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THE EXACT DISCRETE MODEL OF A THIRD-ORDER SYSTEM OF LINEAR STOCHASTIC DIFFERENTIAL EQUATIONS WITH OBSERVABLE STOCHASTIC TRENDS

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The objective of this paper is to develop closed-form formulae for the exact discretization of a third-order system of stochastic differential equations, with fixed initial conditions, driven by observable stochastic trends and white noise innovations. The model provides a realistic alternative to first- and second-order differential equation specifications of the time lag distribution, forming the basis of a testing and estimation procedure. The exact discrete models, derived under two sampling schemes with either stock or flow variables, are put into a system error correction form that preserves the information of the underlying continuous time model regarding the order of integration and the dimension of cointegration space.

Keywords: Continuous Time Models, Exact Discretization, Stochastic Trends

1. INTRODUCTION

Most of the established econometric methods for the parametric estimation of linear systems of stochastic differential equations (SDEs) fall into three main categories: spectral methods [Robinson (1977, 1993)], Kalman filtering methods [Harvey and Stock (1985, 1988)], and methods based on the exact discretization of the underlying continuous time model [Bergstrom (1983, 1985, 1986)]. The relative merits and disadvantages of these methods are discussed in Bergstrom (1985, Section 1).

Notably, the majority of estimation methods in macroeconomic modeling of national economies were based on the exact discrete model (EDM). In a prominent application, Bergstrom, Nowman, and Wymer (1992), henceforth BNW, constructed and estimated a second-order system of partial adjustment equations for the UK economy. Gandolfo and Padoan (1990), following the general framework of the Bergstrom and Wymer (1976) model, estimated a first-order partial adjustment system for the Italian economy. Bergstrom and Nowman (2007), based on the

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results of Bergstrom (1997), extended the BNW model to include unobservable stochastic trends arising from factors such as technological progress and changes in the nonaccelerating level of employment.

In most theoretical work on exact discretization, the underlying continuous time model is a second-order system of SDEs. However, in applied econometric work, it is desirable to have additional modeling flexibility by allowing higher-order dynamics. BNW did acknowledge that a sufficiently realistic dynamic specification requires at least a second-order system. The problem of deriving the EDM of a general kth-order continuous time system from mixed stock and flow samples was addressed by Chambers (1999). McGarry (2003), by applying the method of Chambers (1999), obtained the EDM of a fourth-order system, which is pertinent to modeling seasonally unadjusted quarterly data.

The aim of this paper is to derive the EDM of a third-order system of SDEs with fixed initial conditions driven by white noise innovations as well as by stochastic trends. With regard to nonstationarity, this paper differs from that of Chambers (1999) in that nonstationarity is introduced explicitly through $I(1)$ stochastic trends. It also differs from that of Bergstrom (1997) in that the stochastic trends are assumed to be observable. By modeling the order of integration and the dimension of the cointegration space in the underlying continuous time system, we are able to transform the EDM into an error correction form that embeds this information, so that the optimal inference framework of Phillips (1991a, 1991b) applies. Two sampling schemes are examined closely. In the case of stock variables, it is shown that the EDM can be transformed into an error correction model with moving average innovations of order two, whereas in the case of flow variables, we obtain an error correction model with moving average innovations of order three.

It should be remarked that the EDMs established in this paper cannot be viewed as special cases of Theorems 2 and 3 of Chambers (1999). The EDM for higher-order systems derived by Chambers (1999) is particularly directed toward mixed stock and flow samples. Most EDMs derived for mixed samples exhibit excess differencing.[1] The exact discretization schemes, stated in Theorems 1 and 3, do not suffer from excess differencing, thus preserving the integration properties of the underlying continuous time system. Although excess differencing may not affect the value of the likelihood function, so that it does not invalidate the Gaussian estimation procedure, it complicates the investigation of the large sample properties of the estimates [Bergstrom (1997, p. 485)] and also of the cointegration properties of the EDM.

In the remainder of this section, we provide some examples in which a third-order differential equation or system of equations arises.

1.1. Partial Adjustment Models in Macroeconomics

The partial adjustment mechanism described by the third-order differential equation

$$D^3 y(t) = -\gamma_1 D^2 y(t) - \gamma_2 D y(t) + \gamma_3 [x^*(t) - y(t)] \qquad (\mathbf{1})$$

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is able to generate more complex dynamic behavior than a second-order model. First, notice that the necessary and sufficient condition for the model to be stable around the steady state, that all roots of the characteristic equation have negative real parts, is satisfied when $\gamma_i > 0$, $i = 1, \ldots, 3$, and $\gamma_1\gamma_2 - \gamma_3 > 0$ [Gandolfo (1997, pp. 220–221)]. However, when there exist parameter values such that $\gamma_i > 0$, $i = 1, \ldots, 3$, and $\gamma_1\gamma_2 - \gamma_3 = 0$, the characteristic equation has one real negative root and a pair of complex roots with zero real parts. For this configuration of adjustment parameters, equation (1) generates a stable limit cycle around the steady state, thus providing a testable hypothesis regarding the cyclical behavior of the economy. In the jargon of bifurcation analysis, this dynamic behavior is called Hopf bifurcation. For an application of bifurcation analysis to the BNW model, see Barnett and He (1998).

Moreover, in macroeconometrics, equation (1) may provide an alternative realistic specification of the time lag distribution that can be compared to and tested against second-order systems. For example, in the BNW second-order system, the portmanteau test statistic, which tests the dynamic specification of the model, rejects it, although marginally, at the 5% significance level. In this case, a third-order differential equation for some endogenous variables might be appropriate.

Furthermore, the EDM for the third-order system, derived in this paper, and the corresponding EDM for the second-order system can be the basis for an order selection criterion, based for example on the likelihood ratio, information, or forecast performance [see Lutkepohl (2005, p. 146)]. This test can be the first stage of a three-stage testing strategy proposed by Bergstrom (1990a, p. 154) in the framework of continuous time systems.

The adjustment mechanism (1) can be justified either from macroeconomic dynamics—see Peel and Metcalfe (1979, eq. 18) for a model of third-order dynamics of aggregate prices with adaptive expectations—or by adopting an Euler equation approach. In discrete time, the problem of estimating Euler equations arising from linear quadratic problems with adjustment costs and nonstationary $I(1)$ forcing variables has been investigated by Dolado et al. (1991) and by Gregory et al. (1993). Next, we examine a continuous time model with an optimizing agent facing correlated unobservable shocks and observable exogenous stochastic trends.

