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RESEARCH ARTICLE

Chebyshev systems and Sturm oscillation theory for discrete polynomials

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Abstract

We prove an analogue of Chebyshev's alternation theorem for linearly independent discrete functions $\Phi_n = \{\varphi_k\}_{k=1}^n$ on the interval $[0,q]_{\mathbb{Z}} = [0,q] \cap \mathbb{Z}$. In particular, we establish that the polynomial of best uniform approximation of a discrete function admits a Chebyshev alternance set of length $n+1$ if and only if Φ_n is a Chebyshev $T_{\mathbb{Z}}$-system. We also obtain a discrete version of Sturm's oscillation theorem, according to which the number of discrete zeros of the polynomial $\sum_{k=m}^n a_k\varphi_k$ is no less than $m-1$ and no more than $n-1$. This implies that Φ_n is a $T_{\mathbb{Z}}$-system and a discrete Sturm-Hurwitz spectral gap theorem is valid. As applications, we study the orthogonal polynomials with removed largest zeros. We establish the monotonicity property of coefficients in the Fourier expansions of such polynomials, thereby strengthening the results of H. Cohn and A. Kumar. We apply this to solve a Yudin-type extremal problem for polynomials with spectral gap.

Contents

1 Introduction **2**
1.1 Discrete Chebyshev system . . . 2
1.2 Sturm oscillation theorem, T-systems, and orthogonal polynomials . . . 4
1.3 Monotonicity property of coefficients of polynomials with removed largest zeros . . . 8
1.4 An extremal problem for polynomials with spectral gap . . . 9
1.5 Structure of the paper . . . 10
2 Proofs of Theorem 1.4 and Corollary 1.6 **10**
3 Proofs of Theorems 1.8, 1.10, and 1.11 **15**
3.1 Proof of Theorem 1.11 . . . 17
4 Proof of Theorem 1.14 **22**
5 Proof of Theorem 1.16 **27**
Appendix **28**
A.1 Calculation of determinants for Chebyshev polynomials . . . 28
A.2 Corollaries of Theorem 1.14 for trigonometric polynomials . . . 30
References **31**

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1. Introduction

1.1. Discrete Chebyshev system

Let $C[a,b]$ be the space of continuous real-valued functions on a finite interval $[a,b]$ equipped with the norm $\|f\|_\infty = \max_{t\in[a,b]} |f(t)|$. Given linearly independent functions $\{\varphi_k(t)\}_{k=1}^n \subset C[a,b]$, the set of polynomials is defined by

$$L_n = \Big\{ p = \sum_{k=1}^{n} a_k \varphi_k : a_k \in \mathbb{R} \Big\}.$$

The best approximation of a continuous function f by L_n is given by

$$E(f, L_n)_\infty = \min_{p\in L_n} \|f - p\|_\infty = \|f - p^*\|_\infty$$

and $p^* \in L_n$ is the best approximant.

The Chebyshev systems play the key role in a study of the best approximation of continuous functions by the n-dimensional subsets. Recall that a set of continuous functions $\{\varphi_k\}_{k=1}^n$ is called a T-system (or a Chebyshev system) on $[a,b]$ if any nontrivial polynomial with respect to this system has at most $n-1$ distinct zeros on $[a,b]$. This definition can be equivalently written as follows (cf. [21, Ch. 1, §4]): The set $\{\varphi_k\}_{k=1}^n$ is a T-system if and only if, for any sequence $a \le t_1 < \cdots < t_n \le b$, the determinants

$$\Delta\begin{pmatrix} \varphi_1, & \varphi_2, & \dots, & \varphi_n \\ t_1, & t_2, & \dots, & t_n \end{pmatrix} = \begin{vmatrix} \varphi_1(t_1) & \varphi_2(t_1) & \dots & \varphi_n(t_1) \\ \varphi_1(t_2) & \varphi_2(t_2) & \dots & \varphi_n(t_2) \\ \dots & \dots & \dots & \dots \\ \varphi_1(t_n) & \varphi_2(t_n) & \dots & \varphi_n(t_n) \end{vmatrix} \tag{1.1}$$

have the same sign.

Let us recall two crucial facts on the best approximations in $C[a,b]$.

Haar's theorem [24, Th. 3.4.6]. The best approximant $p^* \in L_n$ of a function $f \in C[a,b]$ is unique if and only if $\{\varphi_k\}_{k=1}^n \subset C[a,b]$ is a Chebyshev system.

Chebyshev's theorem [24, Th. 3.4.7]. Let $\{\varphi_k\}_{k=1}^n \subset C[a,b]$ be the Chebyshev system. A polynomial $p^* \in L_n$ is the best approximant of a function $f \in C[a,b]$ if and only if there exists an ordered set of $n+1$ points $a \le t_1 < \cdots < t_{n+1} \le b$ such that

$$\begin{aligned} &\text{(i)} \quad |p^*(t_i) - f(t_i)| = \|p^* - f\|_\infty, \quad i = 1, \dots, n+1, \\ &\text{(ii)} \quad p^*(t_i) - f(t_i) = -(p^*(t_{i+1}) - f(t_{i+1})), \quad i = 1, \dots, n. \end{aligned} \tag{1.2}$$

Recall that an ordered set of $\{t_i\}$ satisfying properties (1.2) is called the Chebyshev alternance set of length $n+1$.

We are interested in analogues of Haar's and Chebyshev's theorems for the best uniform approximation of discrete functions defined on ordered sets of points, rather than on an interval. Note that some basic results can be partially derived from well-known results on the best uniform approximation of continuous functions on an arbitrary compact set (see, e.g., [11], [12, Ch. 1, §§2, 5], [15, Ch. II], [24, Ch. 3]).

For $m, n \in \mathbb{Z}$, $m \le n$, we define $[m,n]_{\mathbb{Z}} = \{m, m+1, \dots, n\}$. For $q \in \mathbb{N}$, let[1] $C[0,q]_{\mathbb{Z}} = \{f : [0,q]_{\mathbb{Z}} \to \mathbb{R}\}$ be the linear space (of the dimension $q+1$) of real-valued discrete functions on $[0,q]_{\mathbb{Z}}$ equipped with the norm $\|f\|_\infty = \max_{\nu\in[0,q]_{\mathbb{Z}}} |f(\nu)|$. The point ν is called *a zero* of the discrete function f if[2] either

$$f(\nu) = 0 \ \text{ for } \ \nu \in [0,q]_{\mathbb{Z}} \quad \text{or} \quad f(\nu-1)f(\nu) < 0 \ \text{ for } \ \nu \in [1,q]_{\mathbb{Z}}.$$

[1]Here, similarly to the continuous case, we use the notation $C[0,q]_{\mathbb{Z}}$.

[2]Following P. Hartman [19] such zero is sometimes called generalized; see, e.g., [2, Definition 1.2.10] or [13, Definition 7.8].

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It will be important to distinguish these zeros and we will call them the zeros of the *first* or *second* type, respectively. Denote by $N(f, [m, n]_{\mathbb{Z}})$ the number of zeros (of both types) of f on $[m, n]_{\mathbb{Z}}$. Set also $N(f) = N(f, [0, q]_{\mathbb{Z}})$. Moreover, let $N_0(f)$ be the number of zeros of the first type of f on $[0, q]_{\mathbb{Z}}$. Clearly, $N_0(f) \le N(f)$.

F. Gantmacher and M. Krein [15, Ch. II] suggested the following characteristics of oscillatory properties of discrete functions. Let $S^-(f)$ (respectively, $S^+(f)$) be the least (the largest) number of sign changes[3] of f on $[0, q]_{\mathbb{Z}}$ after replacing all zero values of f by arbitrary nonzero values. We claim that

$$S^-(f) \le N(f) \le S^+(f),$$

with the discussion and proof given in Section 3 (see Lemma 3.1).

We study the problem of the characterization and uniqueness of the best uniform approximants of discrete functions. Let $\{\varphi_k\}_{k=1}^n \subset C[0, q]_{\mathbb{Z}}$ be linearly independent discrete functions and L_n be its linear span over $\mathbb{R}$. As in the continuous case,

$$E(f, L_n)_\infty = \min_{p \in L_n} \|f - p\|_\infty = \|f - p^*\|_\infty$$

is the best uniform approximation of f by L_n and $p^* \in L_n$ is the best approximant.

Since, for discrete functions one can distinguish between two types of zeros, we define two corresponding discrete Chebyshev systems. Let $n \in [1, q + 1]_{\mathbb{Z}}$. The set of discrete functions $\{\varphi_k\}_{k=1}^n \subset C[0, q]_{\mathbb{Z}}$ is called the $T_{\mathbb{Z}}$-system (respectively, T_0-system) if for any nontrivial polynomial

$$p(\nu) = \sum_{k=1}^{n} a_k \varphi_k(\nu) \tag{1.3}$$

of degree at most n with real coefficients we have $N(p) \le n - 1$ ($N_0(p) \le n - 1$). It is clear that any $T_{\mathbb{Z}}$-system is also T_0-system but the inverse statement is not valid, see the example in Section 2.

Let $p^* \in L_n$ be the best uniform approximant of the function $f \in C[0, q]_{\mathbb{Z}}$ and set

$$e(\nu) = p^*(\nu) - f(\nu), \quad \nu \in [0, q]_{\mathbb{Z}}.$$

Due to compactness of $[0, q]_{\mathbb{Z}} \subset \mathbb{R}$, we observe that an analogue of Haar's theorem is valid only for T_0-systems.

Theorem 1.1 [24, Th. 3.4.6]. *The best uniform approximant $p^* \in L_n$ of the function $f \in C[0, q]_{\mathbb{Z}}$ is unique if and only if the set $\{\varphi_k\}_{k=1}^n \subset C[0, q]_{\mathbb{Z}}$ is a T_0-system.*

We note that a (nontrivial) polynomial (1.3) vanishes at n integer points $0 \le \nu_1 < \cdots < \nu_n \le q$ if and only if the determinant

$$\Delta \begin{pmatrix} \varphi_1, & \varphi_2, & \dots, & \varphi_n \\ \nu_1, & \nu_2, & \dots, & \nu_n \end{pmatrix} \tag{1.4}$$

defined by (1.1) is zero. Therefore, we arrive at the following obvious statement, which provides a criterion for T_0-systems (see, e.g., [21, Ch. VII, §1]).

Proposition 1.2. *The set $\{\varphi_k\}_{k=1}^n \subset C[0, q]_{\mathbb{Z}}$ is a T_0-system if and only if the determinants (1.4) are nonzero for any integers $0 \le \nu_1 < \cdots < \nu_n \le q$.*

A straightforward analogue of Chebyshev's theorem for T_0-systems reads as follows.

Theorem 1.3 [24, Th. 3.4.7]. *Let the set $\{\varphi_k\}_{k=1}^n \subset C[0, q]_{\mathbb{Z}}$ be a T_0-system. The polynomial $p^* \in L_n$ is the best uniform approximant of a function $f \in C[0, q]_{\mathbb{Z}}$ if and only if there exist distinct*

[3] As usual, f has a sign change on $[0, q]_{\mathbb{Z}}$ if $f(\nu - 1) f(\nu) < 0$ for some $\nu \in [1, q]_{\mathbb{Z}}$.

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points $\nu_1, \dots, \nu_{n+1} \in [0, q]_{\mathbb{Z}}$, $\varepsilon_1, \dots, \varepsilon_{n+1} \in \{-1, 1\}$, *and positive numbers* $\rho_1, \dots, \rho_{n+1}$, *satisfying the conditions*

$$\text{(i)} \quad \varepsilon_i e(\nu_i) = \|e\|_\infty, \quad i = 1, \dots, n+1, \quad e = p^* - f,$$

$$\text{(ii)} \quad \sum_{i=1}^{n+1} \varepsilon_i \rho_i p(\nu_i) = 0 \quad \textit{for all} \quad p \in L_n.$$

We note that the set of points $\nu_1, \dots, \nu_{n+1}$ in Theorem 1.3 may not form the Chebyshev alternance set as we will see in Example 2.7.

Our first main result provides a characterization of the discrete set $\{\varphi_k\}_{k=1}^n \subset C[0, q]_{\mathbb{Z}}$ such that for any $f \in C[0, q]_{\mathbb{Z}}$ the best uniform approximant $p^* \in L_n$ has an alternance set of length $n + 1$, that is, we obtain a complete analogue of Chebyshev's alternance theorem.

We say that the best uniform approximant $p^* \in L_n$ of f admits an alternance set $0 \le \nu_1 < \cdots < \nu_{n+1} \le q$ of length $n + 1$ if the following condition holds (cf. (1.2)):

$$\varepsilon(-1)^i e(\nu_i) = \|e\|_\infty, \quad i = 1, \dots, n+1, \quad \varepsilon = \pm 1.$$

Theorem 1.4. *Let* $n \in [1, q]_{\mathbb{Z}}$. *The following are equivalent*:

(a) *The set* $\{\varphi_k\}_{k=1}^n \subset C[0, q]_{\mathbb{Z}}$ *is a* $T_{\mathbb{Z}}$*-system.*
(b) *For any function* $f \in C[0, q]_{\mathbb{Z}}$, *the best uniform approximant* $p^* \in L_n$ *of* f *admits an alternance set of length* $n + 1$.
(c) *Determinants (1.4) constructed over all sets* $0 \le \nu_1 < \cdots < \nu_n \le q$ *are nonzero and have the same sign.*

Thus, an analogue of Chebyshev's theorem holds only for $T_{\mathbb{Z}}$-systems.

Using Theorem 1.4, one can easily construct different examples of $T_{\mathbb{Z}}$-systems via T-systems on an interval.

Corollary 1.5. *If* $\{\varphi_i(t)\}_{i=1}^n \subset C[0, q]$ *is a* T*-system on* $[0, q]$, *then* $\{\varphi_i(\nu)\}_{i=1}^n \subset C[0, q]_{\mathbb{Z}}$ *is a* $T_{\mathbb{Z}}$*-system.*

Another important property of $T_{\mathbb{Z}}$-systems is given below.

Corollary 1.6. *If* $\{\varphi_k\}_{k=1}^n \subset C[0, q]_{\mathbb{Z}}$ *is a* $T_{\mathbb{Z}}$*-system, then for any nontrivial polynomial* $p \in L_n$ *we have* $S^+(p) \le n - 1$.

Historical comments. S. Karlin and W.J. Studden [21, Ch. VII] introduced $T_{\mathbb{Z}}$-systems via condition (c) and showed that (c) yields (a). In the matrix form the implication (b) $\Leftrightarrow$ (c) can be found in the recent paper [31]. Corollary 1.5 was given in [21, Ch. VII, §1] in the particular case when T-systems are generated by strictly total positive kernels. Also, Corollary 1.6 was proved in [21, Ch. VII, §2] using property (c) of Theorem 1.4. We will give a different proof of Corollary 1.6 based on an oscillation property of discrete polynomials (2.4).

We note that the discrete Chebyshev approximation plays an important role in numerical analysis (see, e.g., [11] and [31, 38] for the best approximation by low-rank matrices). In these problems the use of effective algorithms such as the Remez algorithm is crucial. Let us mention that versions of Theorem 1.3 also allow one to construct effective algorithms for non-Chebyshev systems, see [32]. Moreover, C. Dunham [11] developed a discrete Remez algorithm based on alternance sets. Thus, Theorem 1.4 completes this study as it characterizes *all systems* that admit alternation.

For applications, it is important to construct various discrete Chebyshev systems. To this end, in the next section we consider the eigenfunctions of Sturm-Liouville problems.

