Proc. Camb. Phil. Soc. (1972), **72**, 37
PCPS 72-6
Printed in Great Britain

Convex series

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(*Received* 25 *March* 1971)

1. *Introduction.* Let A be a subset of a Hausdorff topological linear space. By a *convex series* of elements of A we mean a series of the form $\sum_{n=1}^{\infty} \lambda_n a_n$, where $a_n \in A$ and $\lambda_n \geqslant 0$ for each n, and $\sum_{n=1}^{\infty} \lambda_n = 1$. We say that A is:

(i) *CS-closed* if it contains the sum of every convergent convex series of its elements;

(ii) *CS-compact* if every convex series of its elements converges to a point of A (this bold terminology is chosen because sets satisfying this condition turn out to have properties analogous to those of compact sets).

The usefulness of CS-closed sets depends largely on the fact that they satisfy the preservation rules that are lacking for closed sets: for example, if A is CS-compact and B is CS-closed, then $A+B$ and $\mathrm{co}(A \cup B)$ are CS-closed. Also, if R is a CS-closed relation, then the image of A under R is CS-closed. We establish these rules in section 2, after considering the obvious question of which sets are CS-closed and CS-compact.

Let us say that a set A is *semi-closed* if A and its closure $\bar{A}$ have the same interior. Then every CS-closed subset of a metrizable linear space is semi-closed. The proof of this fact (which has already appeared in (4)) is similar to the last part of the proof of the closed-graph theorem for Banach spaces. Section 3 is devoted to justifying the claim that this statement has such a wide range of applications that it is worth isolating it as a 'theorem'. For instance, the closed-graph and open-mapping theorems (for Banach spaces) follow from it with ease. Also, using the preservation rules mentioned above, one deduces that if A and B are closed, convex subsets of a metrizable space, A being bounded and complete, then $A+B$ and $\mathrm{co}(A \cup B)$ are semi-closed. In the case of Banach spaces, this was proved for $A+B$ by Tukey (7); Klee (5) and Ellis (2) observed that Tukey's proof could be adapted to deal with $\mathrm{co}(A \cup B)$.

The main theorem of section 4 states that, without assuming metrizability, the image of a CS-compact set under a closed (not just CS-closed) relation is semi-closed. Tukey's theorem is now seen to hold in all topological linear spaces.

In section 5 we consider the CS-closure of a set (i.e. the smallest CS-closed set containing it). The main result is that, in a metrizable space, the CS-closure of a set is the set of all sums of convergent convex series of its elements. An old example of Banach shows that this is not always true in non-metrizable spaces.

In section 6 we give conditions under which the same sets are CS-closed (or CS-compact) with respect to two different topologies, motivated by the fact that the same convex sets are closed in two locally convex topologies of the same dual pair. We conclude with an example showing that a CS-closed set need not contain all its non-support points.

https://doi.org/10.1017/S0305004100050933 Published online by Cambridge University Press

2. *Elementary properties.* We start with the problem of recognizing CS-closed and CS-compact sets (see section 1 for the definitions). All topological linear spaces are assumed Hausdorff.

Every CS-compact set is CS-closed, and every CS-closed set is convex. If each member of a family of sets is CS-closed (or CS-compact), then so is their intersection.

A set admitting addition and non-negative scalar multiplication will be called a *wedge*. Clearly, a wedge is CS-closed if and only if it contains the sum of every convergent series of its elements (i.e. if it is 'series-closed'). A linear subspace is CS-closed if and only if it is sequentially closed (in a metrizable space, if and only if it is closed).

Given a wedge P, one defines a corresponding partial ordering $\leqslant$ of X by

$$x \leqslant y \text{ if and only if } y - x \in P.$$

In the order terminology, a wedge is CS-closed if and only if it contains the limit of every convergent increasing sequence of its elements.

PROPOSITION 1. *Every sequentially closed convex set is CS-closed*

Proof. Let A be a sequentially closed convex set, and let $\sum_{n=1}^{\infty} \lambda_n a_n$ be a convergent convex series of elements of A, with sum x. Write

$$\mu_n = \sum_{i=1}^{n} \lambda_i, \quad b_n = \sum_{i=1}^{n} \lambda_i a_i.$$

Then $\mu_n \to 1$ and $b_n \to x$, so $\mu_n^{-1} b_n \to x$. But $\mu_n^{-1} b_n \in A$ for all n such that $\mu_n > 0$, so $x \in A$.