1.2. A Model of Factor Demand

Consider the following version of the factor demand model proposed by Hansen and Sargent (1981, p. 5). A competitive firm maximizes the objective function

$$\lim_{T\to\infty} E_0 \int_0^T e^{-rt} \left\{ [\varepsilon(t) + z(t)]' y(t) - \frac{1}{2} y(t)' H_1 y(t) \right.$$
$$\left. - \frac{1}{2} [G(D)y(t)]' H_2 [G(D)y(t)] \right\} dt, \qquad (2)$$

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where E_0 is the expectations operator conditional on the information set up to time $t=0$, $y(t)$ is an $n \times 1$ vector of production factor demands, H_1 is a symmetric $n \times n$ semidefinite matrix, H_2 is a symmetric $n \times n$ positive definite matrix, and $G(D) = G_0 + G_1 D + G_2 D^2$ is a matrix polynomial in the mean square differential operator D. Additionally, it is assumed that $z(t)$, the $n \times 1$ vector of real factor rentals, and $\varepsilon(t)$, the $n \times 1$ vector of unobservable first-order autocorrelated productivity shocks, are described, respectively, by

$$dz(t) = c\,dt + \zeta_1(dt) \tag{3}$$

and

$$d\varepsilon(t) = \mathrm{P}\varepsilon(t)\,dt + \zeta_2(dt). \tag{4}$$

For the purpose of deriving an explicit formula for the rational expectations solution, we assume that P is a diagonal $n \times n$ matrix, whereas the innovations $\zeta_1(dt)$ and $\zeta_2(dt)$ are uncorrelated random measures each having properties described in Bergstrom (1984, Section 2.2). Hansen and Sargent (1980) show that, under the saddlepath property, the demand for production factors is given by

$$D^2 y(t) = \Gamma_1 D y(t) + \Gamma_2 y(t) + \sum\nolimits_{j=1}^{2n} B_j E_t \int_0^\infty e^{\lambda_j s}\,[\varepsilon(t+s) + z(t+s)]\,ds, \tag{5}$$

where Γ_j, B_j, $j = 1, 2$, are $n \times n$ matrices, and λ_j, $j = 1, \ldots, 2n$, are scalars with negative real parts. The coefficient matrices and scalars are defined in Hansen and Sargent (1980, pp. 10–11) and depend upon a more basic structural parameter vector reflecting preferences and technology in general.

The optimal forecasting rule of (3) is given by $E_t z(t+s) = z(t) + cs$. Utilizing this rule yields

$$E_t \int_0^\infty e^{\lambda_j s} z(t+s)\,ds = \left[\int_0^\infty e^{\lambda_j s}\,ds\right] z(t) + \left[\int_0^\infty e^{\lambda_j s}\,s\,ds\right] c = \frac{1}{\lambda_j}\,[c - z(t)]\,. \tag{6}$$

Similarly, from (4), we obtain

$$E_t \varepsilon(t+s) = e^{\mathrm{P}s}\varepsilon(t). \tag{7}$$

Therefore,

$$E_t \int_0^\infty e^{\lambda_j s}\varepsilon(t+s)\,ds = \mathrm{N}_j \varepsilon(t), \tag{8}$$

where $\mathrm{N}_j = [-\lambda_j^{-1} I - P]$.

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Substituting (6) and (8) into (5) and transforming the resulting equation using (4), we obtain

$$d[D^2y(t)] = \{[\Gamma_1 - \Upsilon]\, D^2y(t) + [\Gamma_2 - \Upsilon\Gamma_1]\, Dy(t) - \Upsilon\Gamma_2 y(t)\}\, dt$$
$$- [\Upsilon - I] \sum_{j=1}^{2n} \lambda_j^{-1} B_j c dt + \Upsilon \sum_{j=1}^{2n} \lambda_j^{-1} B_j z(t)\, dt + \Pi\zeta_2(dt)$$
$$- \sum_{j=1}^{2n} \lambda_j^{-1} B_j \zeta_1(dt), \tag{9}$$

where $\Upsilon = \Pi P \Pi^{-1}$ and $\Pi = \sum_{j=1}^{2n} B_j[-\lambda_j^{-1} I - P]$.

Model (9), which incorporates all cross-equation restrictions arising from the rational expectations hypothesis, coupled with (3), belongs to the class of data-generating mechanisms to be investigated in this paper.

1.3. Feedback with Integral Effects on Macroeconomic Models

A third-order system may also arise by controlling a system of second-order differential equations using integral feedback effects. Consider the deterministic version of the optimal control problem investigated in Bergstrom (1990b). It was proved by Bergstrom (1990b) that feedback of the type

$$x(t) = K_1 y(t) + \int_{-\infty}^{t} e^{(t-r)K_3} [K_3K_1 + K_2]\, y(r)\, dr, \tag{10}$$

where $x(t)$ is the vector of control variables, $y(t)$ is the vector of state variables, and K_1, K_2, and K_3, are matrices satisfying the Riccati equations, is optimal in the class of quadratic cost functions with adjustment costs subject to the constraint that the dynamics of the state variables are specified by the second-order system

$$D^2y(t) = A_1 Dy(t) + A_2 y(t) + Bx(t). \tag{11}$$

Substituting the feedback rule from (10) into (11) and differentiating with respect to time, we obtain, under stationarity, the third-order system

$$D^3y(t) = A_1 D^2y(t) + [A_2 + BK_1]\, Dy(t) + B\,[K_3K_1 + K_2]\, y(t), \tag{12}$$

which determines the optimal law of motion for the state variables.

Gandolfo (1997, Section 17.1.2.3) discusses a specific economic example in which integral stabilization policy leads to a third-order differential equation. Feedback rules for models of national economies are quite important, because they may alter the dynamic stability and steady state properties of the model. For example, Bergstrom et al. (1994), by verifying the controllability property, showed

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that fiscal and monetary policy rules of the form (10), derived from optimal control theory, are able to stabilize the BNW model around its steady state.

The paper is organized as follows. The next section lays out the data-generating mechanism and its assumptions. The exact discrete-error-correction representation of the underlying continuous time system is stated in Section 3 for a sample of stock data and in Section 4 for a sample of flow data. Section 5 concludes the paper.