1.2. Sturm oscillation theorem, T-systems, and orthogonal polynomials

We start with the following known fact: T-systems can be obtained using Sturm-Liouville problems. In 1836, C. Sturm proved the following remarkable result, which was obtained in the same year

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by J. Liouville using a different method under less restrictions. See [5, 33, 34] for the historical comments.

Sturm's theorem Let $\{V_k\}_{k=1}^{\infty} \subset C^2[a,b]$ be the set of eigenfunctions associated to eigenvalues $\rho_1 < \rho_2 < \dots$ of the following Sturm-Liouville problem:

$$\begin{gathered}(K(t)V'(t))' + (\rho G(t) - L(t))V(t) = 0, \quad t \in [a,b], \\ (KV' - hV)(a) = 0, \quad (KV' + HV)(b) = 0,\end{gathered} \tag{1.5}$$

where $G, K, L \in C[a,b]$, $K \in C^1(a,b)$, $K, G > 0$ on (a,b), $h, H \in [0,\infty]$, and ρ denotes the spectral parameter. Then

(a) Every k-th eigenfunction V_k has exactly $k-1$ simple zeros in (a,b).
(b) For any nontrivial real polynomial of the form

$$P(t) = \sum_{k=m}^{n} A_k V_k(t), \quad m, n \in \mathbb{N}, \quad m \le n, \tag{1.6}$$

the number of distinct zeros of P on (a,b) is at least $m-1$ and at most $n-1$.

Remark 1.7. (i) Part (a) is sometimes called the weak Sturm oscillation theorem [34] and proved using comparison theorems (see, e.g., [26] and [33]). Part (b) is a much stronger result (the Sturm oscillation theorem). As noted by P. Bérard and B. Helffer [5], while part (a) is a well-known statement, part (b) has been almost forgotten. However, for the trigonometric system, it has been known since the early 20th century as the Sturm-Hurwitz theorem: *If a real Fourier series has a spectral gap, that is*, $f(t) = \sum_{k=m}^{\infty}(a_k \cos kt + b_k \sin kt)$, *then f has at least* $2m$ *sign changes over the period.* Sturm proved this fact for polynomials (cf. (1.6)), while Hurwitz extended it to the general case. Similar results on spectral gap problems are closely related to the Fourier uncertainty principle (see, e.g., [14, 17, 18, 27, 29, 34, 37]) and signal processing (see, e.g., [28]).

(ii) It follows from Sturm's theorem that $\{V_k(t)\}_{k=1}^{n}$ forms a T-system on the interval (a,b). By Corollary 1.5, for $a < 0$ and $b > q$, the set $\{V_k(\nu)\}_{k=1}^{n} \subset C[0,q]_{\mathbb{Z}}$ is a $T_{\mathbb{Z}}$-system.

Our main goal in this section is to obtain the Sturm oscillation theorem for eigenfunctions of a discrete Sturm-Liouville problem on $[0,q]_{\mathbb{Z}}$. This problem is treated in detail in several monographs, for example, [1, Ch. 12], [3, Ch. 4], [22, Ch. 7]. Moreover, discrete comparison and separation theorems in various settings have been studied in, for example, [2, 13, 33, 36]. However, an explicit discrete analogue of Sturm's theorem has been an open question.

To formulate the discrete Sturm-Liouville problem, we use infinite tridiagonal Jacobi matrix, which is, in general, nonsymmetric. Let $\{\alpha_l\}_{l=0}^{\infty}$ be a sequence of real numbers, $\{\gamma_l\}_{l=0}^{\infty}$, $\{\beta_l\}_{l=0}^{\infty}$, $\{\rho_l\}_{l=0}^{\infty}$ be sequences of positive numbers. Define

$$J = \begin{pmatrix} \alpha_0 & \beta_0 & 0 & \dots & 0 & 0 & 0 & \dots \\ \gamma_0 & \alpha_1 & \beta_1 & \dots & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \gamma_{l-1} & \alpha_l & \beta_l & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

Considering the eigenvalue problem

$$JP = \lambda\rho P$$

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with an infinite vector $P = \{P_l(\lambda)\}_{l=0}^{\infty}$ and a diagonal matrix $\rho = \operatorname{diag}(\{\rho_l\}_{l=0}^{\infty})$, we arrive at the Sturm-Liouville problem in the recurrence relation form:

$$\begin{gathered}\gamma_{l-1}P_{l-1}(\lambda) + \alpha_l P_l(\lambda) + \beta_l P_{l+1}(\lambda) = \lambda\rho_l P_l(\lambda), \quad l \in \mathbb{Z}_+, \quad \lambda \in \mathbb{C}, \\ P_{-1}(\lambda) = 0, \quad P_0(\lambda) = 1,\end{gathered} \tag{1.7}$$

where $\gamma_{-1} > 0$ is arbitrary. It is clear that the solution of this problem is given by a family of algebraic polynomials $P_l(\lambda)$ of degree l with positive leading coefficients.

For a nondecreasing function $\mu(\lambda)$ on $\mathbb{R}$, the set

$$\mathcal{S}(\mu) = \{\lambda\colon \mu(\lambda+\delta) - \mu(\lambda-\delta) > 0 \text{ for all } \delta > 0\}$$

is called the spectrum of μ. To attack Problem (1.7), we apply the Favard theorem (see [8, Th. II. 6.4]), which leads to the following statement.

Theorem 1.8.

(a) *Polynomials $P_l(\lambda)$ are orthogonal with respect to a positive measure μ on $\mathbb{R}$, defined by a nondecreasing function of bounded variation with infinite spectrum, and*

$$d_l \int_{\mathbb{R}} P_l(\lambda)P_m(\lambda)\,d\mu(\lambda) = \delta_{lm}, \quad d_0 = \rho_0, \quad d_l = \rho_l\,\frac{\beta_0\cdots\beta_{l-1}}{\gamma_0\cdots\gamma_{l-1}}, \quad l \geq 1. \tag{1.8}$$

(b) *A measure μ has a finite support $[a, b]$, the endpoints of which will be the limit points of zeros of the orthogonal polynomials $P_l(\lambda)$, if and only if the sequences*

$$\frac{\alpha_l}{\rho_l}, \quad \frac{\gamma_{l-1}\beta_{l-1}}{\rho_{l-1}\rho_l} \tag{1.9}$$

are bounded.

Note that positivity of coefficients $\gamma_l, \beta_l, \rho_l$ is a natural condition in Theorem 1.8, see Theorem II.6.4 in [8].

Throughout the paper we assume that $q \in \mathbb{Z}_+$ and $\eta \in \mathbb{R}$. Consider the polynomial

$$\widetilde{P}_{q+1}(\lambda) = P_{q+1}(\lambda) - \eta P_q(\lambda) \tag{1.10}$$

and define its zeros by $\lambda_{q+1} < \cdots < \lambda_1$, see [35, Ch. III, §3.3]. We also define the following set of discrete functions:

$$\psi_k(\nu) = P_\nu(\lambda_k), \quad k \in [1, q+1]_{\mathbb{Z}}, \quad \nu \in [0, q]_{\mathbb{Z}}. \tag{1.11}$$

By virtue of (1.7), λ_k and $\psi_k(\nu)$ are eigenvalues and eigenfunctions of the discrete Sturm-Liouville problem

$$\begin{gathered}\gamma_{\nu-1}\psi(\nu-1) + \alpha_\nu\psi(\nu) + \beta_\nu\psi(\nu+1) = \lambda\rho_\nu\psi(\nu), \quad \nu \in [0, q]_{\mathbb{Z}}, \\ \psi(-1) = 0, \quad \psi(0) = 1, \quad \psi(q+1) = \eta\psi(q).\end{gathered} \tag{1.12}$$

Introducing the Jacobi matrix

$$J_{q+1} = \begin{pmatrix} \alpha_0 & \beta_0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \gamma_0 & \alpha_1 & \beta_1 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & \gamma_{q-2} & \alpha_{q-1} & \beta_{q-1} \\ 0 & 0 & 0 & \dots & 0 & 0 & \gamma_{q-1} & \alpha_q + \eta\beta_q \end{pmatrix},$$

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problem (1.12) corresponds to the eigenvalue problem

$$J_{q+1}\psi = \lambda\rho\psi.$$

Sturm's theory on the zeros of discrete polynomials with respect to the set $\{\psi_k(\nu)\}_{k=1}^{q+1}$ (see (1.11)) is similar to that in the continuous case (see (1.5)), with the additional use of the quantities $S^-(f)$ and $S^+(f)$.

Before stating an analogue of item (a) of Sturm's theorem given in Theorem 1.10 below, we mention the following result by B. Simon.[4]

Theorem 1.9 [33, Th. 2.3]. *For $n \in [1,q]_{\mathbb{Z}}$, there holds*

$$|\{j \in [1,n]_{\mathbb{Z}}\colon \lambda < \lambda_{j,n}\}| = |\{\nu \in [1,n]_{\mathbb{Z}}\colon \operatorname{sign} P_{\nu-1}(\lambda) \neq \operatorname{sign} P_\nu(\lambda)\}|,$$

where $\lambda_{l,l} < \cdots < \lambda_{1,l}$ are the zeros of the polynomials $P_l(\lambda)$, $\lambda \neq \lambda_{j,l}$ for $1 \le j \le l \le n$, and $|I|$ denotes the cardinality of a finite set I.

Taking into account that the function ψ_k cannot have two consecutive zeros of the first type, this theorem implies the following result.

Theorem 1.10. *For all $k \in [1,q+1]_{\mathbb{Z}}$,*

$$S^-(\psi_k) = S^+(\psi_k) = N(\psi_k) = k-1.$$

Note that Theorem 1.10 also follows from Theorem 1 in [15, Ch. II] whose proof is based on the use of the concepts of lines and nodes for the piecewise linear function associated with Jacobi matrices (see Definition 2 in [15, Ch. II]). Moreover, the proof of the fact that $N(\psi_k) = k-1$ based on the discrete Sturm separation theorem can be found in [22, Theorem 7.6]. The corresponding results for self-adjoint systems are given in [9, Theorem 2], [10, Corollary 5.79], [23, Theorem 1]. In Section 3, we present a simple proof of Theorem 1.10 based on the property of the interlacing of zeros of orthogonal polynomials (see (3.2)).

Our main result in this section is the discrete counterpart of item (b) of Sturm's theorem, cf. Remark 1.7.

Theorem 1.11. *Let integers m and n satisfy $1 \le m \le n \le q+1$. For any nontrivial polynomial*

$$V(\nu) = \sum_{k=m}^{n} a_k \psi_k(\nu), \tag{1.13}$$

we have

$$m-1 \le S^-(V) \le N(V) \le S^+(V) \le n-1. \tag{1.14}$$

The proof of Theorem 1.11 is based on Theorem 1.10 and a generalization of Liouville's method (see [5]). Note also that the inequalities $m-1 \le S^-(V)$ and $S^+(V) \le n-1$ for $\eta = 0$ and $\rho_l = 1$ were proven in [15, Ch. II, Th. 6] under the assumption that the Jacobi matrix J_{q+1} is oscillatory. Recall that a square matrix is called oscillatory if it is totally non-negative, and some power of it is totally positive (see [15, Ch. II, Definition 4]). In our case, in general, the matrix J_{q+1} is not oscillatory and it becomes oscillatory only if all its principal minors are positive [15, Ch. II, Th. 11]. It is worth mentioning that the estimate $N(V) \le n-1$ was mentioned in [3, Problem 2, p. 539].

Corollary 1.12. *For any $n \in [1,q+1]_{\mathbb{Z}}$, the set $\{\psi_k(\nu)\}_{k=1}^{n}$ is a $T_{\mathbb{Z}}$-system.*

To conclude, we state the discrete spectral gap theorem (cf. Remark 1.7).

[4] A similar result can be found in [3, Theorem 4.3.5].

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Corollary 1.13. *Let $m \in [1, q+1]_{\mathbb{Z}}$ and $f \in C[0,q]_{\mathbb{Z}}$ be such that $f(\nu) = \sum_{k=m}^{q+1} a_k \psi_k(\nu)$. Then $m - 1 \le S^-(f) \le N(f)$.*

1.3. Monotonicity property of coefficients of polynomials with removed largest zeros

Consider polynomial (1.10) and divide it into $\prod_{j=1}^{m+1}(\lambda - \lambda_j)$ with $m+1$ largest zeros λ_j, $m \le q$. Taking into account (1.8), we expand it into the Fourier sum

$$\frac{\widetilde{P}_{q+1}(\lambda)}{(\lambda-\lambda_1)\cdots(\lambda-\lambda_{m+1})} = \sum_{l=0}^{q-m} d_l a_l P_l(\lambda). \tag{1.15}$$

The question arises whether all coefficients $a_l \ge 0$. It turns out that the answer is affirmative. The case $m = 0$ is simple and corresponds to the Christoffel-Darboux formula, while the case $m = 1$ was proven in [16]. In full generality, this nontrivial fact was proven by H. Cohn and A. Kumar [7, Th. 3.1] while working on the problem on discrete energy on the Euclidean sphere.

In this section, using the discrete Sturm theorem, we take one step further and establish the monotonicity property of the Fourier coefficients for all m, which also implies that $a_l \ge 0$. This fact is crucial to deal with a spectral gap problem in Section 1.4.

Also, we note that the related polynomial $p_m(\lambda) = \frac{P_{q+1}^2(\lambda)}{(\lambda-\lambda_1)\cdots(\lambda-\lambda_{m+1})}$ appeared in various extremal problems. For example, S. Bernstein used p_0 to disprove the existence of Chebyshev's quadrature formula. Moreover, this polynomial is an extremizer in the following question (see [4, 30]): find

$$\inf_{a_k, b_k} \operatorname{mes}\Big\{x \in (-\pi, \pi] \colon \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) > 0\Big\}.$$

V. Yudin used p_m in problems of multiple covering of the torus [40] and the Euclidean sphere [41].

Our main result here is the following theorem.

Theorem 1.14. *Let $q \in \mathbb{N}$ and $m \in [0,q]_{\mathbb{Z}}$. Let also (a,b) be an interval containing all the zeros of the polynomial $P_q(\lambda)$ and $\eta_b = P_{q+1}(b)/P_q(b)$.*

The following inequalities hold for the coefficients a_l in the expansion (1.15):

(a) *if either $m \ge 1$ and $\eta \in \mathbb{R}$ or $m = 0$ and $\eta < \eta_b$, then*

$$\frac{a_0}{P_0(b)} > \frac{a_1}{P_1(b)} > \cdots > \frac{a_{q-m}}{P_{q-m}(b)} > 0; \tag{1.16}$$

(b) *if $m = 0$ and $\eta = \eta_b$, then*

$$\frac{a_0}{P_0(b)} = \frac{a_1}{P_1(b)} = \cdots = \frac{a_q}{P_q(b)} > 0; \tag{1.17}$$

(c) *if $m = 0$ and $\eta > \eta_b$, then*

$$0 < \frac{a_0}{P_0(b)} < \frac{a_1}{P_1(b)} < \cdots < \frac{a_q}{P_q(b)}. \tag{1.18}$$

For polynomials normalized by $P_l(1) = 1$, this result has a simpler form.