PROPOSITION 2. *Let A be a bounded, convex subset of a topological linear space X. Then either of the following conditions implies that A is CS-compact:*

(i) *A is CS-closed and X is sequentially complete, or*

(ii) *A is sequentially complete.*

Conversely, every CS-compact set is CS-closed and bounded.

Proof. Take a convex series $\sum_{n=1}^{\infty} \lambda_n a_n$ of elements of A. Let U be a neighbourhood of 0. Since A is bounded, there exists $\delta > 0$ such that $\lambda A \subseteq U$ whenever $|\lambda| \leqslant \delta$. Take n_0 such that $\sum_{n=n_0}^{\infty} \lambda_n \leqslant \delta$, and suppose that $n_0 \leqslant p < q$. Then

$$\lambda_p a_p + \ldots + \lambda_q a_q \in (\lambda_p + \ldots + \lambda_q) A \subseteq U.$$

Hence the sequence of partial sums of $\Sigma \lambda_n a_n$ is Cauchy. If condition (i) holds, we conclude at once that the series converges, and that its sum is in A. If condition (ii) holds, define μ_n and b_n as in Proposition 1: then $\{\mu_n^{-1} b_n\}$ is a Cauchy sequence in A, so has a limit in A, and $\{b_n\}$ also converges to this limit.

Finally, suppose that A is unbounded. Then there is a neighbourhood U of 0 such that, for each n, there is a point a_n of A that is not in $2^n U$. Then $\{2^{-n} a_n\}$ does not converge to 0, so the series $\sum_{n=1}^{\infty} 2^{-n} a_n$ cannot converge.

The following examples will serve to remind us that there are plenty of sets that are CS-closed but not closed:

(i) Any open, convex set A is CS-closed, since $\lambda A + (1-\lambda)\bar{A} \subseteq A$ for $0 < \lambda < 1$.

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(ii) In the space m of all bounded real sequences, let P be the set of sequences in which the first non-zero term is positive, together with 0 (P is the wedge giving the 'lexicographic' ordering). Then P is CS-closed.

We now turn to preservation rules. The convex cover of a set E will be denoted by $\mathrm{co}(E)$, and the image of E under a relation R by $R(E)$. It is elementary that if A and B are CS-compact, then so are $A+B$ and $\mathrm{co}(A \cup B)$. Also, the image of a CS-compact set under a continuous linear mapping is CS-compact.

PROPOSITION 3. *Let X, Y be topological linear spaces. If A is a CS-compact subset of X, and R is a CS-closed subset of $X \times Y$, then $R(A)$ is CS-closed in Y.*

Proof. Suppose that $\sum_{n=1}^{\infty} \lambda_n b_n$ is a convergent convex series of elements of $R(A)$, with sum b. For each n, take a_n in A such that $(a_n, b_n) \in R$. Then $\sum_{n=1}^{\infty} \lambda_n a_n$ is convergent to an element a of A, and we have

$$(a, b) = \sum_{n=1}^{\infty} \lambda_n (a_n, b_n) \in R,$$

since R is CS-closed. Hence $b \in R(A)$.

PROPOSITION 4. *Suppose that A is CS-compact and B is CS-closed, Then $A+B$ and $\mathrm{co}(A \cup B)$ are CS-closed.*

Proof. See (4), p. 184.

A similar statement holds for a finite number of sets (all but one being CS-compact).

Since the sum of two closed linear subspaces of a Banach space need not be closed, the sets $A+B$, $\mathrm{co}(A \cup B)$ and $R(A)$ can fail to be CS-closed when A and B are CS-closed (but neither is CS-compact).

3. *Applications of the theorem on interior points.* Recall that we are calling a set A *semi-closed* if A and its closure $\bar{A}$ have the same interior. The most basic fact about CS-closed sets is:

THEOREM 1. *Every CS-closed subset of a metrizable topological linear space is semi-closed.*

Proof. See (4), p. 183.

In the present section we list a number of applications of this theorem. The results are either known or slight generalizations of known ones; the point is that, in each case, Theorem 1 facilitates their proof.