2. THE DATA-GENERATING MECHANISM

We shall deal with the following system of stochastic differential equations:

$$d[D^2x(t)] = [A_2(\theta)D^2x(t) + A_1(\theta)Dx(t) + A_0(\theta)x(t) + b(\theta) + C(\theta)z(t)]\,dt + \zeta_1(dt), \tag{13}$$

$$x(0) = \vartheta_0, \begin{bmatrix} Dx(0) \\ D^2x(0) \end{bmatrix} = \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix}, \tag{14}$$

$$dz(t) = \nu(\theta)\,dt + \zeta_2(dt), \tag{15}$$

$$z(0) = \vartheta_3, \tag{16}$$

where $\vartheta_0, \vartheta_1, \vartheta_2$ are $n \times 1$ fixed vectors, ϑ_3 is a $k \times 1$ fixed vector, $A_i(\theta)$, $i = 0, 1, 2$, are $n \times n$ matrices whose elements depend, possibly in a nonlinear fashion, on a more basic p-dimensional vector of parameters θ, $C(\theta)$ is a $n \times k$ matrix, and $b(\theta)$ and $\nu(\theta)$ are $n \times 1$ and $k \times 1$ vectors. The innovation vectors $\zeta_1(dt)$, $\zeta_2(dt)$ are white noise stochastic processes described precisely by the following assumption:

Assumption 1. $\zeta_1(dt)$ and $\zeta_2(dt)$ are, respectively, $n \times 1$ and $k \times 1$ vector random measures defined on all subsets of the half line $0 < t < \infty$ with finite Lebesgue measure and properties $E[\zeta_i(dt)] = 0$, $i = 1, 2$, $E[\zeta_i(dt)\zeta_j(dt)'] = \Sigma_{ij}(\sigma)dt$, $i, j = 1, 2$, where $\Sigma_{ij}(\sigma)$ are matrices depending on a vector σ of parameters, $\Sigma_{11}(\sigma)$ and $\Sigma_{22}(\sigma)$ are positive definite matrices, and $E[\zeta_i(\Delta_1)\zeta_j(\Delta_2)] = 0$ for any two disjoint subsets of the half line Δ_1, Δ_2.

In addition, the following two assumptions, which exclude certain degenerate cases, are imposed. Note that Assumption 3 could be relaxed, allowing a singular system matrix, without affecting the validity of exact discretization.

Assumption 2. The matrix $\Phi(\theta)$ defined in Appendix A is nonsingular.

Assumption 3. The matrix $A_0(\theta)$ is nonsingular.

Assumptions 2 and 3, respectively, extend Assumptions 4 and 2 of Bergstrom (1983) to the case of third-order systems. Because under Assumption 1 neither $D^2x(t)$ nor $z(t)$ is mean square differentiable, (13) and (15) will be interpreted in

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a subset $[0,\ t]$ of the real line as representing the stochastic integral equations

$$D^2x(t) - D^2x(0) = \int_0^t [A_2(\theta)D^2x(r) + A_1(\theta)Dx(r) + A_0(\theta)x(r) + b(\theta) + C(\theta)z(r)]dr + \int_0^t \zeta_1(dr), \tag{17}$$

$$z(t) = z(0) + \kappa t + \int_0^t \zeta_2(dr). \tag{18}$$

Henceforth, to ease notation, I suppress the dependence of the system matrices on the parameter vector θ. Subsequent analysis will be based on (13) formulated as a first-order system,

$$d\bar{y}(t) = \left[A\bar{y}(t) + S_3'b\right]dt + S_3'Cz(t)\,dt + S_3'\zeta_1(dt), \tag{19}$$

where

$$A = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ A_0 & A_1 & A_2 \end{bmatrix},$$

and $S_1 = [I_n\ \ 0_{nxn}\ \ 0_{nxn}]$, $S_2 = [0_{n\times n}\ \ I_n\ \ 0_{nxn}]$, $S_3 = [0_{n\times n}\ \ 0_{nxn}\ \ I_n]$, are partition matrices.

Under Assumption 1, $z(t)$ is integrable in the wide sense [Simos (1996, p. 367)]. Therefore, a solution to (19) over the interval $[0,\ t]$ exists and is unique in the class of mean squared differential processes given by

$$\bar{y}(t) = e^{At}\bar{y}(0) + \int_0^t e^{A(t-s)}S_3'bds + \int_0^t e^{A(t-s)}S_3'Cz(s)\,ds + \int_0^t e^{A(t-s)}S_3'\zeta_1(ds), \tag{20}$$

where

$$e^{At} = \sum\nolimits_{j=0}^{\infty} \frac{1}{j!}(At)^j, \tag{21}$$

$\bar{y}(0) = [\vartheta_0'\ \ \vartheta_1'\ \ \vartheta_2']'$.

In the following sections, for notational convenience, we shall in general denote any $x(t)$ by x_t.

3. THE EXACT DISCRETE MODEL: STOCK VARIABLES

Assume that the data set comprises only stock variables observed at the time points $t = 0, 1, 2, \ldots, T$. Then the EDM is given by the following two theorems.

THEOREM 1. *Let x_t and z_t be the solution of the system* (13) *and* (15)*, under the initial conditions* (14) *and* (16)*, on the time interval [0, T]. Then, under*

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Assumptions 1–3, x_t *and* z_t *satisfy the error correction system*

$$\begin{aligned}\Delta x_t = &-[I-\Pi_1]\Delta x_{t-1}-[I-\Pi_1-\Pi_2]\Delta x_{t-2}\\ &+[T_1+T_2]C\Delta z_{t-1}+[T_1+T_2+T_3]C\Delta z_{t-2}\\ &-[I-\Pi_1-\Pi_2-\Pi_3]x_{t-3}+[T_1+T_2+T_3+T_4]Cz_{t-3}\\ &+\delta+\eta_t,\quad t=3,\ldots,T,\end{aligned} \tag{22}$$

$$\Delta z_t = \nu+\varsigma_t, t=2,\ldots,T, \tag{23}$$

where

$$\eta_t = T_1C\varsigma_t+\varepsilon_{1,t}+\Theta_1\varepsilon_{t-1}+\Theta_2\varepsilon_{t-2},\ t=3,\ldots,T, \tag{24}$$

$$\varsigma_t = \int_{t-1}^{t}\zeta_2(dr),\ t=1,\ldots,\ T, \tag{25}$$

$\delta = T_1C\nu+\beta_1+(\Theta_1+\Theta_2)\beta,$

$$\beta = A^{-1}[(e^A-I)S_3'b+A^{-1}(e^A-I)S_3'C\nu], \tag{26}$$

$$\varepsilon_t = \int_{t-1}^{t}e^{A(t-s)}\,S_3'\zeta_1(ds)+A^{-1}\int_{t-1}^{t}e^{A(t-s)}\,S_3'C\zeta_2(ds), \tag{27}$$

$$\beta_i = S_i\beta,\ i=1,2,3,$$

$$\varepsilon_{i,t} = S_i\varepsilon_t,\ i=1,2,3,$$

and Δ *denotes the first difference operator, defined by* $\Delta x_t = x_t - x_{t-1}$ *for any* x_t.