Corollary 1.15. *Let $P_l(\lambda)$ be orthogonal on $[-1,1]$ and $P_l(1) = 1$. Then*

$$\begin{cases} a_0 > a_1 > \cdots > a_{q-m} > 0, & \text{if either } m = 0 \text{ and } \eta < 1 \text{ or } m \ge 1 \text{ and } \eta \in \mathbb{R}, \\ a_0 = a_1 = \cdots = a_q > 0, & \text{if } m = 0 \text{ and } \eta = 1, \\ 0 < a_0 < a_1 < \cdots < a_q, & \text{if } m = 0 \text{ and } \eta > 1. \end{cases}$$

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1.4. An extremal problem for polynomials with spectral gap

Let $m, n \in \mathbb{Z}_+$, $m < n$. Let also μ be the measure on $[-1,1]$ from Theorem 1.8 and $\{U_l\}$ be the set of polynomials orthogonal on $[-1,1]$ with respect to $d\mu$ with normalization $U_l(1) = 1$, $d_l \int_{-1}^{1} U_l U_m \, d\mu = \delta_{lm}$.

We consider the following problem: find $\lambda \in [-1,1]$ and an algebraic polynomial of degree n on $[-1,1]$ with non-negative Fourier coefficients in $\{U_l\}$ having a spectral gap of the length m (equivalently, with the first m moments being zero), which preserves its sign over the largest possible interval $[-1,\lambda]$.

Let Π be the set of real algebraic polynomials and Π_n consist of those of degree at most n. By $\Pi_+(\{U_l\})$ we denote a subset of $p \in \Pi$ such that all coefficients a_l in the Fourier expansion

$$p(t) = \sum_l d_l a_l U_l(t), \quad a_l = a_l(p) = \int_{-1}^{1} p(t) U_l(t) \, d\mu(t),$$

are non-negative.

Set

$$K_n(\mu, m) = \Big\{ p \in \Pi_n \cap \Pi_+(\{U_l\}) \colon \mu_i(p) = 0,\ i = 0, \dots, m-1,\ \mu_m(p) \geq 0 \Big\},$$

where $\mu_i(p) = \int_{-1}^{1} t^i p(t) \, d\mu(t)$ is the ith moment of the polynomial p. Note that the conditions $\mu_i(p) = 0$ are equivalent to the fact that $a_i(p) = 0$, $i = 0, \dots, m-1$.

Our goal is to find the quantity

$$B_n(\mu, m) = \sup_{p \in K_n(\mu,m) \setminus \{0\}} \{\lambda \in [-1,1] \colon (-1)^{m-1} p(t) \geq 0,\ -1 \leq t \leq \lambda\}.$$

This question turns out to play a crucial role in the study of spherical codes and designs. For $m = 1$ it was posed by V. Yudin [39]. Specifically, for $m = 1$ and $n = 2q$, he found extremizers for this problem, though the positivity of the Fourier coefficients was not proven. Further results in this direction can be found in [16, 17, 18]; see also [7, 25].

Let $\{U_l^{(1)}(t)\}_{l=0}^{\infty}$ be the set of polynomials with normalization $U_l^{(1)}(1) = 1$, orthogonal on $[-1,1]$ with respect to the measure $d\mu^{(1)}(t) = (1+t)\, d\mu(t)$. The zeros of the polynomials $U_{q+1}(t)$ and $U_{q+1}^{(1)}(t)$, numbered in the descending order, are denoted by t_k and $t_k^{(1)}$, $k = 1, \dots, q+1$, respectively.

We say that the set of orthogonal polynomials $\{V_l\}$ satisfies *the Krein property* if $V_m V_n \in \Pi_+(\{V_l\})$, that is, $V_m V_n = \sum_{k=|m-n|}^{m+n} c_{m,n,k} V_k$ with $c_{m,n,k} \geq 0$, for any $m, n \in \mathbb{Z}_+$, see [25].

Theorem 1.16. *Let $m \leq q$ and the set $\{U_l\}$ satisfy the Krein property.*

(a) *If $n = 2q - m + 1$, then*

$$B_n(\mu, m) = t_{m+1} \tag{1.19}$$

and the unique extremal polynomial has the form (up to a positive constant)

$$p_{n,1}^*(t) = \frac{U_{q+1}^2(t)}{(t - t_1) \cdots (t - t_{m+1})}. \tag{1.20}$$

(b) *If $n = 2q - m + 2$ and either the set $\{U_l^{(1)}\}$ satisfies the Krein property or $\mu(t)$ is an odd function, then*

$$B_n(\mu, m) = t_{m+1}^{(1)} \tag{1.21}$$

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and the unique extremal polynomial has the form (up to a positive constant)

$$p^*_{n,2}(t) = \frac{(1+t)(U^{(1)}_{q+1}(t))^2}{(t-t^{(1)}_1)\cdots(t-t^{(1)}_{m+1})}. \tag{1.22}$$

Moreover, we have $\mu_m(p^*_{n,j}) = a_m(p^*_{n,j}) = 0$, $j = 1, 2$.

We note that a version of Yudin-type problem for a wider class of polynomials p (without the condition $p \in \Pi_+(\{U_l\})$) was solved in [20]. Our proof of Theorem 1.16 is based on Corollary 1.15.

1.5. *Structure of the paper*

The rest of the paper is organized as follows. In Section 2, we provide the proofs of Chebyshev's theorem for $T_{\mathbb{Z}}$-systems (Theorem 1.4) and the important Corollary 1.6, which claims that $S^+(p) \le n-1$ for a polynomial $p \in L_n$, where $p(\nu) = \sum_{k=1}^n a_k\varphi_k(\nu)$ and $\{\varphi_k\}_{k=1}^n \subset C[0,q]_{\mathbb{Z}}$ is a $T_{\mathbb{Z}}$-system.

In Section 3, we study the discrete Sturm oscillation theory and its connection with T-systems and prove Theorems 1.8, 1.10, and 1.11.

Section 4 is devoted to the proof of Theorem 1.14. In Section 5, we study polynomials with spectral gaps and solve Yudin's extremal problem (Theorem 1.16).

In the Appendix, we explicitly compute the determinants (4.1) for the Jacobi polynomials corresponding to the classical trigonometric systems. In particular, we verify that, for all $\nu \in [0,q]_{\mathbb{Z}}$ and $T_{\mathbb{Z}}$-systems,

$$\operatorname{sign}\Delta\begin{pmatrix}\varphi_1, & \dots, & \varphi_m, & \varphi_{m+1}\\ \nu_m, & \dots, & \nu_1, & \nu\end{pmatrix} = \operatorname{sign}\Delta\begin{pmatrix}\varphi_1, & \dots, & \varphi_m, & \varphi_{m+1}\\ q, & \dots, & q-m+1, & 0\end{pmatrix}\operatorname{sign}\prod_{j=1}^m(\nu_j-\nu),$$

which in the general case will be shown in Lemma 4.1.

2. Proofs of Theorem 1.4 and Corollary 1.6

Recall that $q \in \mathbb{Z}_+$ and $\eta \in \mathbb{R}$. The proof of Theorem 1.4 is based on Lemmas 2.1–2.5 below. First, in addition to the characteristic property of T_0-systems given by Proposition 1.2, we present another necessary and sufficient condition.

Lemma 2.1 [24, Proposition 3.4.2]. *A set $\{\varphi_k\}_{k=1}^n \subset C[0,q]_{\mathbb{Z}}$ is a T_0-system if and only if, for any integers $0 \le \nu_1 < \cdots < \nu_n \le q$ and any points $\{y_i\}_{i=1}^n \subset \mathbb{R}$, there is a unique polynomial (1.3), for which*

$$p(\nu_i) = y_i, \quad i = 1, \dots, n.$$

Second, we assume that the set $\{\varphi_k\}_{k=1}^n \subset C[0,q]_{\mathbb{Z}}$ is a T_0-system and the determinants (1.4) are nonzero for any integers $0 \le \nu_1 < \cdots < \nu_n \le q$.

Set, for $0 \le \nu_1 < \cdots < \nu_{n+1} \le q$,

$$\Delta_{\nu_1} = \Delta\begin{pmatrix}\varphi_1, & \dots, & \varphi_n\\ \nu_2, & \dots, & \nu_{n+1}\end{pmatrix}, \quad \Delta_{\nu_{n+1}} = \Delta\begin{pmatrix}\varphi_1, & \dots, & \varphi_n\\ \nu_1, & \dots, & \nu_n\end{pmatrix},$$
$$\Delta_{\nu_i} = \Delta\begin{pmatrix}\varphi_1, & \dots, & \varphi_{i-1}, & \varphi_i, & \dots, & \varphi_n\\ \nu_1, & \dots, & \nu_{i-1}, & \nu_{i+1}, & \dots, & \nu_{n+1}\end{pmatrix}, \quad i = 2, \dots, n.$$

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Lemma 2.2. *Let* $0 \le \nu_1 < \cdots < \nu_{n+1} \le q$. *The linear functional*

$$l(p) = \sum_{i=1}^{n+1} \gamma_i p(\nu_i) = 0 \tag{2.1}$$

for any polynomial $p \in L_n$ *if and only if, up to a nonzero constant,* $\gamma_i = (-1)^i \Delta_{\nu_i}$, $i = 1, \dots, n+1$.

Proof. First, we note that $p \in L_n$ only if

$$\Delta \begin{pmatrix} p, & \varphi_1, & \varphi_2, & \dots, & \varphi_n \\ \nu_1, & \nu_2, & \nu_3, & \dots, & \nu_{n+1} \end{pmatrix} = 0.$$

Expanding the determinant along the first column, we obtain

$$\sum_{i=1}^{n+1} (-1)^i \Delta_{\nu_i} p(\nu_i) = 0. \tag{2.2}$$

On the other hand, from (2.1), for any $a_1, \dots, a_k$,

$$0 = \sum_{i=1}^{n+1} \gamma_i \sum_{k=1}^{n} a_k \varphi_k(\nu_i) = \sum_{k=1}^{n} a_k \sum_{i=1}^{n+1} \gamma_i \varphi_k(\nu_i),$$

and therefore,

$$\sum_{i=1}^{n+1} \gamma_i \varphi_k(\nu_i) = 0, \quad k = 1, \dots, n. \tag{2.3}$$

Since $\Delta_{\nu_i} \neq 0$, all the solutions $(\gamma_1, \dots, \gamma_{n+1})$ of the system (2.3) form a one-dimensional space. Taking into account (2.2), we arrive at the statement of the lemma. □

Lemma 2.3. *Let the set* $\{\varphi_k\}_{k=1}^n \subset C[0,q]_{\mathbb{Z}}$ *be a* $T_{\mathbb{Z}}$*-system. The polynomial* $p^* \in L_n$ *is the best uniform approximant of a function* $f \in C[0,q]_{\mathbb{Z}}$ *if and only if* p^* *admits an alternance set of length* $n+1$.

Proof. The sufficiency follows from Theorem 1.3. To prove the necessity, according to Lemma 2.2 and Theorem 1.3, we need to show that for a linear functional $l(p)$ on L_n satisfying property (2.1), there holds $\gamma_i \gamma_{i+1} < 0$ for $i = 1, \dots, n$. We will follow the reasoning given in the proof of Proposition 3.5.1 in [24]. Applying Lemma 2.1, we construct polynomials $p_i \in L_n$ satisfying

$$p_i(\nu_j) = 0, \quad j = 1, \dots, i-1, i+2, \dots, n+1, \quad p_i(\nu_i) = 1.$$

Since p_i has $n-1$ zeros of the first type, it follows from the definition of the $T_{\mathbb{Z}}$-system that p_i has no zeros of the second type and, therefore, $p_i(\nu_{i+1}) > 0$. From this,

$$l(p_i) = \gamma_i p_i(\nu_i) + \gamma_{i+1} p_i(\nu_{i+1}) = 0 \quad \text{and} \quad \gamma_i \gamma_{i+1} < 0,$$

completing the proof. □

Lemma 2.4. (a) *For any* $f \in C[0,q]_{\mathbb{Z}}$, *its best uniform approximant* $p^* \in L_n$ *admits an alternance set of length* $n+1$ *if and only if all determinants (1.4), constructed over all sequences* $0 \le \nu_1 < \cdots < \nu_n \le q$, *have the same sign.*

(b) *If all determinants (1.4) have the same sign, then any sequence* $0 \le \nu_1 < \cdots < \nu_{n+1} \le q$ *is an alternance set of length* $n+1$ *for the polynomial* $p^* \in L_n$, *which is the best approximant of some function* $f \in C[0,q]_{\mathbb{Z}}$.

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Proof. Part (a) follows from Theorem 1.3 and Lemma 2.2.

To obtain part (b), the sequence $0 \le \nu_1 < \cdots < \nu_{n+1} \le q$ is an alternance set of length $n+1$, for example, for the function

$$f(\nu) = \begin{cases} (-1)^{i-1}, & \nu = \nu_i,\ i = 1, \dots, n+1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that, by virtue of Theorem 1.3, the best uniform approximant is $p^*(\nu) \equiv 0$, and $E(f, L_n)_\infty = 1$. The proof is now complete. □

Lemma 2.5. *Let $p \in L_n$ and $N(p) = n$. There exists a collection of points $\Omega = \{0 \le \nu_1 < \cdots < \nu_{n+1} \le q\}$ such that if for some positive sequence $\{\rho_i\}_{i=1}^{n+1}$ there holds*

$$\sum_{i=1}^{n+1} (-1)^i \rho_i p(\nu_i) = 0, \tag{2.4}$$

then $p \equiv 0$.

Proof. Without loss of generality, we can assume that all $\rho_i = 1$. If the polynomial p has n zeros of the first type, then by the definition of T_0-systems, $p \equiv 0$.

Let p have a sign change. An interval $[k, k+l]_{\mathbb{Z}} \subset [0, q]_{\mathbb{Z}}$, $l \ge 1$, is called *a closed segment of sign changes* if $p(\nu - 1)p(\nu) < 0$ for $\nu = k+1, \dots, k+l$ and $p(k-1)p(k) \ge 0$, $p(k+l)p(k+l+1) \ge 0$. It consists of $l+1$ points and contains l zeros of the second type.

Let the polynomial p have t zeros of the first type and s closed segments of sign changes $[k_j, k_j + l_j]_{\mathbb{Z}}$, $j = 1, \dots, s$, where $k_{j+1} - k_j - l_j \ge 1$. Then $l_1 + \cdots + l_s = n - t$. We form the set Ω as follows: Ω consists of all t zeros of the first type and $n+1-t$ points from the closed segments Ω_j, $j = 1, \dots, s$, of sign changes. In more detail, set $\Omega_1 = [k_1, k_1 + l_1]_{\mathbb{Z}}$ and construct Ω_j, $j = 2, \dots, s$, such that either $\Omega_j = [k_j, k_j + l_j - 1]_{\mathbb{Z}}$ or $\Omega_j = [k_j + 1, k_j + l_j]_{\mathbb{Z}}$. Note that the number of points in closed segments is equal to $(l_1 + 1) + l_2 + \cdots + l_s = n + 1 - t$.

If $r_1 + i$, $i = 0, \dots, l_1$, are the indices of the points $k_1 + i$ of the set Ω in Ω_1 and $p(k_1 + i) = (-1)^{i_1 + k_1 + i} |p(k_1 + i)|$, then $k_1 + i = \nu_{r_1 + i}$ and

$$\sum_{i=0}^{l_1} (-1)^{r_1 + i} p(\nu_{r_1 + i}) = (-1)^{i_1 + k_1 + r_1} \sum_{i=0}^{l_1} |p(\nu_{r_1 + i})|.$$

Let the segments Ω_m, for $m = 1, \dots, j-1$, have already been constructed. Let $r_j + i$, $i = 0, \dots, l_j - 1$, be the indices of points of the set Ω in Ω_j and $p(k_j + i) = (-1)^{i_j + k_j + i} |p(k_j + i)|$, $i = 0, \dots, l_j$.