Immediately from the theorem, we have:

COROLLARY 1. *Let A be a dense, CS-closed subset of a metrizable space X. Then $A = X$.*

Proposition 3 gives at once:

COROLLARY 2. *Let X, Y be topological linear spaces, Y being metrizable. If A is a CS-compact subset of X, and R is a CS-closed subset of $X \times Y$, then $R(A)$ is semi-closed.*

The deduction from Corollary 2 of the closed-graph and open-mapping theorems for Banach spaces (though not for metrizable spaces) is standard work.

Proposition 4 yields the following extension of Tukey's theorem to metrizable spaces:

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COROLLARY 3. *Let A, B be subsets of a metrizable linear space X. If A is CS-compact and B is CS-closed, then $A+B$ and $\operatorname{co}(A \cup B)$ are semi-closed.*

Note. In the case when both A and B are bounded, Corollary 3 becomes essentially a result about normed spaces, because the hypothesis that the closure of $A+B$ or of $\operatorname{co}(A \cup B)$ has interior points implies that the space is normable.

COROLLARY 4. *Every separable Banach space X is the image of l_1 under a continuous, open linear mapping.*

Proof. Let U, U_1 denote the closed unit balls in X, l_1 respectively. Let $\{a_n\}$ be dense in U, and for x in l_1, let $T(x) = \sum_{n=1}^{\infty} x(n)a_n$. Then T is continuous, and $\overline{T(U_1)}$ contains U. Since $T(U_1)$ is CS-compact, it contains the open unit ball of X.

Next we derive a result that is important in the theory of partially ordered Banach spaces (and the direct proof of which is somewhat lengthy: cf. (4), 3.5.2, p. 106):

COROLLARY 5. *Let A, B be CS-closed wedges in a Banach space X. Suppose that, for each x in X, there exist bounded sequences $\{a_n\}$ in A and $\{b_n\}$ in B such that $a_n - b_n \to x$. Then $A - B = X$, and there is a positive number δ such that $(A \cap U) - (B \cap U) \supseteq \delta U$, where U denotes the unit ball of X.*

Proof. Write $(A \cap U) - (B \cap U) = C$. It is elementary that C is CS-compact. The hypothesis implies that

$$\bigcup_{n=1}^{\infty} (nD) = X, \quad \text{where} \quad D = \bar{C} \cap (-\bar{C}).$$

By Baire's theorem, D has interior points. Since D is convex and symmetric, it follows that 0 is an interior point of D, and hence of $\bar{C}$. Theorem 1 now shows that 0 is an interior point of C, as stated.

Another example of a proof in the theory of partially ordered Banach spaces that has been substantially shortened using Theorem 1 is (4), 3.6.8, p. 116.

In somewhat similar fashion, we have:

COROLLARY 6. *Let A, B be closed linear subspaces of a Banach space X and let U denote the closed unit ball in X. Then $A+B$ is closed if and only if there is a positive number δ such that $(A \cap U) + (B \cap U) \supseteq (A+B) \cap (\delta U)$.*

Proof. (i) Suppose that the condition holds, and take $x_n \in A+B$ with $\|x_n\| \leqslant 2^{-n}$. Express x_n as $a_n + b_n$, where the norms of a_n and b_n are not greater than $1/(\delta 2^n)$. One deduces that Σx_n is convergent to an element of $A+B$.

(ii) Suppose that $A+B$ is closed, and let $C = (A \cap U) + (B \cap U)$. Then

$$\bigcup_{n=1}^{\infty} (nC) = A+B,$$

so by Baire's theorem, $\bar{C}$ is a neighbourhood of 0 in $A+B$. By Theorem 1, it follows that C is a neighbourhood of 0 in $A+B$.

A simple example shows that the metrizability condition is not dispensable in theorem 1. Let X be the space of all mappings of the real field into itself, with the

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topology of pointwise convergence. Let A be the subspace consisting of functions f such that $\{x: f(x) \neq 0\}$ is countable. Then A is a dense, CS-closed linear subspace.