The rest of the coefficient matrices of the EDM (22)–(25) are stated in Appendix B.1. Bergstrom (1986) noted that, in obtaining the parameter estimates, simply treating the initial observable vectors $x_1,\ x_2$ as fixed does not lead to the conditional likelihood function. Therefore, for a proper implementation of an exact Gaussian estimation method, we need to encompass the information originated from the equations that relate $x_1,\ x_2$ to the initial conditions $\vartheta_0, \vartheta_1, \vartheta_2, \vartheta_3$ in the likelihood function. The following theorem completes the EDM by stating these equations.

THEOREM 2. *Let* x_t *and* z_t *be the solution of the system* (13) *and* (15) *under the initial conditions* (14) *and* (16) *on the time interval* $[0,\ T]$. *Then, under Assumptions* 1–3, *the observable vectors* x_1 *and* x_2 *satisfy the system of equations*

$$x_1 = \phi_{11}\vartheta_0+\phi_{12}\vartheta_1+\phi_{13}\vartheta_2-[A^{-1}]_{13}Cz_1+[A^{-1}e^A]_{13}C\vartheta_3+\beta_1+\eta_1, \tag{28}$$

$$\eta_1 = S_1\varepsilon_1, \tag{29}$$

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$$x_2 = \phi_{11}x_1 + \Phi_1\vartheta_0 + \Phi_2\vartheta_1 + \Phi_3\vartheta_2 + [\phi_{12}[A^{-1}e^A]_{23} + \phi_{13}[A^{-1}e^A]_{33}]C\vartheta_0 + [\phi_{13}[A^{-1}]_{33} + [A^{-1}e^A]_{13} - \phi_{12}[A^{-1}]_{23}]Cz_1 - [A^{-1}]_{13}Cz_2 + \beta_1 + \phi_{12}\beta_2 + \phi_{13}\beta_3 + \eta_2, \quad (30)$$

$$\eta_2 = S_\phi\varepsilon_1 + S_1\varepsilon_2, \quad (31)$$

$$z_1 = \vartheta_3 + \kappa + \varsigma_1, \quad (32)$$

where $\Phi_1 = \phi_{12}\phi_{21} + \phi_{13}\phi_{31}$, $\Phi_2 = \phi_{12}\phi_{22} + \phi_{13}\phi_{32}$, $\Phi_3 = \phi_{12}\phi_{23} + \phi_{13}\phi_{33}$, *and* $S_\phi = \phi_{12}S_2 + \phi_{13}S_3$, *and all other coefficient matrices and vectors are defined as in Theorem* 1.

A necessary input for the construction and numerical evaluation of the exact Gaussian likelihood function, as well as in procedures such as testing and post-sample forecasting, is the autocovariance function of discrete time innovations. Closed-form formulae, derived under Assumptions 1–3, for the autocovariance function of discrete innovations η_j, ς_j, $j = 1, \ldots, T$, generated by (24), (25), (29), and (31), are stated in Appendix B.2.

These formulae establish that $\eta_t, t = 3, \ldots, T$, is a vector moving average process of order two and so the whole EDM, equations (22) and (23), is in the form of an error correction model with a vector moving average innovation of order at most two. Except for the moving average structure of the innovation process, (22) and (23) differ in two other respects from standard error correction models [utilized, for example, by Johansen (1995)]. First, an important characteristic is that a certain number of unit roots are imposed by construction via the underlying continuous time model.[2] In this way we are able to bring (22) and (23) into the class of models where full system maximum likelihood estimation methods are optimal, in the sense of Phillips (1991a, 1991b), and therefore, in contrast to unrestricted VAR models estimated in levels, conventional asymptotic theory is valid. Second, the EDM is highly nonlinear in the parameters. All the coefficient matrices in (22) and the error term covariance matrices are nonlinear functions of the underlying parameter vector θ, even if the continuous time model is linear in the parameters.

4. THE EXACT DISCRETE MODEL: FLOW VARIABLES

In this section we assume that $x(t), z(t), t \in [0, T]$, are flow variables; that is, they are observed as integrals $\int_{j-1}^{j} x(r)\,dr$, $\int_{j-1}^{j} z(r)\,dr$ over the intervals $[j-1, j]$, $j = 1, \ldots, T$. In this case we state the EDM in the following theorem.

THEOREM 3. *Let* x_t *and* z_t *be the solution of the system* (13) *and* (15) *under the initial conditions* (14) *and* (16), *on the time interval* $[0, T]$. *Then, under Assumptions* 1–3, *the observable vectors* $\int_{t-1}^{t} x(r)\,dr$, $\int_{t-1}^{t} z(r)\,dr$,

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$t = 4, \ldots, T$, satisfy the error correction system

$$\begin{aligned}\Delta \int_{t-1}^{t} x(r)\,dr &= -[I - \Pi_1]\,\Delta \int_{t-2}^{t-1} x(r)\,dr \\ &\quad - [I - \Pi_1 - \Pi_2]\,\Delta \int_{t-3}^{t-2} x(r)\,dr + [T_1 + T_2]\,C\Delta \int_{t-2}^{t-1} z(r)\,dr \\ &\quad + [T_1 + T_2 + T_3]\,C\Delta \int_{t-3}^{t-2} z(r)\,dr - [I - \Pi_1 - \Pi_2 - \Pi_3] \int_{t-4}^{t-3} x(r)\,dr \\ &\quad + [T_1 + T_2 + T_3 + T_4]\,C \int_{t-4}^{t-3} z(r)\,dr + \delta + \lambda_t, \quad t = 4, \ldots, T, \end{aligned} \tag{33}$$