If $(-1)^{i_j + k_j + r_j} = (-1)^{i_1 + k_1 + r_1}$, then we set $\nu_{r_j + i} = k_j + i$, $i = 0, \dots, l_j - 1$. We have

$$\sum_{i=0}^{l_j - 1} (-1)^{r_j + i} p(\nu_{r_j + i}) = \sum_{i=0}^{l_j - 1} (-1)^{r_j + i} p(k_j + i) = (-1)^{i_1 + k_1 + r_1} \sum_{i=0}^{l_j - 1} |p(\nu_{r_j + i})|.$$

If $(-1)^{i_j + k_j + r_j + 1} = (-1)^{i_1 + k_1 + r_1}$, then we put $\nu_{r_j + i} = k_j + i + 1$, $i = 0, \dots, l_j - 1$. Again, we have

$$\sum_{i=0}^{l_j - 1} (-1)^{r_j + i} p(\nu_{r_j + i}) = \sum_{i=0}^{l_j - 1} (-1)^{r_j + i} p(k_j + i + 1) = (-1)^{i_1 + k_1 + r_1} \sum_{i=0}^{l_j - 1} |p(\nu_{r_j + i})|.$$

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Thus, the set $\Omega = \{0 \le \nu_1 < \cdots < \nu_{n+1} \le q\}$ is constructed. For this set, in view of (2.4), there holds

$$0 = \sum_{i=1}^{n+1} (-1)^i p(\nu_i) = \sum_{i=0}^{l_1} (-1)^{r_1+i} p(\nu_{r_1+i}) + \sum_{j=2}^{s} \sum_{i=0}^{l_j-1} (-1)^{r_j+i} p(\nu_{r_j+i})$$
$$= (-1)^{i_1+k_1+r_1} \Big\{ \sum_{i=0}^{l_1} |p(\nu_{r_1+i})| + \sum_{j=2}^{s} \sum_{i=0}^{l_j-1} |p(\nu_{r_j+i})| \Big\}.$$

Hence, $p(\nu_i) = 0$ for $i = 1, \dots, n+1$. Then we deduce that $p \equiv 0$, completing the proof. □

Proof of Theorem 1.4. By Lemma 2.3, (a) implies (b) and by Lemma 2.4, items (b) and (c) are equivalent. Finally, applying Lemmas 2.2 and 2.5, (a) follows from (b). The proof is now complete. □

Proof of Corollary 1.6. To prove this result, it is sufficient to show the following analogue of Lemma 2.5:

If $S^+(p) = n$, then there exists a set $\Omega = \{0 \le \nu_1 < \cdots < \nu_{n+1} \le q\}$ such that if for some positive numbers $\{\rho_i\}_{i=1}^{n+1}$ property (2.4) holds, then $p \equiv 0$.

The construction of the set Ω is similar to the one in Lemma 2.5, so we will only sketch the proof. As above, we assume that each $\rho_i = 1$.

Let us consider four different types of intervals. An interval of the *first* type is a closed segment of sign changes (see the proof of Lemma 2.5). An interval of the *second* type is $I = [k, k+l]_{\mathbb{Z}}$, for which

$$p(k)p(k+l) > 0, \quad p(k+1) = \cdots = p(k+l-1) = 0 \quad \text{and} \quad l-1 \text{ is odd.}$$

An interval of the *third* type is $I = [k, k+l]_{\mathbb{Z}}$, for which

$$p(k)p(k+l) < 0, \quad p(k+1) = \cdots = p(k+l-1) = 0 \quad \text{and} \quad l-1 \text{ is even.}$$

It is worth mentioning that intervals of the second and third types contribute to an increase in the number of zeros of the polynomial when zero values are changed to nonzero values, as $N(p, [k, k+l]_{\mathbb{Z}}) = l-1$ and $S^+(p, [k, k+l]_{\mathbb{Z}}) = l$.

Intervals of the *fourth* type are $I = [k, k+l]_{\mathbb{Z}}$, consisting of zeros of the first type, for which one of the following conditions holds: (i) $k = 0$, (ii) $k + l = q$, (iii) $l+1$ is odd and $p(k-1)p(k+l+1) < 0$, (iv) $l+1$ is even and $p(k-1)p(k+l+1) > 0$. Note that changing zero values of p to any nonzero numbers at points of intervals of the fourth type cannot increase the number of its zeros.

Denote the sets of intervals of the first, second, third, and fourth types by Σ_1, Σ_2, Σ_3, and Σ_4, respectively. Set $\Sigma = \cup_{i=1}^{3} \Sigma_i$. Note that intervals from different sets Σ_i can intersect only at their endpoints.

We have

$$n = \sum_{I \in \Sigma_1} (|I| - 1) + \sum_{I \in \Sigma_2} (|I| - 1) + \sum_{I \in \Sigma_3} (|I| - 1) + \sum_{I \in \Sigma_4} |I|.$$

By Ω we define a set of all[5] points of the intervals from Σ and Σ_4. Then the cardinality of Ω is

$$1 + \sum_{I \in \Sigma_1} (|I| - 1) + \sum_{I \in \Sigma_2} (|I| - 1) + \sum_{I \in \Sigma_3} (|I| - 1) + \sum_{I \in \Sigma_4} |I| = n + 1.$$

Let now $[\nu_{r_1}, \nu_{r_1} + l_{r_1}]_{\mathbb{Z}} \subset \Omega$ be the first interval from Σ and, for some integer i_0, $(-1)^{i_0+r_1} p(\nu_{r_1}) > 0$. Note that, for $r_1 \ge 2$, $0 \le \nu_1 < \cdots < \nu_{r_1-1}$ are the points of the set $\Omega \cap \Sigma_4$. Then $(-1)^{i_0+r_1+i} p(\nu_{r_1}+i) \ge 0$, $i = 0, \dots, l_{r_1}$.

[5]Here the left or right endpoint of each interval from Σ except the first one is excluded, but each interval from Σ_4 is included completely; cf. the proof of Lemma 2.5.

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Proceeding similarly, let $0 \le \nu_1 < \cdots < \nu_{r_2-1}$ be the points of the set Ω that have been constructed, and the second interval $I = [k, k+l]_{\mathbb{Z}} \in \Sigma$ is such that $k = \nu_{r_2}$ or $k = \nu_{r_2} - 1$. If $(-1)^{i_0+r_2} p(k) > 0$, then we include in Ω the points of the interval $[\nu_{r_2}, \nu_{r_2} + l_{r_2}]_{\mathbb{Z}}$, where $\nu_{r_2} = k$ and $l_{r_2} = l - 1$. Otherwise, we include in Ω the points of the interval $[\nu_{r_2}, \nu_{r_2} + l_{r_2}]_{\mathbb{Z}}$, where $\nu_{r_2} = k + 1$ and $l_{r_2} = l$. In both cases, $|[\nu_{r_2}, \nu_{r_2} + l_{r_2}]_{\mathbb{Z}}| = |I| - 1$ and $(-1)^{i_0+r_2+i} p(\nu_{r_2} + i) \ge 0$, $i = 0, \dots, l_{r_2}$.

Continuing in the same way, we complete the construction of the set Ω. Then, according to (2.4),

$$0 = \sum_{i=1}^{n+1} (-1)^i p(\nu_i) = (-1)^{i_0} \sum_{i=1}^{n+1} |p(\nu_i)|.$$

From this, $p(\nu_i) = 0$ for $i = 1, \dots, n+1$. Therefore, $p \equiv 0$, completing the proof. □

The proof of Corollary 1.6 immediately implies the following result.

Corollary 2.6. *If the set $\{\varphi_k\}_{k=1}^n \subset C[0,q]_{\mathbb{Z}}$ is a $T_{\mathbb{Z}}$-system, $p \in L_n$, and $N(p) = n - 1$, then the polynomial p has no intervals of the second and third types.*

To conclude this section, we give an example of a T_0-system which is not a $T_{\mathbb{Z}}$-system.

Example 2.7. For $k = 1, \dots, q-1$, define

$$\varphi_k(\nu) = \begin{cases} 0, & \nu = 0, \dots, k-2, k, \dots, q-1, \\ 1, & \nu = k-1, q, \end{cases} \qquad \varphi_q(\nu) = \begin{cases} 0, & \nu = 0, \dots, q-2, \\ 1, & \nu = q-1, \\ -1, & \nu = q. \end{cases}$$

Then $\dim L_q = q$ and $N_0(\varphi_k) = q - 1$, $k = 1, \dots, q$. Note that the function $\varphi_q(\nu)$ has a zero of the second type at the point q and hence $N(\varphi_q) = q$.

For an arbitrary polynomial $p(\nu) = \sum_{k=1}^q a_k \varphi_k(\nu)$, the following equalities hold:

$$p(\nu) = \begin{cases} a_{\nu+1}, & \nu = 0, \dots, q-1, \\ a_1 + \cdots + a_{q-1} - a_q, & \nu = q. \end{cases} \tag{2.5}$$

If p has q zeros of the first type, then either all coefficients of the polynomial are zero, or $q-1$ coefficients are zero and $a_1 + \cdots + a_{q-1} - a_q = 0$, that is, in any case all coefficients of p are zeros. Therefore, if p is nontrivial, $N_0(p) \le q - 1$. Thus, the set $\{\varphi_k\}_{k=1}^q$ is a T_0-system but not a $T_{\mathbb{Z}}$-system.

Further, define for any $f \in C[0,q]_{\mathbb{Z}}$

$$p^*(\nu) = \sum_{k=1}^{q-1} (f(k-1) + \delta)\varphi_k(\nu) + (f(q-1) - \delta)\varphi_q(\nu),$$

where

$$\delta = -\frac{1}{q+1}\Big(\sum_{k=1}^{q-1} f(k-1) - f(q-1) - f(q) \Big).$$

By (2.5), we have

$$p^*(\nu) = \begin{cases} f(\nu) + \delta, & \nu = 0, \dots, q-2, \\ f(\nu) - \delta, & \nu = q-1, q. \end{cases}$$

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If $\delta = 0$, then $f \equiv p^* \in L_q$ and $E_q(f, L_q)_\infty = 0$. If $\delta \neq 0$, $e(\nu) = p^*(\nu) - f(\nu)$,

$$\varepsilon_\nu = \begin{cases} \operatorname{sign}\delta, & \nu = 0, \dots, q-2, \\ -\operatorname{sign}\delta, & \nu = q-1, q, \end{cases}$$

then, by Theorem 1.3, for $\nu \in [0, q]_{\mathbb{Z}}$ and all polynomials $p \in L_q$

$$\varepsilon_\nu e(\nu) = \|e\|_\infty = |\delta|, \qquad \sum_{\nu=0}^{q} \varepsilon_\nu p(\nu) = 0.$$

Consequently, $p^* \in L_q$ is the unique best uniform approximant of the function f and $E_q(f, L_q)_\infty = |\delta|$. Note that the metric projection operator $Pf = p^*$ onto the q-dimensional subspace L_q is linear (see, e.g., [6]), and for any function $f \notin L_q$, the polynomial Pf does not admit a Chebyshev alternance set of length $q+1$.

3. Proofs of Theorems 1.8, 1.10, and 1.11

Proof of Theorem 1.8. Let $k_l > 0$ be the leading coefficient of the polynomial $P_l(\lambda)$. Then, by (1.7), we have $\beta_l k_{l+1} = \rho_l k_l$. Using this, for the monic polynomials $Q_l(\lambda) = k_l^{-1} P_l(\lambda)$, we obtain the recurrence relation

$$B_l Q_{l-1}(\lambda) + A_l Q_l(\lambda) + Q_{l+1}(\lambda) = \lambda Q_l(\lambda),$$

where $B_0 > 0$ is arbitrary and

$$A_l = \frac{\alpha_l}{\rho_l}, \quad B_l = \frac{\gamma_{l-1}\beta_{l-1}}{\rho_{l-1}\rho_l} = \frac{\gamma_{l-1}\rho_{l-1}k_{l-1}^2}{\beta_{l-1}\rho_l k_l^2}. \tag{3.1}$$

We have $A_l \in \mathbb{R}$ and $B_l > 0$ for any $l \geq 0$. These conditions are necessary and sufficient in Favard's theorem [8, Ch. II, Th. 6.4] on the existence of a nondecreasing function of bounded variation with infinite spectrum which generates a positive measure μ such that

$$\int_{\mathbb{R}} Q_l(\lambda) Q_m(\lambda)\, d\mu(\lambda) = B_0 B_1 \cdots B_l \delta_{lm}.$$

Now, to show (1.8), it is enough to set $B_0 = \rho_0^{-1}$.

According to [8, Ch. IV, Th. 2.2], the measure μ has a finite support $[a, b]$ if and only if the sequences A_l and B_l are uniformly bounded. In this case, the endpoints of the interval $[a, b]$ will be the limit points of the zeros of $Q_l(\lambda)$, and, therefore, of $P_l(\lambda)$ as well. By (3.1), the boundedness of A_l and B_l is equivalent to the boundedness of the sequences (1.9), which completes the proof. □

Proof of Theorem 1.10. Recall that $\lambda_{l,l} < \cdots < \lambda_{1,l}$ are the zeros of the polynomials $P_l(\lambda)$, $l \in \mathbb{N}$, and $\lambda_{q+1} < \cdots < \lambda_1$ are zeros of the polynomial $\widetilde{P}_{q+1}(\lambda)$, cf. (1.10).

In what follows, we will use the fact that the zeros of orthogonal polynomials $P_{l-1}(\lambda)$ and $P_l(\lambda)$, $l \in [2, q]_{\mathbb{Z}}$, as well as $P_q(\lambda)$ and $\widetilde{P}_{q+1}(\lambda)$, interlace (see [35, Ch. III, §3.3]):

$$\begin{gathered} \lambda_{l,l} < \lambda_{l-1,l-1} < \lambda_{l-1,l} < \lambda_{l-2,l-1} < \cdots < \lambda_{2,l-1} < \lambda_{2,l} < \lambda_{1,l-1} < \lambda_{1,l}, \\ \lambda_{q+1} < \lambda_{q,q} < \lambda_q < \lambda_{q-1,q} < \cdots < \lambda_{2,q} < \lambda_2 < \lambda_{1,q} < \lambda_1. \end{gathered} \tag{3.2}$$

Let the interval (a, b) contain the zeros of the polynomial $P_q(\lambda)$ and let $k \in [1, q+1]_{\mathbb{Z}}$. If $s = 0$, then $\psi_k(0) = P_0(\lambda_k) = 1$, so further $s \in [1, q]_{\mathbb{Z}}$.