4. *A closed-relation theorem.* We now show that it is possible, in Corollary 2 of Theorem 1, to omit the metrizability condition provided that the relation is closed (and not just CS-closed). It is instructive to compare the resulting theorem with the usual closed-relation theorem for topological linear spaces. In the latter, one assumes that $\overline{R(U)}$ is a neighbourhood of 0 for each neighbourhood U, and deduces that $R(U)$ is a neighbourhood of 0. In our theorem, we are concerned with the image of only one bounded, convex set. It is not surprising, in the context of topological linear spaces, that there are different closed-relation theorems applying to neighbourhoods of 0 and to bounded, convex sets. It does not appear to be possible to deduce either theorem from the other.

THEOREM 2. *Let X, Y be topological linear spaces. Suppose that A is a CS-compact subset of X, and that R is a closed, convex subset of $X \times Y$. Then $R(A)$ is semi-closed.*

Proof. Let y be an interior point of $\overline{R(A)}$. Write $R_0 = R - (0, y)$, $B = R_0(A)$. Then $\bar{B}$ is a neighbourhood of 0. We show that $0 \in B$, from which we can deduce that $y \in R(A)$ to obtain the result.

Exactly as in the proof of Theorem 1 (see (4), p. 183), we can construct a sequence $\{b_n\}$ in B such that, for each n, we have $-y_n \in 2^{-n-1}\bar{B}$, where

$$y_n = 2^{-1}b_1 + \ldots + 2^{-n}b_n.$$

For each n, take a_n in A such that $(a_n, b_n) \in R_0$. Since A is CS-compact, $\sum_{n=1}^{\infty} 2^{-n}a_n$ converges to a point a of A. We use the fact that R_0 is closed and convex to establish that $(a, 0) \in R_0$. Take neighbourhoods U, V of the zero elements in X, Y respectively. For some n, it is true that $2^{-n}A \subseteq U$ (since A is bounded), and that

$$a - (2^{-1}a_1 + \ldots + 2^{-n}a_n) \in U.$$

Now $\bar{B}$ is convex and contains 0, so $\frac{1}{2}\bar{B} \subseteq \bar{B}$. Hence $-y_n \in 2^{-n}\bar{B}$, so there exist points a' of A and t of Y such that $(a', t) \in R_0$ and $y_n + 2^{-n}t \in V$. Since R_0 is convex, it contains the element

$$(2^{-1}a_1 + \ldots + 2^{-n}a_n + 2^{-n}a', \quad y_n + 2^{-n}t).$$

But this element is in $(a + U - U) \times V$, showing that $(a, 0) \in R_0$, as required.

Letting R be the identity mapping in Theorem 2, one deduces that every CS-compact set is semi-closed. This, however, is no advance on Theorem 1, because the topology will always be normable in non-trivial cases.

Theorem 2 enables us to extend Tukey's theorem to non-metrizable spaces:

COROLLARY 1. *Let A, B be convex subsets of a topological linear space X. Suppose that A is CS-compact and B is closed. Then $A + B$ is semi-closed.*

Proof. Let $R = \{(x, x+b): x \in X \text{ and } b \in B\}$. Then R is a closed, convex subset of $X \times X$, and $R(A) = A + B$.

The author does not see how a similar argument can be used to prove that $\operatorname{co}(A \cup B)$ is semi-closed, but this can be deduced from Corollary 1 as follows:

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COROLLARY 2. *Let A, B be non-empty, convex subsets of a topological linear space X. Suppose that A is CS-compact and B is closed. Then $\mathrm{co}(A \cup B)$ is semi-closed.*

Proof. It is sufficient to show that $\mathrm{co}(A \cup B)$ has interior points whenever its closure has, since any convex set with interior points is semi-closed. Take b_0 in B, and let $A_1 = \mathrm{co}(A \cup \{b_0\})$. Then A_1 is CS-compact, and $\mathrm{co}(A \cup B) = \mathrm{co}(A_1 \cup B)$. Let

$$A_1' = A_1 - b_0, \quad B' = B - b_0.$$

Then A_1' and B' contain 0, and $\mathrm{co}(A_1' \cup B') = \mathrm{co}(A \cup B) - b_0$. Denote this set by C, and suppose that $\bar{C}$ has interior points. Since $C \subseteq A_1' + B'$, it follows that the closure of $A_1' + B'$ has interior points. Corollary 1 now shows that $A_1' + B'$ has interior points. But $\frac{1}{2}(A_1' + B') \subseteq C$, so C has interior points, as required.