$$\lambda_t = T_1 C \psi_t + S_1 \xi_t + \Theta_1 \xi_{t-1} + \Theta_2 \xi_{t-2}, \quad t = 4, \ldots, T, \tag{34}$$

$$\begin{aligned}\xi_t &= \int_{t-1}^{t} [A^{-1} e^{A(t-r)} - A^{-1}]\,S_3' \zeta_1(dr) \\ &\quad + \int_{t-2}^{t-1} \left[A^{-1} e^{A} - A^{-1} e^{A(t-1-r)}\right] S_3' \zeta_1(dr) + A^{-1}\left[\int_{t-1}^{t} \left[A^{-1} e^{A(t-r)}\right.\right. \\ &\quad \left.\left. - A^{-1}\right] S_3' C \zeta_2(dr) + \int_{t-2}^{t-1} \left[A^{-1} e^{A} - A^{-1} e^{A(t-1-r)}\right] S_3' C \zeta_2(dr)\right], \\ &\quad t = 4, \ldots, T, \end{aligned} \tag{35}$$

$$\Delta \int_{t-1}^{t} z(r)\,dr = \nu + \psi_t, \; t = 2, \ldots, T, \tag{36}$$

$$\psi_t = \int_{t-1}^{t} (t - r)\,\zeta_2(dr) + \int_{t-2}^{t-1} (r - t + 2)\,\zeta_2(dr), \; t = 2, \ldots, T, \tag{37}$$

where all coefficient matrices and vectors are defined as in Theorem 1.

Using Assumptions 1–3, it is a straightforward matter to establish the autocovariance function of the discrete innovation vectors λ_t, ξ_t and ψ_t generated by (34), (35), and (37). The autocovariance function of ξ_t implies that it is a vector moving average process of order one. Therefore, noting (34), (33) is an error correction model with a vector moving average innovation of order three. Similarly, ψ_t, $t = 2, \ldots, T$, given by (37), is a moving average innovation of order two. Therefore, the whole EDM, (33) and (36), is in an error correction form with a vector moving average innovation of order at most three.

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5. CONCLUDING REMARKS

The main contribution of this paper has been the derivation of the EDM of a third-order system of stochastic differential equations with fixed initial conditions, white noise innovations, and observable stochastic trends. A white noise innovation process is more easily justified in higher-order dynamic models. Therefore, a third-order system may provide a viable and useful alternative to second-order systems, taking into consideration that higher-order systems come at the cost of a parameter space of larger dimension. The results presented in Theorems 1–3 expand the range of alternative EDMs that are available for estimation and testing, whereas the inclusion of stochastic trends provides a more realistic alternative to simple deterministic trends. Based on the current results, the exact discretization method can be extended to more general forms of stochastic trends. For example, we can specify stable first- or second-order SDEs in either $Dz(t)$ or $D^2z(t)$, implying, respectively, either $I(1)$ or $I(2)$ time series properties.

NOTES

1. By excess differencing, we mean the fact that the stock variables enter into the observable vectors of the EDM in first differences. An important exception is Chambers (2008), who constructed the EDM of a first-order triangular cointegrated system of SDEs with observable stochastic trends under a mixed sampling scheme.

2. Under the assumption that the eigenvalues of the $3n \times 3n$ system matrix A of the stochastic differential equation (13) have negative real parts, the only source of nonstationarity, apart from initial conditions effects, is the $I(1)$ stochastic trends $z(t)$. If we further assume that the $n \times k$ matrix C has rank k, then all elements of the vector $x(t)$ are $I(1)$, ensuring that there are exactly n cointegrating vectors and k stochastic trends. These characteristics are preserved in the EDM, because it inherits the stability properties of the underlying continuous time model. Consequently, a version of Granger's representation theorem [see, for example, Theorem 4.2 in Johansen (1995)] is expected to hold in a framework that is nonlinear in the parameters VARMA(3, 2).

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APPENDIX A

A.1. PROOF OF THEOREM 1

First, we observe that equation (23) can be verified by integrating (15) in the time intervals $[t-1,\ t]$, $t = 2, \ldots, T$. Next, from (20), we obtain

$$\bar{y}(t) = e^{A}\bar{y}(t-1) + \int_{t-1}^{t} e^{A(t-s)} S_3' b ds + \int_{t-1}^{t} e^{A(t-s)}\ S_3' C z(s)\, ds\ + \int_{t-1}^{t} e^{A(t-s)}\ S_3'\zeta_1(ds). \tag{A.1}$$

Applying formula (22) in Simos (1996) and using the definition (15) establishes that

$$\int_{t-1}^{t} e^{A(t-s)}\ S_3' C z(s)\, ds = -A^{-1}\left[S_3' C z(t) - e^{A} S_3' C z(t-1) - \int_{t-1}^{t} e^{A(t-s)}\ S_3' C \nu\, ds - \int_{t-1}^{t} e^{A(t-s)}\ S_3' C \zeta_2(ds)\right]. \tag{A.2}$$

Then substituting from (A.2) into (A.1), under Assumption 3, yields

$$\bar{y}(t) = e^{A}\bar{y}(t-1) + \beta - A^{-1}\left[S_3' C z(t) - e^{A} S_3' C z(t-1)\right] + \varepsilon_t, \tag{A.3}$$

with β and ε_t stated in (26) and (27), respectively.

Now, separate system (A.3) into the following two subsystems:

$$x_t = \phi_{11} x_{t-1} + \phi_{12} D x_{t-1} + \phi_{13} D^2 x_{t-1} - [A^{-1}]_{13} C z_t + [A^{-1} e^{A}]_{13} C z_{t-1} + \beta_1 + \varepsilon_{1,t}, \tag{A.4}$$

$$\begin{bmatrix} Dx_t \\ D^2 x_t \end{bmatrix} = \begin{bmatrix} \phi_{21} \\ \phi_{31} \end{bmatrix} x_{t-1} + \begin{bmatrix} \phi_{22} & \phi_{23} \\ \phi_{32} & \phi_{33} \end{bmatrix} \begin{bmatrix} Dx_{t-1} \\ D^2 x_{t-1} \end{bmatrix} - \begin{bmatrix} [A^{-1}]_{23} \\ [A^{-1}]_{33} \end{bmatrix} C z_t + \begin{bmatrix} [A^{-1} e^{A}]_{23} \\ [A^{-1} e^{A}]_{33} \end{bmatrix} C z_{t-1} + \begin{bmatrix} \beta_2 \\ \beta_3 \end{bmatrix} + \begin{bmatrix} \varepsilon_{2,t} \\ \varepsilon_{3,t} \end{bmatrix}. \tag{A.5}$$