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We will also use the following observation. If $s \in [1,q]_{\mathbb{Z}}$, $l \in [1,s]_{\mathbb{Z}}$, $\lambda_{s+1,s} = a$, $\lambda_{0,s} = b$, and $\lambda_{l,s} < \lambda_k < \lambda_{l-1,s}$, then

$$\operatorname{sign}\psi_k(s) = \operatorname{sign}P_s(\lambda_k) = (-1)^{l-1}, \tag{3.3}$$

and if $\lambda_k = \lambda_{l,s}$ or $\lambda_k = \lambda_{l-1,s}$, then

$$\psi_k(s) = P_s(\lambda_k) = P_s(\lambda_{l,s}) = P_s(\lambda_{l-1,s}) = 0. \tag{3.4}$$

Further, for $k = 1$ and $s \in [1,q]_{\mathbb{Z}}$, the inequality $\lambda_{1,s} < \lambda_1$ yields

$$\psi_1(s) > 0, \quad s \in [0,q]_{\mathbb{Z}}. \tag{3.5}$$

Therefore, $N(\psi_1) = 0$. For $k = q+1$ and $s \in [1,q]_{\mathbb{Z}}$, from the inequalities (3.2) we have $\lambda_{q+1} < \lambda_{s,s}$, so $\operatorname{sign}\psi_{q+1}(s) = (-1)^s$ and $\psi_{q+1}(0) = 1$. Thus, $N(\psi_{q+1}) = q$.

Let $k \in [2,q]_{\mathbb{Z}}$. Since, according to (3.2), $\lambda_k < \lambda_{k-1,q}$, there exists a smallest $i_1 \le q$ for which

$$\lambda_{k,i_1} < \lambda_{k-1,i_1-1} \le \lambda_k < \lambda_{k-1,i_1}, \quad i_1 \ge k. \tag{3.6}$$

If $i_1 \le s \le q$, then, by (3.2) and (3.6),

$$\lambda_{k,i_1} \le \lambda_{k,s} \le \lambda_{k,q} < \lambda_k < \lambda_{k-1,i_1} \le \lambda_{k-1,s} \le \lambda_{k-1,q}.$$

Using (3.3), we then derive

$$\operatorname{sign}\psi_k(s) = (-1)^{k-1}. \tag{3.7}$$

Since, according to (3.2), $\lambda_k < \lambda_{k-1,i_1} < \lambda_{k-2,i_1-1}$, there exists a smallest $i_2 \le i_1 - 1$ for which

$$\lambda_{k-1,i_2} < \lambda_{k-2,i_2-1} \le \lambda_k < \lambda_{k-2,i_2}, \quad i_2 \ge k-1. \tag{3.8}$$

If $i_2 \le s < i_1 - 1$, then, by (3.2), (3.6) and (3.8),

$$\lambda_{k-1,i_2} \le \lambda_{k-1,s} < \lambda_{k-1,i_1-1} \le \lambda_k < \lambda_{k-2,i_2} \le \lambda_{k-2,s} < \lambda_{k-2,i_1-1}.$$

Therefore, using (3.3) and (3.4), we obtain

$$\operatorname{sign}\psi_k(s) = (-1)^{k-2}, \quad \operatorname{sign}\psi_k(i_1-1) = (-1)^{k-2} \vee 0.$$

Proceeding similarly for $j \in [2,k-1]_{\mathbb{Z}}$, we construct points $\{i_j\}_{j=1}^{k-1}$ such that

$$k-j+1 \le i_j \le i_{j-1} - 1, \quad \lambda_{k-j+1,i_j} < \lambda_{k-j,i_j-1} \le \lambda_k < \lambda_{k-j,i_j},$$

and for $i_j \le s < i_{j-1} - 1$

$$\begin{gathered}\lambda_{k+1-j,i_j} \le \lambda_{k+1-j,s} < \lambda_{k+1-j,i_{j-1}-1} \le \lambda_k < \lambda_{k-j,i_j} \le \lambda_{k-j,s} < \lambda_{k-j,i_{j-1}-1},\\ \operatorname{sign}\psi_k(s) = (-1)^{k-j}, \quad \operatorname{sign}\psi_k(i_{j-1}-1) = (-1)^{k-j} \vee 0.\end{gathered} \tag{3.9}$$

In the case $1 \le s < i_{k-1} - 1$, we have $\lambda_{1,s} < \lambda_{1,i_{k-1}-1} \le \lambda_k$. Hence, in light of (3.2),

$$\operatorname{sign}\psi_k(s) = 1, \quad \operatorname{sign}\psi_k(i_{k-1}-1) = 1 \vee 0. \tag{3.10}$$

If for all $j \in [2,k-1]_{\mathbb{Z}}$, $\lambda_{k-j,i_j-1} \ne \lambda_k$, then, according to (3.7), (3.9), (3.10), only the points $\nu_j = i_j$, $j \in [1,k-1]_{\mathbb{Z}}$, are zeros of $\psi_k(\nu)$ and, moreover, $N(\psi_k) = k-1$.

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Let $\psi_k(i_j - 1) = 0$. Since $\psi_k(i_{j+1}) \neq 0$, it follows that $i_{j+1} \leq i_j - 2 < i_j - 1$ and

$$\operatorname{sign}\left(\psi_k(i_j - 2)\psi_k(i_j)\right) = (-1)^{k-j-1}(-1)^{k-j} = -1.$$

Therefore, the zero $\nu_j = i_j$ is replaced by $\nu_j = i_j - 1$ and we again have $N(\psi_k) = k - 1$. Along the way, we proved that if $\psi_k(\nu) = 0$, $k \in [2, q]_{\mathbb{Z}}$, then $\nu \in [1, q-1]_{\mathbb{Z}}$ and $\psi_k(\nu - 1)\psi_k(\nu + 1) < 0$. This implies that $S^-(\psi_k) = S^+(\psi_k) = k - 1$, completing the proof. □

3.1. *Proof of Theorem 1.11*

Step 1. First, let us rewrite the discrete Sturm-Liouville problems (1.7) and (1.12) and then derive the Christoffel-Darboux formulas.

The self-adjoint form of the problem (1.7) is given by

$$\nabla(w_l \Delta P_l(\lambda)) + c_l P_l(\lambda) = \lambda d_l P_l(\lambda), \quad l \in \mathbb{Z}_+, \quad \lambda \in \mathbb{C},$$

where $\Delta b_l = b_{l+1} - b_l$, $\nabla b_l = b_l - b_{l-1}$ are the forward and backward differences, respectively, d_l is defined in (1.8), and

$$w_l = \frac{\beta_l d_l}{\rho_l}, \quad c_l = \frac{(\alpha_l + \beta_l + \gamma_{l-1}) d_l}{\rho_l}. \tag{3.11}$$

Indeed, using (1.7), (1.8) and (3.11), we obtain

$$w_{l-1} = \frac{\gamma_{l-1} d_l}{\rho_l}, \quad \nabla(w_l \Delta P_l(\lambda)) = w_l P_{l+1}(\lambda) + w_{l-1} P_{l-1}(\lambda) - (w_l + w_{l-1}) P_l(\lambda),$$

and

$$\nabla(w_l \Delta P_l(\lambda)) + c_l P_l(\lambda) - \lambda d_l P_l(\lambda) = \frac{d_l}{\rho_l}\left\{\beta_l P_{l+1}(\lambda) + \alpha_l P_l(\lambda) + \gamma_{l-1} P_{l-1}(\lambda) - \lambda \rho_l P_l(\lambda)\right\}.$$

Now we can easily derive the Christoffel-Darboux formula

$$\sum_{s=0}^{l} d_s P_s(x) P_s(y) = w_l \frac{P_{l+1}(x) P_l(y) - P_l(x) P_{l+1}(y)}{x - y}, \quad x \neq y. \tag{3.12}$$

To see this, using the Lagrange formula [2]

$$(x - y) d_s P_s(x) P_s(y) = P_s(y) \nabla(w_s \Delta P_s(x)) - P_s(x) \nabla(w_s \Delta P_s(y)),$$

we have

$$\begin{aligned}
(x - y) \sum_{s=0}^{l} d_s P_s(x) P_s(y) &= \sum_{s=0}^{l} \left\{P_s(y) \nabla(w_s \Delta P_s(x)) - P_s(x) \nabla(w_s \Delta P_s(y))\right\} \\
&= \sum_{s=0}^{l} \left\{w_s P_{s+1}(x) P_s(y) + w_{s-1} P_s(y) P_{s-1}(x) - w_s P_{s+1}(y) P_s(x) - w_{s-1} P_s(x) P_{s-1}(y)\right\} \\
&= w_l \left\{P_{l+1}(x) P_l(y) - P_l(x) P_{l+1}(y)\right\}.
\end{aligned}$$

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Further, by (3.12), it follows that $w_l = k_l d_l / k_{l+1}$, where k_l is the leading coefficient of the polynomial P_l. Applying (1.8) and (3.11), we get

$$k_0 = 1, \quad k_l = \frac{\rho_0 \cdots \rho_{l-1}}{\beta_0 \cdots \beta_{l-1}}, \quad l \geq 1.$$

Moreover, the Sturm-Liouville problem (1.12) can also be written in an equivalent self-adjoint form as follows:

$$\begin{aligned} &\nabla(w_l \Delta\psi(\nu)) + c_l \psi(\nu) = \lambda d_l \psi(\nu), \\ &\psi(-1) = 0, \quad \psi(q+1) - \eta\psi(q) = 0. \end{aligned} \tag{3.13}$$

From (3.13), as in the proof of (3.12), we obtain the following Lagrange equality:

$$d_\nu(\lambda_l - \lambda_k)\psi_l(\nu)\psi_k(\nu) = \nabla(w_\nu\{\psi_l(\nu)\Delta\psi_k(\nu) - \psi_k(\nu)\Delta\psi_l(\nu)\}).$$

It is important to mention that the Christoffel-Darboux formula for the eigenfunctions ψ_k has the form

$$\sum_{\nu=0}^{l} d_\nu \psi_k(\nu)\psi_s(\nu) = w_l \frac{\psi_k(l+1)\psi_s(l) - \psi_k(l)\psi_s(l+1)}{\lambda_k - \lambda_s}, \tag{3.14}$$

which immediately implies the discrete orthogonality of the eigenfunctions. Indeed, for $k \neq s$, due to the boundary condition in (1.12),

$$\begin{aligned} (\psi_k, \psi_s) &= \sum_{\nu=0}^{q} d_\nu \psi_k(\nu)\psi_s(\nu) = w_q \frac{\psi_k(q+1)\psi_s(q) - \psi_k(q)\psi_s(q+1)}{\lambda_k - \lambda_s} \\ &= w_q \frac{(\psi_k(q+1) - \eta\psi_k(q))\psi_s(q) - (\psi_s(q+1) - \eta\psi_s(q))\psi_k(q)}{\lambda_k - \lambda_s} = 0. \end{aligned}$$

Step 2. Now we will prove several lemmas on zeros and sign changes of discrete functions.

Lemma 3.1. *If $f \in C[0, q]_{\mathbb{z}}$, then*

$$S^-(f) \leq N(f) \leq S^+(f). \tag{3.15}$$

Proof. It suffices to compare $S^-(f, [m, n]_{\mathbb{z}})$, $N(f, [m, n]_{\mathbb{z}})$, and $S^+(f, [m, n]_{\mathbb{z}})$ at interior points of the intervals $[m, n]_{\mathbb{z}}$, where f equals zero. We may assume that $n - m \geq 2$.

- If $m = 0$, $f(0) = f(1) = \cdots = f(n-1) = 0$, and $f(n) \neq 0$, then $S^-(f, [0, n]_{\mathbb{z}}) = 0$, $N(f, [0, n]_{\mathbb{z}}) = n$, $S^+(f, [0, n]_{\mathbb{z}}) = n$ and thus (3.15) is valid.
- If $n = q$, $f(m) \neq 0$, $f(m+1) = \cdots = f(q) = 0$, and $f(n) \neq 0$, then $S^-(f, [m, q]_{\mathbb{z}}) = 0$, $N(f, [m, q]_{\mathbb{z}}) = q - m$, $S^+(f, [m, q]_{\mathbb{z}}) = q - m$ and (3.15) follows.
- If $f(m)f(n) < 0$ (for example, $f(m) > 0$, $f(m+1) = \cdots = f(n-1) = 0$, $f(n) < 0$), then $S^-(f, [m, q]_{\mathbb{z}}) = 1$, $N(f, [m, q]_{\mathbb{z}}) = n - m - 1$, $S^+(f, [m, n]_{\mathbb{z}}) = n - m - 1$ for even $n - m$ and $S^+(f, [m, n]_{\mathbb{z}}) = n - m$ for odd $n - m$. Hence, (3.15) is valid.
- If $f(m)f(n) > 0$ (for example, $f(m) > 0$, $f(m+1) = \cdots = f(n-1) = 0$, $f(n) > 0$), then $S^-(f, [m, q]_{\mathbb{z}}) = 1$, $N(f, [m, q]_{\mathbb{z}}) = n - m - 1$, $S^+(f, [m, n]_{\mathbb{z}}) = n - m - 1$ for odd $n - m$ and $S^+(f, [m, n]_{\mathbb{z}}) = n - m$ for even $n - m$, implying (3.15).

The proof is now complete. □

The next few results correspond to discrete analogues of Rolle's theorem. Below, for integers $m < n$, we say (m, n) is a pair of *generalized sign change* of f if $f(m)f(n) < 0$ and $f(m+1) = \ldots = f(n-1) = 0$. Clearly, an ordinary sign-change pair is also generalized (for $n - m = 1$). Note that the number of pairs of generalized sign changes of f on $[a, b]_{\mathbb{z}}$ is exactly $S^-(f, [a, b]_{\mathbb{z}})$.

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Lemma 3.2. *If $m < n$ and $f(m)f(n) < 0$, then the function f has a zero on $[m+1, n]_{\mathbb{Z}}$ and a generalized sign change on $[m, n]_{\mathbb{Z}}$.*

Proof. We can assume that $f(m) < 0$ and $f(n) > 0$. If $n = m + 1$, then n is a zero of the second type and (m, n) is the pair of a sign change. Let $n > m + 1$ and $s_1 \le n$ be the largest integer such that $f(m) < 0, f(m + 1) < 0, \dots, f(s_1 - 1) < 0$. Then either $f(s_1) > 0$ or $f(s_1) = 0$. If $f(s_1) > 0$, then $s_1 \ge m+1$ is a zero of the second type and $(s_1 - 1, s_1)$ is the pair of a sign change. Suppose $f(s_1) = 0$ and $s_2 < n$ is the largest integer such that $f(s_1) = f(s_1 + 1) = \cdots = f(s_2 - 1) = 0$. Then either $f(s_2) > 0$ or $f(s_2) < 0$. If $f(s_2) > 0$, then $s_1, s_1+1, \dots, s_2-1$ are zeros of the first type and $(s_1 - 1, s_2)$ is the pair of a generalized sign change. If $f(s_2) < 0$, using similar arguments, we arrive at the required statement. □

Lemma 3.3. *Let $m < n$ be zeros of f. Then the differences ∇f on $[m + 1, n]_{\mathbb{Z}}$ and Δf on $[m, n - 1]_{\mathbb{Z}}$ have zeros.*

Proof. First, let us prove the lemma for ∇f. Consider the possible values f at the zeros m and n. We will repeatedly use the trivial formula

$$f(l) - f(k) = \sum_{\nu=k+1}^{l} \nabla f(\nu). \tag{3.16}$$

- If $f(m) = f(n) = 0$, then according to (3.16), either $\nabla f(\nu) = 0$ for all $\nu \in [m + 1, n]_{\mathbb{Z}}$, or $\nabla f(\nu_1)\nabla f(\nu_2) < 0$ for some $\nu_1, \nu_2 \in [m + 1, n]_{\mathbb{Z}}$, and by Lemma 3.2 ∇f has a zero on $[m + 1, n]_{\mathbb{Z}}$.
- If $f(m) = 0$, $f(n - 1) > 0$, $f(n) < 0$, then $f(n - 1) - f(m) > 0$, $\nabla f(n) < 0$. According to (3.16), for some $s \in [m + 1, n - 1]_{\mathbb{Z}}$, $\nabla f(s) > 0$, so $\nabla f(s)\nabla f(n) < 0$. By Lemma 3.2, ∇f on $[m + 1, n]_{\mathbb{Z}}$ has a zero. The case $f(m) = 0$, $f(n - 1) < 0$, $f(n) > 0$ is considered similarly.
- If $f(m - 1) > 0$, $f(m) < 0$, $f(n) = 0$, then $\nabla f(m) < 0$, $f(n) - f(m) > 0$. According to (3.16), for some $s \in [m + 1, n]_{\mathbb{Z}}$, $\nabla f(s) > 0$, therefore $\nabla f(m)\nabla f(s) < 0$. In light of Lemma 3.2, ∇f on $[m + 1, n]_{\mathbb{Z}}$ has a zero. The case $f(m - 1) < 0$, $f(m) > 0$, $f(n) = 0$ is treated similarly.
- If $f(m - 1) > 0$, $f(m) < 0$, $f(n - 1) > 0$, $f(n) < 0$, then $\nabla f(m) < 0$, $f(n - 1) - f(m) > 0$. According to (3.16), for some $s \in [m + 1, n - 1]_{\mathbb{Z}}$, $\nabla f(s) > 0$, so $\nabla f(m)\nabla f(s) < 0$. By Lemma 3.2, ∇f on $[m + 1, n]_{\mathbb{Z}}$ has a zero.
- If $f(m - 1) > 0$, $f(m) < 0$, $f(n - 1) < 0$, $f(n) > 0$, then $\nabla f(m) < 0$, $\nabla f(n) < 0$. Using again Lemma 3.2, ∇f on $[m + 1, n]_{\mathbb{Z}}$ has a zero. The remaining two cases are considered similarly.