Next, we give an application of Theorem 2 to linear mappings. Let X, Y be topological linear spaces, and let T be a linear mapping from X to Y. We shall say that T is *strongly bounded* if there is a neighbourhood U of 0 in X such that $T(U)$ is bounded. This is a stronger condition than continuity (equivalent when Y is a normed space), but a weaker one than compactness.

COROLLARY 3. *Let X, Y be locally convex spaces, Y being sequentially complete. Let T be a linear mapping from X to Y with closed graph. If there is a bounded subset B of Y such that the closure of $T^{-1}(B)$ is a neighbourhood of 0 in X, then T is strongly bounded.*

Proof. Let C be the closed, convex cover of B. Since Y is locally convex, C is bounded. Hence C is CS-compact, so, by Theorem 2, $T^{-1}(C)$ is a neighbourhood of 0.

COROLLARY 4. *Let X be a barrelled space, and Y a sequentially complete, locally convex space. Let T be a linear mapping from X to Y with closed graph. Suppose that there is a bounded subset B of Y such that $T(X) \subseteq \bigcup_{n=1}^{\infty} (nB)$. Then T is strongly bounded.*

Proof. Let $C = \mathrm{co}(B)$. Then $T^{-1}(C)$ is convex and absorbent in X. Hence its closure is a neighbourhood of 0.

A particular case of Corollary 4 is the well-known fact that a closed-graph linear mapping of a barrelled space into a Banach space is continuous.

There are no corresponding results for compact, instead of strongly bounded, linear mappings, because the image of a compact set under a closed relation is always closed.

It is not sufficient in Theorem 2 if R is only CS-closed. For if A is the dense, CS-closed linear subspace defined at the end of section 3, and $R = \{0\} \times A$, then R is CS-closed, and $R(0)$ is A, which is not semi-closed.

5. *The CS-closure.* Since the intersection of a family of CS-closed sets is CS-closed, there is a smallest CS-closed set containing a given set, which we call its CS-closure.

PROPOSITION 5. *The CS-closure of a wedge or linear subspace is again a wedge or linear subspace.*

Proof. Let $\hat{P}$ be the CS-closure of a wedge P. Take x in $\hat{P}$ and $\lambda > 0$. For each CS-closed set A containing P, $\lambda^{-1}A$ is also a CS-closed set containing P, so $x \in \lambda^{-1}A$, or

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$\lambda x \in A$. Hence $\lambda x \in \hat{P}$, so $\hat{P}$ (being convex) is a wedge. If P is a linear subspace, similar reasoning shows that $-\hat{P} = \hat{P}$, so $\hat{P}$ is a linear subspace.

The main result of this section is that, in a metrizable space, the CS-closure of a set is the set of all sums of convergent convex series of its elements. Before embarking on this, we prove (by much shorter and more elegant methods) two results about the CS-closure of a sequence. In both cases, the proof uses a linear mapping defined on l_1 as in Corollary 4 of Theorem 1. We denote by $x(n)$ the nth term of a numerical sequence x.

PROPOSITION 6. *Let $\{a_n\}$ be a bounded sequence in a sequentially complete, locally convex space. Then the CS-closure of $\{a_n\}$ is the set of all sums of convex series $\Sigma\lambda_n a_n$.*

Proof. For any x in l_1, the series $\sum_{n=1}^{\infty} x(n)a_n$ is convergent: let its sum be $T(x)$. Then T is continuous (local convexity is used here). Let V be the set of sequences x in l_1 such that $x(n) \geqslant 0$ for each n and $\sum_{n=1}^{\infty} x(n) = 1$. Then V is CS-compact, so $T(V)$ is CS-compact.

The second result is the only one in which we conclude that the CS-closure of a set is closed. The proof is by a neat device first employed (as far as the author knows) by Dr N. J. Kalton. If (X, Y) is a dual pair, we denote by $\sigma(Y)$ the weak topology induced by Y on X.