Then, lagging (A.5) by one through two periods and combining the resulting systems yields a system of $6n$ equations and $6n$ unknowns. Under Assumption 2, which ensures the existence of the inverse of

$$\Phi = \begin{bmatrix} 0 & 0 & -\phi_{12} & -\phi_{13} & 0 & 0 \\ I & 0 & -\phi_{22} & -\phi_{23} & 0 & 0 \\ 0 & I & -\phi_{32} & -\phi_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\phi_{12} & -\phi_{13} \\ 0 & 0 & I & 0 & -\phi_{22} & -\phi_{23} \\ 0 & 0 & 0 & I & -\phi_{32} & -\phi_{33} \end{bmatrix},$$

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TABLE A.1. Analytical inverse of matrix

$$[\Phi^{-1}]_{11} = (\phi_{22} - \phi_{23}\Lambda K^{-1})M^{-1}, [\Phi^{-1}]_{21} = (\phi_{32} - \phi_{33}\Lambda K^{-1})M^{-1}, [\Phi^{-1}]_{31} = M^{-1},$$
$$[\Phi^{-1}]_{41} = -\Lambda K^{-1}M^{-1}, [\Phi^{-1}]_{51} = -\phi_{13}K^{-1}M^{-1},$$
$$[\Phi^{-1}]_{61} = \phi_{12}K^{-1}M^{-1}, [\Phi^{-1}]_{14} = (\phi_{22} - \phi_{23}\Lambda K^{-1})M^{-1}\phi_{13}ZK^{-1} + \phi_{23}ZK^{-1},$$
$$[\Phi^{-1}]_{24} = (\phi_{32} - \phi_{33}\Lambda K^{-1})M^{-1}\phi_{13}ZK^{-1} + \phi_{33}ZK^{-1},$$
$$[\Phi^{-1}]_{34} = M^{-1}\phi_{13}ZK^{-1}, [\Phi^{-1}]_{44} = ZK^{-1} - \Lambda K^{-1}M^{-1}\phi_{13}ZK^{-1},$$
$$[\Phi^{-1}]_{54} = -\phi_{23}K^{-1} - \phi_{13}K^{-1}M^{-1}\phi_{13}ZK^{-1},$$
$$[\Phi^{-1}]_{64} = \phi_{22}K^{-1} - \phi_{12}K^{-1}M^{-1}\phi_{13}ZK^{-1}, [\Phi^{-1}]_{15} = (\phi_{22} - \phi_{23}\Lambda K^{-1})M^{-1}\phi_{13}\Lambda K^{-1} + \phi_{23}\Lambda K^{-1},$$
$$[\Phi^{-1}]_{25} = (\phi_{32} - \phi_{33}\Lambda K^{-1})M^{-1}\phi_{13}\Lambda K^{-1} + \phi_{33}\Lambda K^{-1}, [\Phi^{-1}]_{35} = M^{-1}\phi_{13}\Lambda K^{-1},$$
$$[\Phi^{-1}]_{45} = \Lambda K^{-1} - \Lambda K^{-1}M^{-1}\phi_{13}\Lambda K^{-1},$$
$$[\Phi^{-1}]_{55} = \phi_{13}K^{-1} - \phi_{13}K^{-1}M^{-1}\phi_{13}\Lambda K^{-1}, [\Phi^{-1}]_{65} = -\phi_{12}K^{-1} - \phi_{12}K^{-1}M^{-1}\phi_{13}\Lambda K^{-1},$$
$$[\Phi^{-1}]_{16} = (\phi_{22} - \phi_{23}\Lambda K^{-1})M^{-1}\phi_{13} + \phi_{23},$$
$$[\Phi^{-1}]_{26} = (\phi_{32} - \phi_{33}\Lambda K^{-1})M^{-1}\phi_{13} + \phi_{33}, [\Phi^{-1}]_{36} = M^{-1}\phi_{13}, [\Phi^{-1}]_{46} = I - \Lambda K^{-1}M^{-1}\phi_{13},$$
$$[\Phi^{-1}]_{56} = -\phi_{13}K^{-1}M^{-1}\phi_{13},$$
$$[\Phi^{-1}]_{66} = -\phi_{12}K^{-1}M^{-1}\phi_{13}, [\Phi^{-1}]_{12} = [\Phi^{-1}]_{23} = I_n,$$
$$[\Phi^{-1}]_{22} = [\Phi^{-1}]_{32} = [\Phi^{-1}]_{42} = [\Phi^{-1}]_{52} = [\Phi^{-1}]_{62} = 0,$$
$$[\Phi^{-1}]_{13} = [\Phi^{-1}]_{33} = [\Phi^{-1}]_{43} = [\Phi^{-1}]_{53} = [\Phi^{-1}]_{63} = 0,$$
$$K = \phi_{12}\phi_{23} - \phi_{13}\phi_{22}, Z = \phi_{33}\phi_{22} - \phi_{32}\phi_{23}, \quad \Lambda = \phi_{32}\phi_{13} - \phi_{33}\phi_{12}, M = -\phi_{12} + \phi_{13}\Lambda K^{-1}$$

we can solve the system with respect to the unknowns Dx_{t-j}, D^2x_{t-j}, $j = 1, 2, 3$. Then, eliminating Dx_{t-1} and D^2x_{t-1} from (A.4), we obtain the VARMA(3,2) representation of the EDM,

$$x_t = \Pi_1 x_{t-1} + \Pi_2 x_{t-2} + \Pi_3 x_{t-3} + T_1 C z_t + T_2 C z_{t-1} + T_3 C z_{t-2}$$
$$+ T_4 C z_{t-3} + \beta_1 + (\Theta_1 + \Theta_2)\beta + \eta_t, \; t = 3, \ldots, \; T, \tag{A.6}$$

where all coefficient matrices and vectors are stated in Appendices A and B. Note that the formulae for the coefficient matrices and vectors make use of the formulae for the inverse Φ^{-1} (derived by repeated application of block matrix inversion) stated in Table A.1. Moreover, using standard operations [see, for example, Johansen (1995, p. 45)], the following identity:

$$T_1 C z_t + T_2 C z_{t-1} + T_3 C z_{t-2}$$
$$= T_1 C \Delta z_t + (T_1 + T_2) C \Delta z_{t-1} + (T_1 + T_2 + T_3) C \Delta z_{t-2}$$
$$+ (T_1 + T_2 + T_3) C z_{t-3}, \tag{A.7}$$

and (23), the EDM (A.6) can be formulated in the error correction format (22). ■

A.2. PROOF OF THEOREM 2

Equations (28) and (29) are derived by putting $t = 1$ in (A.4) and using the definitions (14) and (16). Moreover, putting $t = 2$ in (A.4) and $t = 1$ in (A.5) yields

$$x_2 = \phi_{11} x_1 + [\phi_{12} \;\; \phi_{13}] \begin{bmatrix} Dx_1 \\ D^2 x_1 \end{bmatrix}$$
$$- [A^{-1}]_{13} C z_2 + [A^{-1} e^A]_{13} C z_1 + \beta_1 + S_1 \varepsilon_2 \tag{A.8}$$