Finally, using the equality $f(l) - f(k) = \sum_{\nu=k}^{l-1} \Delta f(\nu)$ one similarly proves the statement for Δf. The proof is now complete. □

Lemma 3.4. *Let $f \in C[0, q]_{\mathbb{Z}}$, $0 \le m < n \le m' < n' \le q$, and the pairs (m, n) and (m', n') be consecutive generalized sign changes of f. Then the differences ∇f on $[m + 1, m' + 1]_{\mathbb{Z}}$ and Δf on $[n - 1, n' - 1]_{\mathbb{Z}}$ have generalized sign changes.*

Proof. Let $f(m) < 0$ and $f(n) > 0$. Then

$$f(\nu) = 0, \quad \nu \in [m + 1, n - 1]_{\mathbb{Z}}, \quad f(\nu) > 0, \quad \nu \in [n, m']_{\mathbb{Z}},$$

$$f(\nu) = 0, \quad \nu \in [m' + 1, n' - 1]_{\mathbb{Z}}, \quad f(n') < 0.$$

Since $\nabla f(m + 1) = f(m + 1) - f(m) > 0$ and $\nabla f(m' + 1) = f(m' + 1) - f(m') < 0$, by Lemma 3.2, ∇f has a generalized sign change on $[m + 1, m' + 1]_{\mathbb{Z}}$. The case $f(m) > 0$ and $f(n) < 0$ and the fact that the difference Δf has a generalized sign change on $[n - 1, n' - 1]_{\mathbb{Z}}$ can be obtained similarly. □

Lemma 3.5. *Let $f \in C[0, q]_{\mathbb{Z}}$, $f(-1) = f(q + 1) = 0$, $0 \le m < n \le q$, and the pair (m, n) be a generalized sign change of f. Then the difference ∇f on $[0, m + 1]_{\mathbb{Z}}$ and $[n, q + 1]_{\mathbb{Z}}$ has generalized sign changes.*

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Proof. Let $f(\nu) = 0$ for $\nu \in [-1, m-1]_{\mathbb{Z}}$. If $f(m) > 0$ and $f(n) < 0$, then $\nabla f(m) > 0$. Since $\nabla f(m+1) < 0$, $(m, m+1)$ is the pair of a sign change of ∇f. The case $f(m) < 0$ and $f(n) > 0$ is treated similarly.

Suppose the function $f(\nu)$ has a nonzero value on $[0, m-1]_{\mathbb{Z}}$, $m \geq 1$, for example, $f(0) > 0$. If $f(m) > 0$ and $f(n) < 0$, then $\nabla f(0) > 0$ and $\nabla f(m+1) < 0$. By Lemma 3.2, ∇f has a generalized sign change on $[0, m+1]_{\mathbb{Z}}$. If $f(m) < 0$ and $f(n) > 0$, then $f(m) - f(0) < 0$. Then for some $\nu_1 \in [1, m]_{\mathbb{Z}}$, $\nabla f(\nu_1) < 0$ and $\nabla f(0) > 0$. By Lemma 3.2, ∇f has a generalized sign change on $[0, m]_{\mathbb{Z}}$. The case $f(0) < 0$ is treated similarly. Therefore, ∇f on $[0, m+1]_{\mathbb{Z}}$ always has a generalized sign change.

The fact that the difference ∇f has a generalized sign change on $[n, q+1]_{\mathbb{Z}}$ can be shown similarly. The proof is now complete. □

Definition 3.6. By $K(\cdot)$ we denote any one of the following quantities: $N(\cdot)$, $S^-(\cdot)$, or $S^+(\cdot)$.

We will use the simple facts that $K(f, [a, b]_{\mathbb{Z}}) = K(f, [a, c]_{\mathbb{Z}}) + K(f, [c, b]_{\mathbb{Z}})$ whenever $a \leq c \leq b$ with $f(c) \neq 0$, and that $K(fg) = K(f)$ if g preserves the sign.

Proposition 3.7. *If $f \in C[0, q]_{\mathbb{Z}}$, $f \neq 0$, then*

$$K(\nabla f, [1, q]_{\mathbb{Z}}) \geq K(f, [0, q]_{\mathbb{Z}}) - 1, \tag{3.17}$$

$$K(\Delta f, [0, q-1]_{\mathbb{Z}}) \geq K(f, [0, q]_{\mathbb{Z}}) - 1. \tag{3.18}$$

If, in addition, $f(-1) = f(q+1) = 0$, then

$$K(\nabla f, [0, q+1]_{\mathbb{Z}}) \geq K(f, [0, q]_{\mathbb{Z}}) + 1. \tag{3.19}$$

Remark 3.8. Note that (3.18) was proved in [19, Proposition 5.1] (see also the monograph [1, Theorem 1.8.1]) for $K(\cdot) = N(\cdot)$.

Proof of Proposition 3.7. As usual, we denote $K(f) = K(f, [0, q]_{\mathbb{Z}})$. The cases $f > 0$ and $f < 0$ on $[0, q]_{\mathbb{Z}}$ are straightforward and may be omitted.

(a) The cases $K(\cdot) = N(\cdot)$ or $K(\cdot) = S^-(\cdot)$. To prove inequality (3.17) in the first case, let $\{\nu_1 < \cdots < \nu_s\} \subset [0, q]_{\mathbb{Z}}$ be the zeros of f. By Lemma 3.3 on the intervals $[\nu_1 + 1, \nu_2]_{\mathbb{Z}}, \ldots, [\nu_{s-1} + 1, \nu_s]_{\mathbb{Z}}$, the function ∇f has at least $s-1$ zeros. Therefore, $N(\nabla f, [1, q]_{\mathbb{Z}}) \geq N(f) - 1$.

In the second case, let $\{(m_i, n_i)\}_{i=1}^{s} \subset [0, q]_{\mathbb{Z}} \times [0, q]_{\mathbb{Z}}$, $m_i < n_i \leq m_{i+1} < n_{i+1}$, be the generalized sign changes of f on $[0, q]_{\mathbb{Z}}$. By Lemma 3.4, ∇f has generalized sign changes on the intervals $[m_1 + 1, m_2 + 1]_{\mathbb{Z}}, \ldots, [m_{s-1} + 1, m_s + 1]_{\mathbb{Z}}$. Since this number is not less than $s-1$, $S^-(\nabla f, [1, q]_{\mathbb{Z}}) \geq S^-(f) - 1$.

Inequality (3.18) can be established similarly.

Suppose $f(-1) = f(q+1) = 0$. By Lemma 3.3, the difference ∇f has at least two additional zeros on $[0, \nu_1]_{\mathbb{Z}}$ and $[\nu_s + 1, q+1]_{\mathbb{Z}}$. Similarly, by Lemma 3.5, ∇f has at least two generalized sign changes on the intervals $[0, m_1 + 1]_{\mathbb{Z}}$ and $[\nu_s, q+1]_{\mathbb{Z}}$. This establishes (3.19).

(b) The case $K(\cdot) = S^+(\cdot)$. For the sake of brevity, we will prove the intermediate inequality

$$S^+(\nabla f) \geq S^+(f), \quad f(-1) = 0, \tag{3.20}$$

by induction on q. Inequalities (3.17) and (3.19) can be proved in a similar way.

Let $n = q = 1$ and $f(0) = \nabla f(0) > 0$. If $f(1) > 0$, then $S^+(\nabla f) = 0$ and (3.20) is valid. If $f(1) \leq 0$, then $\nabla f(1) < 0$, so $S^+(\nabla f) = S^+(f) = 1$ and (3.20) holds. The case $f(0) < 0$ can be treated similarly. Let $f(0) = \nabla f(0) = 0$. If $f(1) \neq 0$, then $S^+(\nabla f) = S^+(f) = 1$ and (3.20) is valid.

Assume that inequality (3.20) holds for all $n \leq q-1$. To prove it for $n = q$, if $n \leq q-1$ is chosen so that $f(n) \neq 0$, $\nabla f(n) \neq 0$ and $S^+(\nabla f, [n, q]_{\mathbb{Z}}) \geq S^+(f, [n, q]_{\mathbb{Z}})$, then by the inductive assumption we obtain (3.20) as follows:

$$S^+(\nabla f) = S^+(\nabla f, [0, n]_{\mathbb{Z}}) + S^+(\nabla f, [n, q]_{\mathbb{Z}}) \geq S^+(f, [0, n]_{\mathbb{Z}}) + S^+(f, [n, q]_{\mathbb{Z}}) = S^+(f).$$

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- Let $f(q-1) > 0$. If $f(q) > 0$, then, by the inductive assumption,

$$S^+(\nabla f) \geq S^+(\nabla f, [0, q-1]_{\mathbb{Z}}) \geq S^+(f, [0, q-1]_{\mathbb{Z}}) = S^+(f).$$

- Let $f(q-1) > 0$, $f(q) \leq 0$, and $s \leq q+1$ be such that

$$f(q-s) \leq 0,\ f(q-s+1) > 0,\ \ldots,\ f(q-1) > 0.$$

Since $\nabla f(q-s+1) > 0$ and $\nabla f(q) < 0$, Lemma 3.2 implies that $S^+(\nabla f, [q-s+1, q]_{\mathbb{Z}}) \geq 1$ and

$$S^+(\nabla f, [q-s+1, q]_{\mathbb{Z}}) \geq 1 = S^+(f, [q-s+1, q]_{\mathbb{Z}}).$$

The case $f(q-1) < 0$ can be treated similarly.

Assume now that $f(q-1) = \cdots = f(q-s_1+1) = 0$ and

$$f(q-s_1) > 0,\ \ldots,\ f(q-s_1-s_2+1) > 0,\ f(q-s_1-s_2) \leq 0.$$

- If $f(q) = 0$, then

$$S^+(\nabla f, [q-s_1-s_2+1, q]_{\mathbb{Z}}) \geq S^+(f, [q-s_1-s_2+1, q]_{\mathbb{Z}}) = s_1.$$

- If $f(q) > 0$, then

$$S^+(\nabla f, [q-s_1-s_2+1, q]_{\mathbb{Z}}) \geq S^+(f, [q-s_1-s_2+1, q]_{\mathbb{Z}}) = \begin{cases} s_1 - 1, & s_1 \text{ odd}, \\ s_1, & s_1 \text{ even}. \end{cases}$$

- If $f(q) < 0$, then

$$S^+(\nabla f, [q-s_1-s_2+1, q]_{\mathbb{Z}}) \geq S^+(f, [q-s_1-s_2+1, q]_{\mathbb{Z}}) = \begin{cases} s_1, & s_1 \text{ odd}, \\ s_1 - 1, & s_1 \text{ even}. \end{cases}$$

The remaining cases can be reduced to the ones above by replacing f with $-f$. The proof is now complete. □

In order to prove Theorem 1.11, we employ the discrete version of the Liouville method (see, e.g., [5] for the continuous case). In addition to the polynomial $V(\nu) = \sum_{k=m}^{n} a_k \psi_k(\nu)$ (see (1.13)), we define

$$V_r(\nu) = \sum_{s=m}^{n} (\lambda_1 - \lambda_s)^r a_s \psi_s(\nu), \quad r \in \mathbb{Z}.$$

Lemma 3.9. *We have*

$$K(V_1) \geq K(V).$$

Proof. Multiplying both sides of (3.14) with $k = 1$ by $(\lambda_1 - \lambda_s)a_s$ and summing over s, for $l \in [0, q-1]_{\mathbb{Z}}$, we derive

$$\begin{aligned} g(l) &= \sum_{\nu=0}^{l} d_\nu \psi_1(\nu) V_1(\nu) = w_l \{\psi_1(l+1)V(l) - \psi_1(l)V(l+1)\} \\ &= -w_l \psi_1(l)\psi_1(l+1)\left\{\frac{V(l+1)}{\psi_1(l+1)} - \frac{V(l)}{\psi_1(l)}\right\} = -w_l \psi_1(l)\psi_1(l+1)\Delta\left(\frac{V(l)}{\psi_1(l)}\right). \end{aligned} \tag{3.21}$$

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Due to the boundary condition in (1.12), we have $g(-1) = 0$ and

$$\begin{aligned} g(q) &= w_q\{\psi_1(q+1)V(q) - \psi_1(q)V(q+1)\} \\ &= w_q\{(\psi_1(q+1) - \eta\psi_1(q))V(q) - \psi_1(q)(V(q+1) - \eta V(q))\} = 0. \end{aligned}$$

Applying Proposition 3.7, formula (3.21), and taking into account the positivity of $\psi_1(l)$ (see (3.5)), d_l, w_l, we obtain

$$\begin{aligned} K(V_1) = K\Big(\frac{\nabla g(l)}{d_l\psi_1(l)}\Big) &= K(\nabla g, [0,q]_{\mathbb{Z}}) \geq K(g, [0,q-1]_{\mathbb{Z}}) + 1 \\ &= K\Big(-w_s\psi_1(l)\psi_1(l+1)\Delta\Big(\frac{V(l)}{\psi_1(l)}\Big), [0,q-1]_{\mathbb{Z}}\Big) + 1 \\ &= K\Big(\Delta\Big(\frac{V(l)}{\psi_1(l)}\Big), [0,q-1]_{\mathbb{Z}}\Big) + 1 \geq K\Big(\frac{V(l)}{\psi_1(l)}, [0,q]_{\mathbb{Z}}\Big) = K(V), \end{aligned}$$

completing the proof of the lemma. □

Step 3 of the proof of Theorem 1.11. Note that in the case $m = n = 1$ Theorem 1.11 holds since ψ_1 is positive. Thus, we may assume that $n \geq 2$. Moreover, the lower bound in (1.14) for $m = 1$ holds trivially.