PROPOSITION 7. *Let (X, τ) be a sequentially complete, locally convex space, with dual X^*. Let $\{a_n\}$ be a sequence in X that converges to 0 with respect to $\sigma(X^*)$. Then the set of all elements of the form $\sum_{n=1}^{\infty} \lambda_n a_n$, where all $\lambda_n \geqslant 0$ and $\sum_{n=1}^{\infty} \lambda_n \leqslant 1$, is $\sigma(X^*)$-compact. Hence it is closed with respect to τ and $\sigma(X^*)$, and is the CS-closure of $\{a_n\} \cup \{0\}$ in either topology.*

Proof. Let T be defined as in Proposition 6, and let W be the set of all sequences x in l_1 such that $x(n) \geqslant 0$ for each n and $\Sigma x(n) \leqslant 1$. Then W is $\sigma(c_0)$-compact. We show that T is continuous with respect to the topologies $\sigma(c_0)$ and $\sigma(X^*)$; it then follows that $T(W)$ is $\sigma(X^*)$-compact, as stated. Choose $f \in X^*$. Then $\{f(a_n)\}$ is an element h of c_0. If $x \in l_1$, then

$$\langle Tx, f \rangle = \sum_{n=1}^{\infty} x(n) f(a_n) = \langle h, x \rangle.$$

The required continuity of T follows.

Elementary examples show that the CS-closure of $\{a_n\}$, excluding 0, need not be closed.

THEOREM 3. *Let X be a metrizable topological linear space. Then the CS-closure of a subset of X is the set of all sums of convergent convex series of its elements.*

Proof. Let A_0 denote the set of sums of convergent convex series of elements of A. Let $\sum_{n=1}^{\infty} \lambda_n x_n$ be a convergent convex series of elements of A_0, with sum x. For each n, there are elements $a_{j,n}$ of A and $\alpha_{j,n}$ of $[0, 1]$ such that

$$\sum_{j=1}^{\infty} \alpha_{j,n} = 1, \quad \sum_{j=1}^{\infty} \alpha_{j,n} a_{j,n} = x_n.$$

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Let $\{U_k\colon k = 1, 2, \ldots\}$ be a base of neighbourhoods of 0 such that $U_k + U_k \subseteq U_{k-1}$ for each $k \geqslant 2$. For each k, choose an integer $N(k) > N(k-1)$ such that

$$x - \sum_{n=1}^{N} \lambda_n x_n \in U_{k+1} \quad \text{for} \quad N \geqslant N(k)$$

and

$$\sum_{n=1}^{N(k)} \lambda_n > 1 - \frac{1}{k}.$$

Then take a circled neighbourhood V_k of 0 such that the sum of any $N(k)$ elements of V_k is in U_{k+1}, and choose $J(k) > J(k-1)$ such that for $n = 1, 2, \ldots, N(k)$, we have

$$x_n - \sum_{j=1}^{J} \alpha_{j,n} a_{j,n} \in V_k \quad \text{for} \quad J \geqslant J(k)$$

and

$$\sum_{j=1}^{J(k)} \alpha_{j,n} > 1 - \frac{1}{k}.$$

Arrange the elements $\lambda_n \alpha_{j,n} a_{j,n}$ as a series by adding at stage k those with

$$J(k-1) < j \leqslant J(k) \quad \text{and} \quad N(k-1) < n \leqslant N(k)$$

(in some order). All partial sums occurring after stage k are in $x + U_k$, so the series converges to x. Also,

$$\Sigma\{\lambda_n \alpha_{j,n} \colon (j,n) \in D\} > \left(1 - \frac{1}{k}\right)^2$$

if D is any set containing

$$\{(j,n)\colon 1 \leqslant j \leqslant J(k) \quad \text{and} \quad 1 \leqslant n \leqslant N(k)\},$$

so the series is convex.

In a non-metrizable space, the set of limits of sequences in a linear subspace need not be sequentially closed ((1), pp. 209–213). Hence metrizability is not dispensable in Theorem 3.

6. *Different topologies.* We shall say that two locally convex topologies on X are *compatible* if they make the same linear functionals continuous. Motivated by the fact that the same convex sets are closed with respect to two compatible topologies, we shall seek to prove similar results for CS-compact and CS-closed sets.

First, we remark that if σ is a smaller topology than τ, then every σ-CS-closed set is clearly τ-CS-closed, and every τ-CS-compact set is clearly σ-CS-compact.