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and

$$\begin{bmatrix} Dx_1 \\ D^2x_1 \end{bmatrix} = \begin{bmatrix} \phi_{21} \\ \phi_{31} \end{bmatrix} x_0 + \begin{bmatrix} \phi_{22} & \phi_{23} \\ \phi_{32} & \phi_{33} \end{bmatrix} \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix} - \begin{bmatrix} [A^{-1}]_{23} \\ [A^{-1}]_{33} \end{bmatrix} Cz_1$$
$$+ \begin{bmatrix} [A^{-1}e^A]_{23} \\ [A^{-1}e^A]_{33} \end{bmatrix} Cz_0 + \begin{bmatrix} \beta_2 \\ \beta_3 \end{bmatrix} + \begin{bmatrix} S_2 \\ S_3 \end{bmatrix} \varepsilon_1. \tag{A.9}$$

Substituting from (A.9) into (A.8), for $[Dx_1' \;\; D^2x_1']'$, establishes (30) and (31).

To obtain (32) we integrate (15) over the interval [0, 1] and use the definition (14). ■

A.3. PROOF OF THEOREM 3

The EDM in this case can be obtained by integrating (A.7) over the interval $[t-1,\ t]$, $t = 4, \ldots, T$. We obtain the VARMA(3,3) representation

$$\int_{t-1}^{t} x(r)\,dr = \Pi_1 \int_{t-2}^{t-1} x(r)\,dr + \Pi_2 \int_{t-3}^{t-2} x(r)\,dr$$
$$+\Pi_3 \int_{t-4}^{t-3} x(r)\,dr + T_1C \int_{t-1}^{t} z(r)\,dr + T_2C \int_{t-2}^{t-1} z(r)\,dr + T_3C \int_{t-3}^{t-2} z(r)\,dr$$
$$+T_4C \int_{t-4}^{t-3} z(r)\,dr + \beta_1 + (\Theta_1 + \Theta_2)\beta + \lambda_t, \quad t = 4, \ldots, T, \tag{A.10}$$

where all matrices and vectors are defined as in Theorems 1 and 3. Note that the error term ξ_t stated in (35) was obtained by applying Bergstrom (1997, equation A.8). Moreover, the error correction representation (33) can be deduced from the EDM (A.10) in the same way as (22) was deduced from (A.7).

Now, I turn to the exact discretization of stochastic trends. Integrating (23) over the interval $[t-1,\ t]$, $t = 2, \ldots, T$, we obtain (36) with ψ_t given by

$$\psi_t = \int_{t-1}^{t} \int_{\tau-1}^{\tau} \zeta_2(dr)\,d\tau.$$

Equation (37) is obtained by applying Theorem 2.4 in Rozanov (1967, p. 12). Let $\varphi(\tau, r)$ be the indicator function; that is,

$$\varphi(\tau, r) = \begin{cases} 1 & r \in [\tau - 1,\ \tau] \\ 0 & r \notin [\tau - 1,\ \tau] \end{cases}.$$

Then

$$\psi_t = \int_{t-1}^{t} \int_{\tau-1}^{\tau} \varphi(\tau, r)\,\zeta_2(dr)\,d\tau = \int_{t-1}^{t} \left[\int_{t-2}^{t} \varphi(\tau, r)\,\zeta_2(dr) \right] d\tau$$
$$= \int_{t-2}^{t} \left[\int_{t-1}^{t} \varphi(\tau, r)\ d\tau \right] \zeta_2(dr) = \int_{t-2}^{t-1} \left[\int_{t-1}^{r+1} \varphi(\tau, r)\ d\tau \right] \zeta_2(dr)$$
$$+ \int_{t-1}^{t} \left[\int_{r}^{t} \varphi(\tau, r)\ d\tau \right] \zeta_2(dr) = \int_{t-2}^{t-1} (r - t + 2)\,\zeta_2(dr)$$
$$+ \int_{t-1}^{t} (t - r)\,\zeta_2(dr).$$

■

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APPENDIX B

B.1. COEFFICIENT MATRICES RELATED TO THE EXACT DISCRETE MODEL OF THEOREM 1

$$\Theta_1 = \left[\Theta_1^1 \quad \Theta_1^2 \quad \Theta_1^3\right],$$

$$\Theta_1^1 = \phi_{12}[\Phi^{-1}]_{11} + \phi_{13}\left[\Phi^{-1}\right]_{21},$$

$$\Theta_1^2 = \phi_{12},$$

$$\Theta_1^3 = \phi_{13},$$

$$\Theta_2 = \left[\Theta_2^1 \quad \Theta_2^2 \quad \Theta_2^3\right],$$

$$\Theta_2^1 = \phi_{12}[\Phi^{-1}]_{14} + \phi_{13}[\Phi^{-1}]_{24},$$

$$\Theta_2^2 = \phi_{12}[\Phi^{-1}]_{15} + \phi_{13}[\Phi^{-1}]_{25},$$

$$\Theta_2^3 = \phi_{12}\left[\Phi^{-1}\right]_{16} + \phi_{13}[\Phi^{-1}]_{26},$$

$$\Pi_1 = \phi_{11} - \Theta_1^1,$$

$$\Pi_2 = \Theta_1^1\phi_{11} + \Theta_1^2\phi_{21} + \Theta_1^3\phi_{31} - \Theta_2^1,$$