Applying Lemma 3.9 repeatedly, we obtain

$$\begin{aligned} K(V_r/(\lambda_1 - \lambda_n)^r) = K(V_r) \geq K(V), \quad r \geq 1, \quad m \geq 1, \\ K(V_r/(\lambda_1 - \lambda_m)^r) = K(V_r) \leq K(V), \quad r \leq -1, \quad m \geq 2. \end{aligned}$$

Since

$$\lim_{r\to+\infty} \|\psi_n - V_r/(\lambda_1 - \lambda_n)^r)\|_\infty = \lim_{r\to-\infty} \|\psi_m - V_r/(\lambda_1 - \lambda_m)^r)\|_\infty = 0,$$

by Theorem 1.10 for sufficiently large $N > 0$ we deduce

$$\begin{aligned} K(\psi_n) = K(V_r/(\lambda_1 - \lambda_n)^r), \quad r > N, \\ K(\psi_m) = K(V_r/(\lambda_1 - \lambda_m)^r), \quad r < -N. \end{aligned}$$

Thus,

$$\begin{aligned} n - 1 = K(\psi_n) = \lim_{r\to+\infty} K(V_r/(\lambda_1 - \lambda_n)^r) = \lim_{r\to+\infty} K(V_r) \geq K(V), \\ m - 1 = K(\psi_m) = \lim_{r\to-\infty} K(V_r/(\lambda_1 - \lambda_m)^r) = \lim_{r\to-\infty} K(V_r) \leq K(V). \end{aligned}$$

This and Lemma 3.1 complete the proof.

4. Proof of Theorem 1.14

Let $\nu \in [0,q]_{\mathbb{Z}}$, $1 \leq \nu_1 < \cdots < \nu_m \leq q$, and

$$R_m(\nu) = R_m(\nu, \nu_1, \dots, \nu_m) = \prod_{j=1}^{m} (\nu_j - \nu).$$

Let also $\{\varphi_k\}_{k=1}^{m+1}$ be a $T_{\mathbb{Z}}$-system on $[0,q]_{\mathbb{Z}}$ and

$$D_{m+1}(\nu) = D_{m+1}(\nu, \nu_1, \dots, \nu_m) = \Delta\begin{pmatrix} \varphi_1, & \dots, & \varphi_m, & \varphi_{m+1} \\ \nu_m, & \dots, & \nu_1, & \nu \end{pmatrix}. \tag{4.1}$$

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Observe that ν_j, $j = 1, \dots, m$, are the only zeros of the polynomial D_{m+1}. By applying Theorem 1.4 and basic properties of determinants, we obtain the precise distribution of the signs of $D_{m+1}(\nu)$.

Lemma 4.1. *For all* $\nu \in [0, q]_{\mathbb{Z}}$,

$$\operatorname{sign} D_{m+1}(\nu) = \operatorname{sign} D_{m+1}(0, q-m+1, \dots, q) \operatorname{sign} R_m(\nu).$$

Proof of Theorem 1.14. Let the zeros of $P_q(\lambda)$ lie on the interval (a, b) and let $\eta_b = P_{q+1}(b)/P_q(b)$. Then the zeros of the polynomial $\widetilde{P}_{q+1}(\lambda) = P_{q+1}(\lambda) - \eta P_q(\lambda)$ are such that $\lambda_2, \dots, \lambda_q \in (a, b)$, $\lambda_1 < b$ for $\eta < \eta_b$, $\lambda_1 = b$ for $\eta = \eta_b$, and $\lambda_1 > b$ for $\eta > \eta_b$ (see [35, Ch. III, §3.3]).

(b) In this case $m = 0$. From the interlacing of zeros of $P_\nu(\lambda)$ (see (3.2)), there holds

$$\lambda_{1,1} < \dots < \lambda_{1,q} < \lambda_1,$$

and therefore, $P_\nu(\lambda_1) > 0$, $\nu \in [0, q]_{\mathbb{Z}}$. Applying the Christoffel-Darboux formula (3.12), we obtain

$$\begin{aligned}\sum_{l=0}^{q} d_l P_l(\lambda_1) P_l(\lambda) &= w_q \frac{P_{q+1}(\lambda)P_q(\lambda_1) - P_{q+1}(\lambda_1)P_q(\lambda)}{\lambda - \lambda_1} \\ &= w_q \frac{\widetilde{P}_{q+1}(\lambda)P_q(\lambda_1) - \widetilde{P}_{q+1}(\lambda_1)P_q(\lambda)}{\lambda - \lambda_1} = w_q P_q(\lambda_1) \frac{\widetilde{P}_{q+1}(\lambda)}{\lambda - \lambda_1}.\end{aligned}$$

Therefore, the coefficients a_ν in expansion (1.15) satisfy

$$a_\nu = \frac{P_\nu(\lambda_1)}{w_q P_q(\lambda_1)} > 0, \quad \nu \in [0, q]_{\mathbb{Z}}. \tag{4.2}$$

For $\eta = \eta_b$, we have $\lambda_1 = b$ and (1.17) is valid.

(a, c): the case $m = 0$ and $\eta \neq \eta_b$. By the Christoffel-Darboux formula,

$$\frac{P_\nu(\lambda_1)}{P_\nu(b)} - \frac{P_{\nu+1}(\lambda_1)}{P_{\nu+1}(b)} = \frac{b - \lambda_1}{w_\nu P_\nu(b) P_{\nu+1}(b)} \sum_{l=0}^{\nu} d_l P_l(b) P_l(\lambda_1).$$

Since $\operatorname{sign}(b - \lambda_1) = \operatorname{sign}(\eta_b - \eta)$, we have, for $\nu \in [0, q-1]_{\mathbb{Z}}$,

$$\operatorname{sign}\left(\frac{P_\nu(\lambda_1)}{P_\nu(b)} - \frac{P_{\nu+1}(\lambda_1)}{P_{\nu+1}(b)}\right) = \operatorname{sign}(\eta_b - \eta).$$

From this and (4.2) we obtain (1.16) and (1.18) for $m = 0$.

(a): the case $m \geq 1$ and $\eta \neq \eta_b$. Let us calculate the coefficients a_ν in expansion (1.15) in terms of $D_{m+1}(\nu, \nu_1, \dots, \nu_m)$, $1 \leq \nu_1 < \dots < \nu_m \leq q$.

Letting $\omega_{m+1}(\lambda) = \prod_{j=1}^{m+1}(\lambda - \lambda_j)$, we have

$$\frac{\widetilde{P}_{q+1}(\lambda)}{\omega_{m+1}(\lambda)} = \sum_{i=1}^{m+1} \frac{\widetilde{P}_{q+1}(\lambda)}{\omega'_{m+1}(\lambda_i)(\lambda - \lambda_i)},$$

where

$$\omega'_{m+1}(\lambda_i) = \prod_{j \neq i}(\lambda_i - \lambda_j), \quad \operatorname{sign} \omega'_{m+1}(\lambda_i) = (-1)^{i-1}.$$

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Applying the Christoffel-Darboux formula, we obtain as above

$$\sum_{\nu=0}^{q} d_\nu P_\nu(\lambda_i)P_\nu(\lambda) = w_q \frac{P_{q+1}(\lambda)P_q(\lambda_i) - P_q(\lambda)P_{q+1}(\lambda_i)}{\lambda - \lambda_i} = w_q \frac{\widetilde{P}_{q+1}(\lambda)P_q(\lambda_i)}{\lambda - \lambda_i}. \tag{4.3}$$

Therefore,

$$\frac{\widetilde{P}_{q+1}(\lambda)}{\omega_{m+1}(\lambda)} = \frac{1}{w_q} \sum_{\nu=0}^{q-m} d_\nu \sum_{i=1}^{m+1} \frac{P_\nu(\lambda_i)}{\omega'_{m+1}(\lambda_i)P_q(\lambda_i)} P_\nu(\lambda)$$

and

$$a_\nu = \frac{1}{w_q} \sum_{i=1}^{m+1} \frac{P_\nu(\lambda_i)}{\omega'_{m+1}(\lambda_i)P_q(\lambda_i)}. \tag{4.4}$$

Since

$$\Delta(\lambda_1, \dots, \lambda_{m+1}) = \prod_{1 \le j < s \le m+1} (\lambda_s - \lambda_j)$$

is the Vandermonde determinant, we derive

$$\frac{1}{\omega'_{m+1}(\lambda_i)} = \frac{1}{\Delta(\lambda_1, \dots, \lambda_{m+1})} \frac{\Delta(\lambda_1, \dots, \lambda_{m+1})}{\prod_{j \ne i}(\lambda_i - \lambda_j)} = \frac{(-1)^{i-1}\Delta(\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_{m+1})}{\Delta(\lambda_1, \dots, \lambda_{m+1})}.$$

Taking into account (4.4), this gives

$$a_\nu = \frac{1}{w_q \Delta(\lambda_1, \dots, \lambda_{m+1})} \sum_{i=1}^{m+1} (-1)^{i-1} \Delta(\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_{m+1}) \frac{P_\nu(\lambda_i)}{P_q(\lambda_i)}$$
$$= \frac{1}{w_q \Delta(\lambda_1, \dots, \lambda_{m+1})} \begin{vmatrix} 1 & \dots & 1 \\ \lambda_1 & \dots & \lambda_{m+1} \\ \dots & \dots & \dots \\ \lambda_1^{m-1} & \dots & \lambda_{m+1}^{m-1} \\ \frac{P_\nu(\lambda_1)}{P_q(\lambda_1)} & \dots & \frac{P_\nu(\lambda_{m+1})}{P_q(\lambda_{m+1})} \end{vmatrix}. \tag{4.5}$$

We now rewrite formula (1.7) as follows:

$$P_{l-1}(\lambda) = (A_l\lambda + B_l)P_l(\lambda) - C_l P_{l+1}(\lambda),$$

where

$$A_l = \frac{\rho_l}{\gamma_{l-1}} > 0, \quad B_l = -\frac{\alpha_l}{\gamma_{l-1}}, \quad C_l = \frac{\beta_l}{\gamma_{l-1}} > 0.$$

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This implies the recurrence relation

$$\frac{P_{q-j}(\lambda_i)}{P_q(\lambda_i)} = (A_{q-j+1}\lambda_i + B_{q-j+1})\,\frac{P_{q-j+1}(\lambda_i)}{P_q(\lambda_i)} - C_{q-j+1}\,\frac{P_{q-j+2}(\lambda_i)}{P_q(\lambda_i)}, \quad j \in [1,q]_{\mathbb{Z}},$$

which yields

$$\frac{P_{q-j}(\lambda_i)}{P_q(\lambda_i)} = \sum_{s=0}^{j} \alpha_{s,j}\lambda_i^s, \quad \alpha_{j,j} = A_q A_{q-1}\cdots A_{q-j+1} > 0,$$

where the coefficients $\alpha_{s,j}$ are independent of λ_i. Substituting these relations into (4.5), we get

$$\begin{aligned} a_\nu &= \frac{1}{w_q \Delta(\lambda_1,\dots,\lambda_{m+1}) \prod_{j=1}^{m-1}\alpha_{j,j}} \begin{vmatrix} \frac{P_q(\lambda_1)}{P_q(\lambda_1)} & \cdots & \frac{P_q(\lambda_{m+1})}{P_q(\lambda_{m+1})} \\ \frac{P_{q-1}(\lambda_1)}{P_q(\lambda_1)} & \cdots & \frac{P_{q-1}(\lambda_{m+1})}{P_q(\lambda_{m+1})} \\ \dots & \dots & \dots \\ \frac{P_{q-m+1}(\lambda_1)}{P_q(\lambda_1)} & \cdots & \frac{P_{q-m+1}(\lambda_{m+1})}{P_q(\lambda_{m+1})} \\ \frac{P_\nu(\lambda_1)}{P_q(\lambda_1)} & \cdots & \frac{P_\nu(\lambda_{m+1})}{P_q(\lambda_{m+1})} \end{vmatrix} \\ &= \frac{(P_q(\lambda_1)\cdots P_q(\lambda_{m+1}))^{-1}}{w_q \Delta(\lambda_1,\dots,\lambda_{m+1}) \prod_{j=1}^{m-1}\alpha_{j,j}} \begin{vmatrix} P_q(\lambda_1) & \dots & P_q(\lambda_{m+1}) \\ P_{q-1}(\lambda_1) & \dots & P_{q-1}(\lambda_{m+1}) \\ \dots & \dots & \dots \\ P_{q-m+1}(\lambda_1) & \dots & P_{q-m+1}(\lambda_{m+1}) \\ P_\nu(\lambda_1) & \dots & P_\nu(\lambda_{m+1}) \end{vmatrix}. \end{aligned}$$

From the interlacing of zeros of the polynomials P_q and $\widetilde{P}_{q+1}$, we have $\operatorname{sign} P_q(\lambda_i) = (-1)^{i-1}$,

$$\operatorname{sign}\Delta(\lambda_1,\dots,\lambda_{m+1}) = \operatorname{sign}\left(P_q(\lambda_1)\cdots P_q(\lambda_{m+1})\right) = (-1)^{\frac{m(m+1)}{2}},$$

and

$$a_\nu = c(q,m) \begin{vmatrix} P_q(\lambda_1) & \dots & P_q(\lambda_{m+1}) \\ P_{q-1}(\lambda_1) & \dots & P_{q-1}(\lambda_{m+1}) \\ \dots & \dots & \dots \\ P_{q-m+1}(\lambda_1) & \dots & P_{q-m+1}(\lambda_{m+1}) \\ P_\nu(\lambda_1) & \dots & P_\nu(\lambda_{m+1}) \end{vmatrix} = c(q,m) D_{m+1}(\nu, q-m+1,\dots,q) \tag{4.6}$$

with $c(q,m) > 0$.