The following lemma is well known:

LEMMA. *Let σ, τ be two topologies for a linear space X such that σ is smaller than τ and such that τ has a local base consisting of σ-closed sets. If a sequence $\{x_n\}$ is τ-Cauchy and σ-convergent, then it is τ-convergent.*

THEOREM 4. *Let τ_1, τ_2 be two compatible locally convex topologies for a linear space X. Then the same sets are CS-compact with respect to τ_1 and τ_2.*

Proof. It is sufficient to show that if τ is any locally convex topology for X, and σ is the associated weak topology, then every σ-CS-compact set A is τ-CS-compact. Let

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$\Sigma\lambda_n a_n$ be a convex series of elements of A. The series has a σ-limit x in A. Now A is σ-bounded, so τ-bounded, and hence (as in Proposition 2) the sequence of partial sums of $\Sigma\lambda_n a_n$ is τ-Cauchy. By the lemma, the series is therefore τ-convergent to x.

The same method can be applied to give an analogue of the Banach–Mackey theorem. If (X, τ) is a locally convex space, then the Lebesgue–Toeplitz proof ((6), pp. 254–255) shows that every τ-CS-compact subset of X is bounded with respect to the strong topology $\beta(X, X^*)$ induced on X by X^*. It is not necessary to assume that (X, τ) is sequentially complete: this hypothesis is effectively replaced by CS-compactness. Reasoning as in Theorem 4, we therefore conclude:

PROPOSITION 8. *If (X, τ) is a locally convex space, then the same sets are CS-compact with respect to τ and $\beta(X, X^*)$.*

The above lemma also enables us to solve the problem for *bounded* CS-closed sets (this was pointed out by Dr J. D. Pryce).

PROPOSITION 9. *Let τ_1, τ_2 be two compatible locally convex topologies for a linear space X. Then the same bounded sets are CS-closed with respect to τ_1 and τ_2.*

Proof. The statement is meaningful because the same sets are bounded with respect to τ_1 and τ_2. It is sufficient to show that if τ is any locally convex topology for X, and σ is the associated weak topology, then every bounded, τ-CS-closed set A is σ-CS-closed. Let $\Sigma\lambda_n a_n$ be a convex series of elements of A that is σ-convergent to x. Since A is τ-bounded, the sequence of partial sums is τ-Cauchy. By the lemma, the series is τ-convergent to x. Hence $x \in A$.

We now show (with considerably more work) that the problem of unbounded CS-closed sets has a satisfactory solution in the presence of metrizability.

THEOREM 5. *Let σ, τ be compatible locally convex topologies for a linear space X, τ being metrizable. Then*:

(i) *If $A \subseteq X$ and x is the σ-sum of a convex series of elements of A, then x is the τ-sum of (another) convex series of elements of A.*

(ii) *The same sets are CS-closed with respect to σ and τ.*

Proof. It is elementary that σ is not larger than τ, and consequently that (i) implies (ii). To prove (i), let x be the σ-sum of $\Sigma\lambda_n a_n$. Write $b_n = \lambda_n a_n$, and $c_1 = x$,

$$c_n = x - (b_1 + \ldots + b_{n-1}) \quad (n \geqslant 2).$$

Then $b_n = c_n - c_{n+1}$ $(n \geqslant 1)$, and $c_n \to 0$ with respect to σ. Therefore, for each k, 0 is in the closure of $\operatorname{co}\{c_n : n \geqslant k\}$ (which is the same with respect to σ and τ). Let $\{U_n\}$ be a decreasing base of τ-neighbourhoods of 0. Let $x_1 = c_1 = x$, and for each $n \geqslant 2$, let $x_n = \sum_{i \in K_n} \alpha_i c_i$ be a convex combination of c_i $(i \in K_n)$ such that $x_n \in U_n$, where K_n is a block of integers of the form

$$\{k(n-1)+1, \ldots, k(n)\}.$$

Then $x_n \to 0$ with respect to τ. For $i \in K_n$, write

$$\beta_i = \Sigma\{\alpha_j : j \in K_n \quad \text{and} \quad j \leqslant i\}.$$

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Then $\beta_{k(n)} = 1$, and

$$\sum_{i \in K_n} \beta_i b_i = \sum_{i \in K_n} \alpha_i c_i - c_{k(n)+1}$$

$$= x_n - c_{k(n)+1}.$$

Hence also

$$\sum_{i \in K_n} (1-\beta_i) b_i = c_{k(n-1)+1} - x_n,$$

and

$$x_n - x_{n+1} = \sum_{i \in K_n} \beta_i b_i + \sum_{i \in K_{n+1}} (1-\beta_i) b_i \quad (n \geqslant 1),$$

so we can write $x_n - x_{n+1} = \mu_n a_n'$, where $a_n' \in A$ and

$$\mu_n = \sum_{i \in K_n} \beta_i \lambda_i + \sum_{i \in K_{n+1}} (1-\beta_i) \lambda_i.$$

