) is a weak morphism.

The above considerations can now be used to establish our main result.

Theorem 4.8. Let (χ,υ) and (α,β) be congruences for the stochastic relations $\mathscr{K} = (X,Y,K)$ and $\mathscr{L} = (A,B,L)$, respectively, and assume that there exists a weak morphism $(F,G) : \mathscr{K} \rightsquigarrow \mathscr{L}$ such that (χ,α) and (υ,β) are simulation equivalent congruences for (X,A,F) and (Y,B,G), respectively. Then $\mathscr{K}$ and $\mathscr{L}$ are weakly bisimilar.

Proof. Define the relations $M_1 : X \rightsquigarrow A\times B$ and $M_2 : X \rightsquigarrow Y\times B$ through

$$\begin{aligned}M_1(x) &:= F(x)\otimes L\\ M_2(x) &:= K(x)\otimes G.\end{aligned}$$

The above discussion shows that the congruences $(\chi,\alpha\times\beta)$ and $(\chi,\upsilon\times\beta)$ are simulation equivalent. Thus we obtain from the general criterion in Doberkat (2006, Theorem 4.8) that M_1 and M_2 are strongly bisimilar. We can now infer the claim from Proposition 4.7. □

In the category of stochastic relations with strong morphisms, a bisimulation is constructed through a cospan. This construction is helpful, for example, when we need to show that Kripke models for a modal logic are bisimilar, provided they are behavioural equivalent (Doberkat 2007b, Section 6.2.3). Thus, the question arises as to whether a similar construction is also possible when strong morphisms are replaced by weak ones.

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In Lemma 4.3, we reduced a cospan of weak morphisms to a weak morphism on a coproduct, and this construction will now be helpful for answering the above question.

Proposition 4.9. Let $(F, G) : \mathscr{K} \rightsquigarrow \mathscr{N}$ and $(F', G') : \mathscr{K}' \rightsquigarrow \mathscr{N}$ be a cospan of weak morphisms, and assume that (χ, υ) and (χ', υ') are congruences on $\mathscr{K}$ and $\mathscr{K}'$, respectively, and (α, β) is a congruence on $\mathscr{N}$ such that (χ, α) is simulation equivalent to (υ, β), and (χ', α) is simulation equivalent to (υ', β).

Then $\mathscr{K}$ and $\mathscr{K}'$ are weakly bisimilar.

Proof.

(1) We can infer from Lemma 3.8 that $(\chi + \chi', \alpha + \alpha)$ is simulation equivalent to $(\upsilon + \upsilon', \beta + \beta')$, so we find through Theorem 4.8 a stochastic relation $\mathscr{M} = (C, D, M)$ and weak morphisms that makes the following diagram commute in the Kleisli category:

$$\begin{array}{ccccc} X + X' & \overset{I}{\prec\!\cdots} & C & \overset{I'}{\cdots\!\succ} & A + A \\ {\scriptstyle K+K'}\downarrow & & \downarrow{\scriptstyle M} & & \downarrow{\scriptstyle N+N} \\ Y + Y' & \underset{J}{\prec\!\cdots} & D & \underset{J'}{\cdots\!\succ} & B + B \end{array}$$

(using the notation from Theorem 4.8 with $\mathscr{N} = (A, B, C)$). Discarding the right-hand side of this diagram, we define the stochastic relation $I_1 : C \rightsquigarrow X$ through $I_1(c)(G) := I(c)(G \cap X)$ with $G \in \mathscr{B}(X)$. The relations $I_2 : C \rightsquigarrow X'$, $J_1 : D \rightsquigarrow Y$ and $J_2 : D \rightsquigarrow Y'$ are defined similarly.

(2) Now let $c \in C$ and $H \subseteq Y$ be a Borel set. Then

$$\begin{aligned} (K * I_1)(c)(H) &= \int_X K(x)(H)\ I_1(c)(dx) \\ &= \int_{X+X'} (K + K')(x)(H)\ I(c)(dx) \\ &= \big((K + K') * I\big)(c)(H) \\ &= (J * M)(c)(H) \\ &= \int_D J(d)(H)\ M(c)(dd) \\ &= \int_D J_1(d)(H)\ M(c)(dd) \\ &= (J_1 * M)(c)(H). \end{aligned}$$

We can similarly establish $K' * I_2 = J_2 * M$, so we have now constructed the mediating relation. □

In this way we have obtained a general criterion for two stochastic relations being weakly bisimilar. It is technically more involved than the corresponding criterion for strong morphisms (which simply states that two relations are strongly bisimilar if there exists simulation equivalent congruences for them). Both statements have in common that they do not need an external instance (like a modal logic) to decide whether or not they are bisimilar, and both use the existence of a cospan, albeit in very different ways.

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5. Conclusion

We have derived a criterion for weak bisimilarity in the Giry monad. Technically, this was done through a reduction argument together with an appliction of a general criterion for strong bisimilarity. This shows a close relationship between Kleisli morphisms and morphisms in the base category, and it appears that this has not been sufficiently exploited yet in either this special case or the general case. To this end, looking again at tensored categories (Abramsky *et al.* 1999) may be interesting.

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