$$\Pi_3 = \Theta_2^1\phi_{11} + \Theta_2^2\phi_{21} + \Theta_2^3\phi_{31},$$

$$T_1 = -[A^{-1}]_{13},$$

$$T_2 = [A^{-1}e^A]_{13} - \Theta_1^1[A^{-1}]_{13} - \Theta_1^2[A^{-1}]_{23} - \Theta_1^3[A^{-1}]_{33},$$

$$T_3 = \Theta_1^1[A^{-1}e^A]_{13} + \Theta_1^2[A^{-1}e^A]_{23} + \Theta_1^3[A^{-1}e^A]_{33} - \Theta_2^1[A^{-1}]_{13} - \Theta_2^2[A^{-1}]_{23} - \Theta_2^3[A^{-1}]_{33},$$

$$T_4 = \Theta_2^1[A^{-1}e^A]_{13} + \Theta_2^2[A^{-1}e^A]_{23} + \Theta_2^3[A^{-1}e^A]_{33},$$

$$e^A = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{bmatrix},$$

$$A^{-1} = \begin{bmatrix} [A^{-1}]_{11} & [A^{-1}]_{12} & [A^{-1}]_{13} \\ [A^{-1}]_{21} & [A^{-1}]_{23} & [A^{-1}]_{33} \\ [A^{-1}]_{31} & [A^{-1}]_{32} & [A^{-1}]_{33} \end{bmatrix},$$

$$A^{-1}e^A = \begin{bmatrix} [A^{-1}e^A]_{11} & [A^{-1}e^A]_{12} & [A^{-1}e^A]_{13} \\ [A^{-1}e^A]_{21} & [A^{-1}e^A]_{23} & [A^{-1}e^A]_{33} \\ [A^{-1}e^A]_{31} & [A^{-1}e^A]_{32} & [A^{-1}e^A]_{33} \end{bmatrix}.$$

Note that $\phi_{ij}, [A^{-1}e^A]_{ij}, [A^{-1}]_{ij}, i, j = 1, 2, 3$, are $n \times n$ matrices. The matrices $[\Phi^{-1}]_{ij}$, $i, j = 1, \ldots, 6$, and K, Z, Λ, M are defined in Table A.1.

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B.2. SECOND-ORDER PROPERTIES OF THE DISCRETE TIME INNOVATIONS

$[\eta_t' \ \varsigma_t']', t = 1, \ldots, T$

$$E[\eta_1\eta_1'] = S_1 V_\varepsilon(0) S_1',$$

$$E[\eta_1\eta_2'] = S_1 V_\varepsilon(0) S_\phi',$$

$$E[\eta_1\eta_3'] = S_1 V_\varepsilon(0)\Theta_2',$$

$$E[\eta_1\eta_t'] = 0, \quad t = 4, \ldots, T,$$

$$E[\eta_2\eta_2'] = S_\phi V_\varepsilon(0) S_\phi' + S_1 V_\varepsilon(0) S_1',$$

$$E[\eta_2\eta_3'] = S_\phi V_\varepsilon(0)\Theta_2' + S_1 V_\varepsilon(0)\Theta_1',$$

$$E[\eta_2\eta_4'] = S_1 V_\varepsilon(0)\Theta_2',$$

$$E[\eta_2\eta_t'] = 0, \quad t = 5, \ldots, T,$$

$$E[\eta_t\eta_t'] = S_1 V_\varepsilon(0) S_1' + \Theta_1 V_\varepsilon(0)\Theta_1' + \Theta_2 V_\varepsilon(0)\Theta_2' + T_1 C \Sigma_{22} C' T_1', \quad t = 3, \ldots, T,$$

$$E[\eta_t\eta_{t+1}'] = S_1 V_\varepsilon(0)\Theta_1' + \Theta_1 V_\varepsilon(0)\Theta_2', \quad t = 3, \ldots, T,$$

$$E[\eta_t\eta_{t+2}'] = S_1 V_\varepsilon(0)\Theta_2', \quad t = 3, \ldots, T,$$

$$E[\eta_t\eta_{t+\tau}'] = 0, \quad \tau \geq 3, \quad t = 3, \ldots, T,$$

$$E[\varsigma_1\eta_1'] = \Sigma_{21} S_3 \int_0^1 e^{A'r} dr S_1' + \Sigma_{22} C' S_3 \int_0^1 e^{A'r} dr A'^{-1} S_1',$$

$$E[\varsigma_1\eta_2'] = \left[\Sigma_{21} S_3 \int_0^1 e^{A'r} dr + \Sigma_{22} C' S_3 \int_0^1 e^{A'r} dr A'^{-1}\right] S_\phi',$$

$$E[\varsigma_t\eta_t'] = \Sigma_{21} S_3 \int_0^1 e^{A'r} dr S_1' + \Sigma_{22} C' S_3 \int_0^1 e^{A'r} dr A'^{-1} S_1' + \Sigma_{22} C' T_1', \quad t = 3, \ldots, T,$$

$$E[\varsigma_t\eta_{t+1}'] = \Sigma_{21} S_3 \int_0^1 e^{A'r} dr \Theta_1' + \Sigma_{22} C' S_3 \int_0^1 e^{A'r} dr A'^{-1}\Theta_1', \quad t = 3, \ldots, T,$$

$$E[\varsigma_t\eta_{t+2}'] = \Sigma_{21} S_3 \int_0^1 e^{A'r} dr \Theta_2' + \Sigma_{22} C' S_3 \int_0^1 e^{A'r} dr A'^{-1}\Theta_2', \quad t = 3, \ldots, T,$$

$$E[\varsigma_t\eta_{t+\tau}'] = 0, \quad \tau \geq 3, \quad t = 3, \ldots, T,$$

$$V_\varepsilon(0) = \int_0^1 e^{Ar} S_3' \Sigma_{11} S_3 e^{A'r} dr + A^{-1} \int_0^1 e^{Ar} S_3' C \Sigma_{22} C' S_3 e^{A'r} dr A'^{-1} + \int_0^1 e^{Ar} S_3' \Sigma_{12} C' S_3 e^{A'r} dr A'^{-1} + A^{-1} \int_0^1 e^{Ar} S_3' C \Sigma_{21} S_3 e^{A'r} dr,$$

$$V_\varepsilon(\tau) = 0, \quad \tau = 1, \ldots, T,$$

where $V_\varepsilon(\tau) = E[\varepsilon_t \varepsilon_{t+\tau}']$ and S_ϕ is defined as in Theorem 2.

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