Then

$$\mu_1 + \ldots + \mu_n = \sum_{i=1}^{k(n)} \lambda_i + \sum_{i \in K_{n+1}} (1-\beta_i) \lambda_i,$$

so $\sum_{n=1}^{\infty} \mu_n = 1$. Also $\sum_{j=1}^{n} \mu_j a_j' = x - x_{n+1}$, so $\sum_{n=1}^{\infty} \mu_n a_n'$ is τ-convergent to x.

This theorem enables us to prove that Theorem 3 holds for topologies compatible with a metrizable one:

COROLLARY. *Let σ, τ be compatible locally convex topologies, τ being metrizable. Then the σ-CS-closure of a set is the set of all sums of σ-convergent convex series of its elements.*

Proof. By statement (ii) of Theorem 5, the σ-CS-closure of A is the same as the τ-CS-closure. By Theorem 3, this is the set of all sums of τ-convergent convex series of its elements. By statement (i) of Theorem 5, this is the set of all sums of σ-convergent convex series of its elements.

Theorem 1 also holds for topologies compatible with a metrizable one. For if σ and τ are compatible locally convex topologies, and σ is smaller than τ, then every τ-semi-closed set is σ-semi-closed (any σ-interior points of the closure are τ-interior points, so are in the set).

Metrizability is not dispensable in theorem 5. To show this let τ be the Mackey topology induced on l_1 by c_0. The same sequences converge with respect to τ and the usual norm of l_1 ((3), Theorem 12, p. 977). Consequently, the τ-CS-closed linear subspaces of l_1 are the norm-closed ones. However, the Krein–Šmul'yan theorem shows that the $\sigma(c_0)$-CS-closed linear subspaces are the $\sigma(c_0)$-closed ones.

7. *A counter-example on non-support points.* Suppose that A is CS-closed and $x_0 \in A$. Pursuing the analogy with closed, convex sets, one might suppose that x_0 can be separated from A in the weak sense that there is a non-zero continuous linear functional f such that $f(x_0) \geqslant \sup f(A)$, in other words, that A contains all its non-support points. The following example shows that this is not the case.

In l_1, let A be the set of sequences $\{\alpha_n\}$ such that $\alpha_{2n} = \alpha_{2n+1}$ for all n and

$$\alpha_1 = \sum_{n=2}^{\infty} \alpha_n, \quad \sum_{n=1}^{\infty} |\alpha_n| \leqslant 1.$$

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Let B be the set of sequences $\{\beta_n\}$ such that $\beta_{2n-1} = \beta_{2n}$ for all n and $\sum_{n=1}^{\infty} |\beta_n| \leqslant 1$. Then A and B are bounded, closed, convex sets, so $A+B$ is CS-compact. Let u be the sequence having $\frac{1}{4}$ in place 1 and 0 elsewhere. Then it is a straightforward matter to verify that $u \notin A+B$ and that u is a non-support point of $A+B$.

REFERENCES

(1) Banach, S. *Théorie des Opérations Linéaires* (Warsaw, 1932).

(2) Ellis, A. J. The duality of partially ordered normed linear spaces. *J. London Math. Soc.* **39** (1964), 730–744.

(3) Garling, D. J. H. The β- and γ-duality of sequence spaces. *Proc. Cambridge Philos. Soc.* **63** (1967), 963–981.

(4) Jameson, G. J. O. *Ordered linear spaces.* Springer Lecture Notes no. 141 (Berlin, 1970).

(5) Klee, V. L. Convex sets in linear spaces II. *Duke Math. J.* **18** (1951), 875–883.

(6) Köthe, G. *Topologische lineare Räume I* (Berlin, 1960).

(7) Tukey, J. W. Some notes on the separation of convex sets. *Portugal. Math.* **3** (1942), 95–102.

https://doi.org/10.1017/S0305004100050933 Published online by Cambridge University Press