By Lemma 4.1, $D_{m+1}(\nu, q-m+1,\dots,q)$ preserves the sign for $\nu \in [0, q-m]_{\mathbb{Z}}$ and then, according to (4.6), all a_ν have the same sign. Further, since the polynomial (1.15) is positive for sufficiently large λ, it follows that $a_{q-m} > 0$ and thus all $a_\nu > 0$. In particular, we have $D_{m+1}(0, q-m+1,\dots,q) > 0$, which yields, by Lemma 4.1,

$$D_{m+1}(\nu,\nu_1,\dots,\nu_m) > 0 \quad \text{for all sequences} \quad 0 \le \nu < \nu_1 < \cdots < \nu_m \le q. \tag{4.7}$$

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To show (1.16), let us write the difference of coefficients a_ν in (1.15) using (4.6):

$$J_\nu = \frac{a_\nu}{P_\nu(b)} - \frac{a_{\nu+1}}{P_{\nu+1}(b)} = c(q,m)\begin{vmatrix} P_q(\lambda_1) & \dots & P_q(\lambda_{m+1}) \\ P_{q-1}(\lambda_1) & \dots & P_{q-1}(\lambda_{m+1}) \\ \dots & \dots & \dots \\ P_{q-m+1}(\lambda_1) & \dots & P_{q-m+1}(\lambda_{m+1}) \\ \frac{P_\nu(\lambda_1)}{P_\nu(b)} - \frac{P_{\nu+1}(\lambda_1)}{P_{\nu+1}(b)} & \dots & \frac{P_\nu(\lambda_{m+1})}{P_\nu(b)} - \frac{P_{\nu+1}(\lambda_{m+1})}{P_{\nu+1}(b)} \end{vmatrix}$$

$$= c(q,m)\Big(\prod_{j=q-m+1}^{q} P_j(b)\Big)\begin{vmatrix} \frac{P_q(\lambda_1)}{P_q(b)} & \dots & \frac{P_q(\lambda_{m+1})}{P_q(b)} \\ \frac{P_{q-1}(\lambda_1)}{P_{q-1}(b)} - \frac{P_q(\lambda_1)}{P_q(b)} & \dots & \frac{P_{q-1}(\lambda_{m+1})}{P_{q-1}(b)} - \frac{P_q(\lambda_{m+1})}{P_q(b)} \\ \dots & \dots & \dots \\ \frac{P_{q-m+1}(\lambda_1)}{P_{q-m+1}(b)} - \frac{P_{q-m+2}(\lambda_1)}{P_{q-m+2}(b)} & \dots & \frac{P_{q-m+1}(\lambda_{m+1})}{P_{q-m+1}(b)} - \frac{P_{q-m+2}(\lambda_{m+1})}{P_{q-m+2}(b)} \\ \frac{P_\nu(\lambda_1)}{P_\nu(b)} - \frac{P_{\nu+1}(\lambda_1)}{P_{\nu+1}(b)} & \dots & \frac{P_\nu(\lambda_{m+1})}{P_\nu(b)} - \frac{P_{\nu+1}(\lambda_{m+1})}{P_{\nu+1}(b)} \end{vmatrix}.$$

Applying formula (4.3) with $\lambda = b$ and $\widetilde{P}_{q+1}(b) = P_q(b)(\eta_b - \eta)$, and also the Christoffel-Darboux formula with $x = b$ and $y = \lambda_i$, we derive

$$J_\nu \frac{(\eta_b - \eta)P_\nu(b)P_{\nu+1}(b)P_q(b)\big(\prod_{j=q-m+2}^{q} P_j(b)\big)w_\nu \prod_{j=q-m+1}^{q} w_j}{c(q,m)\prod_{j=1}^{m+1}(b-\lambda_j)}$$

$$= \begin{vmatrix} \sum_{j=0}^{q} d_j P_j(b)P_q(\lambda_1) & \dots & \sum_{j=0}^{q} d_j P_j(b)P_j(\lambda_{m+1}) \\ \sum_{j=0}^{q-1} d_j P_j(b)P_j(\lambda_1) & \dots & \sum_{j=0}^{q-1} d_j P_j(b)P_j(\lambda_{m+1}) \\ \dots & \dots & \dots \\ \sum_{j=0}^{q-m+1} d_j P_j(b)P_j(\lambda_1) & \dots & \sum_{j=0}^{q-m+1} d_j P_j(b)P_j(\lambda_{m+1}) \\ \sum_{j=0}^{\nu} d_j P_j(b)P_j(\lambda_1) & \dots & \sum_{j=0}^{\nu} d_j P_j(b)P_j(\lambda_{m+1}) \end{vmatrix}$$

$$= \Big(\prod_{j=q-m+2}^{q} d_j P_j(b)\Big)\begin{vmatrix} P_q(\lambda_1) & \dots & P_q(\lambda_{m+1}) \\ P_{q-1}(\lambda_1) & \dots & P_{q-1}(\lambda_{m+1}) \\ \dots & \dots & \dots \\ P_{q-m+2}(\lambda_1) & \dots & P_{q-m+2}(\lambda_{m+1}) \\ \sum_{j=\nu+1}^{q-m+1} d_j P_j(b)P_j(\lambda_1) & \dots & \sum_{j=\nu+1}^{q-m+1} d_j P_j(b)P_j(\lambda_{m+1}) \\ \sum_{j=0}^{\nu} d_j P_j(b)P_j(\lambda_1) & \dots & \sum_{j=0}^{\nu} d_j P_j(b)P_j(\lambda_{m+1}) \end{vmatrix}.$$

Letting

$$C_1 = \frac{\eta_b - \eta}{b - \lambda_1}, \quad C_2 = \frac{P_\nu(b)P_{\nu+1}(b)P_q(b)w_\nu \prod_{j=q-m+1}^{q} w_j}{c(q,m)\prod_{j=2}^{m+1}(b-\lambda_j)\prod_{j=q-m+2}^{q} d_j},$$

for $\nu \in [0, q-m-1]_{\mathbb{Z}}$, we have

$$C_1 C_2 J_\nu = \sum_{\nu_2=\nu+1}^{q-m+1} \sum_{\nu_1=0}^{\nu} d_{\nu_1} d_{\nu_2} P_{\nu_1}(b) P_{\nu_2}(b) D_{m+1}(\nu_1, \nu_2, q-m+2, \dots, q). \tag{4.8}$$

Moreover, for $m = 1$, $D_2(\nu_1, \nu_2, q-m+2, \dots, q) = D_2(\nu_1, \nu_2)$.

We note that the right-hand side of (4.8) (due to (4.7)) as well as the factor C_2 are positive. We have already mentioned that, for $\eta \neq \eta_b$, the factor C_1 is also positive, therefore $J_\nu > 0$ and (1.16) holds for $m \geq 1$.

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(a): the case $m \geq 1$ and $\eta = \eta_b$. Since, according to [35, Ch. III, §3.3], the function $f(\lambda) = P_{q+1}(\lambda)/P_q(\lambda)$ on the interval $(\lambda_{1,q}, \infty)$ increases and $f'(b) > 0$, by the inverse function rule,

$$\lim_{\eta \to \eta_b} \frac{b - \lambda_1(\eta)}{\eta_b - \eta} = \frac{1}{f'(b)} > 0.$$

Thus, $C_1 > 0$ in (4.8) and (1.16) is also valid for $\eta = \eta_b$.

The proof of Theorem 1.14 is now complete. □

5. Proof of Theorem 1.16

As proven in [20], polynomials (1.20) and (1.22) are the only extremal polynomials, up to a positive constant, in a version of Yudin's problem, where the polynomial coefficients are not restricted to be non-negative. Thus, [20] provides the upper bounds for $B_n(\mu, m)$ in (1.19) and (1.21). To prove the lower bounds for $B_n(\mu, m)$, it remains to check that polynomials (1.20) and (1.22) belong to $\Pi_n \cap \Pi_+(\{U_l\})$.

(a) In this case $n = 2q - m + 1$. According to Theorem 1.14 with $\eta = 0$, we have

$$\frac{U_{q+1}(t)}{(t - t_1) \cdots (t - t_{m+1})} \in \Pi_+(\{U_l\}).$$

Applying the Krein property, we note that

$$\frac{U_{q+1}^2(t)}{(t - t_1) \cdots (t - t_{m+1})} \in \Pi_n \cap \Pi_+(\{U_l\}).$$

(b): the case $n = 2q - m + 2$ and the polynomials $U_l^{(1)}(t)$ satisfy the Krein condition. Since (see [25])

$$U_l^{(1)}(t) = \frac{\theta_l U_l(t) + (2 - \theta_l) U_{l+1}(t)}{1 + t}, \quad \theta_l \in (0, 2),$$

applying Theorem 1.14 for $U_{q+1}^{(1)}(t)$ with $\eta = 0$, we derive that the polynomial

$$\begin{aligned}
\frac{(1 + t)(U_{q+1}^{(1)}(t))^2}{(t - t_1^{(1)}) \cdots (t - t_{m+1}^{(1)})} &= \sum_{l=0}^{q-m} d_l^{(1)} b_l (1 + t) U_{q+1}^{(1)}(t) U_l^{(1)}(t) \\
&= \sum_{l=0}^{q-m} d_l^{(1)} b_l \sum_{k=q+1-l}^{q+1+l} c_{q+1,l,k} (1 + t) U_k^{(1)}(t) \\
&= \sum_{l=0}^{q-m} d_l^{(1)} b_l \sum_{k=q+1-l}^{q+1+l} c_{q+1,l,k} (\theta_k U_k(t) + (2 - \theta_k) U_{k+1}(t))
\end{aligned}$$

belongs to $\Pi_n \cap \Pi_+(\{U_l\})$.

(b): the case $n = 2q - m + 2$ and $\mu(t)$ be an odd function. Then the measure $d\mu(t)$ is even, $U_l(-1) = (-1)^l$, $\theta_l = 1$, and

$$U_l^{(1)}(t) = \frac{U_l(t) + U_{l+1}(t)}{1 + t}. \tag{5.1}$$

Hence, according to the Christoffel-Darboux formula (2.1),

$$U_l^{(1)}(t) = w_l^{-1} \sum_{\nu=0}^{l} d_\nu (-1)^{l+\nu} U_\nu(t). \tag{5.2}$$

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Applying (5.1) and (5.2), we get

$$\int_{-1}^{1}(U_l^{(1)}(t))^2\,(1+t)\,d\mu(t)=\int_{-1}^{1}w_l^{-1}\sum_{\nu=0}^{l}d_\nu(-1)^{l+\nu}U_\nu(t)(U_l(t)+U_{l+1}(t))\,d\mu(t)=w_l^{-1}.$$

This, Theorem 1.14, and (5.2) imply that in the expansion

$$\frac{U_{q+1}^{(1)}(t)}{(t-t_1^{(1)})\cdots(t-t_{m+1}^{(1)})}=\sum_{l=0}^{q-m}w_lb_lU_l^{(1)}(t)$$

the coefficients are monotone, that is, $b_0>b_1>\cdots>b_{q-m}>0$. Applying (5.2), we obtain

$$\sum_{l=0}^{q-m}w_lb_lU_l^{(1)}(t)=\sum_{l=0}^{q-m}b_l\sum_{\nu=0}^{l}d_\nu(-1)^{l+\nu}U_\nu(t)=\sum_{\nu=0}^{q-m}d_\nu U_\nu(t)\sum_{l=\nu}^{q-m}b_l(-1)^{l+\nu}=\sum_{\nu=0}^{q-m}d_\nu\delta_\nu U_\nu(t),$$

where

$$\delta_\nu=\sum_{l=\nu}^{q-m}b_l(-1)^{l+\nu}=b_\nu-b_{\nu+1}+\cdots+(-1)^{\nu+q-m}b_{q-m}\geq 0.$$

Thus,

$$\frac{(1+t)(U_{q+1}^{(1)}(t))^2}{(t-t_1^{(1)})\cdots(t-t_{m+1}^{(1)})}=\sum_{\nu=0}^{q-m}d_\nu\delta_\nu U_\nu(t)(U_{q+1}(t)+U_{q+2}(t))\in\Pi_n\cap\Pi_+(\{U_l\}),$$

completing the proof.

Appendix

A.1. Calculation of determinants for Chebyshev polynomials

In this subsection, we calculate the determinants $D(\nu)$ (see (4.1)) for the Jacobi polynomials [35, Ch. IV, §4.1] given by

$$U_l^{(\alpha,\beta)}(t)=\frac{P_l^{(-\frac12+\alpha,-\frac12+\beta)}(t)}{P_l^{(-\frac12+\alpha,-\frac12+\beta)}(1)},\quad \alpha,\beta\in\{0,1\}.$$

It will be convenient to rearrange ν_i as follows: $1\leq\nu_m<\cdots<\nu_1\leq q$.

(i) If $\alpha=\beta=0$, $U_{q+1}^{(0,0)}(t)=P_{q+1}^{(-\frac12,-\frac12)}(t)=\cos((q+1)\arccos t)$ is the Chebyshev polynomial of the first kind and $t_j=\cos\frac{\pi(2j-1)}{2q+2}$, $j=1,\dots,q+1$, are its zeros. If $x_i=\frac{\pi\nu_i}{2q+2}$, then

$$D(\nu)=D(\nu,\nu_m,\dots,\nu_1)=\det\left(\cos(2j-1)x_i\right)_{i,j=1}^{m+1}.$$

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Since $\cos(2j-1)x_i = \sum_{s=0}^{j-1} c_s \cos^{2s+1} x_i$, $c_{j-1} = 2^{2j-2}$, the determinant $D(\nu)$ reduces to the Vandermonde determinant

$$\begin{aligned} D(\nu) &= \det\left(2^{2j-2}(\cos x_i)^{2j-1}\right)_{i,j=1}^{m+1} = 2^{m(m+1)} \prod_{i=1}^{m+1}(\cos x_i) \det\left((\cos^2 x_i)^{j-1}\right)_{i,j=1}^{m+1} \\ &= 2^{m(m+1)} \prod_{i=1}^{m+1}(\cos x_i) \prod_{1\le k<l\le m+1} (\cos^2 x_l - \cos^2 x_k) \\ &= 2^{m(m+1)} \prod_{i=1}^{m+1}(\cos x_i) \prod_{1\le k<l\le m} (\cos^2 x_l - \cos^2 x_k) \prod_{1\le k\le m} \left(\cos^2\left(\frac{\pi\nu}{2q+2}\right) - \cos^2\left(\frac{\pi\nu_k}{2q+2}\right)\right). \end{aligned}$$

Taking into account that $0 < \cos x_1 < \cdots < \cos x_{m+1} < 1$, we obtain

$$\operatorname{sign} D(\nu) = \operatorname{sign} \prod_{k=1}^{m} \left(\cos\left(\frac{\pi\nu}{2q+2}\right) - \cos\left(\frac{\pi\nu_k}{2q+2}\right)\right) = \operatorname{sign} \prod_{k=1}^{m}(\nu_k - \nu).$$

(ii) The polynomial $U_{q+1}^{(0,1)}(t)$ is given by

$$U_{q+1}^{(0,1)}(t) = P_{q+1}^{(-\frac{1}{2},\frac{1}{2})}(t) = \frac{\sqrt{2}\cos\left((q+3/2)\arccos t\right)}{\sqrt{1+t}}$$

and $t_j = \cos\frac{\pi(2j-1)}{2q+3}$, $j = 1,\dots,q+1$, are its zeros. If $x_i = \frac{\pi(\nu_i+1/2)}{2q+3}$, then as above

$$\begin{aligned} D(\nu) &= \det\left(\frac{\sqrt{2}\cos(2j-1)x_i}{\sqrt{1+t_j}}\right)_{i,j=1}^{m+1} = \prod_{j=1}^{m+1}\left(\frac{2}{1+t_j}\right)^{1/2} \det\big(\cos(2j-1)x_i\big)_{i,j=1}^{m+1} \\ &= 2^{m(m+1)} \prod_{j=1}^{m+1}\left(\frac{2\cos^2 x_j}{1+t_j}\right)^{1/2} \prod_{0\le k<l\le m+1} (\cos^2 x_l - \cos^2 x_k). \end{aligned}$$

(iii) The polynomial

$$U_{q+1}^{(1,1)}(t) = P_{q+1}^{(\frac{1}{2},\frac{1}{2})}(t) = \frac{\sin((q+2)\arccos t)}{(q+2)\sqrt{1-t^2}},$$

is the Chebyshev polynomial of the second kind with the zeros $t_j = \cos\frac{\pi j}{q+2}$, $j = 1,\dots,q+1$. If $x_i = \frac{\pi(\nu_i+1)}{q+2}$, then

$$D(\nu) = \det\left(\frac{\sin jx_i}{(\nu_i+1)\sin\frac{\pi j}{q+2}}\right)_{i,j=1}^{m+1} = \frac{\det(\sin jx_i)_{i,j=1}^{m+1}}{\prod_{j=1}^{m+1}(\nu_j+1)\sin\frac{\pi j}{q+2}}.$$

Since $\sin jx_i = \sin x_i \sum_{s=0}^{j-1} c_s \cos^s x_i$ with $c_{j-1} = 2^{j-1}$, we arrive at

$$\begin{aligned} D(\nu) &= 2^{\frac{m(m+1)}{2}} \prod_{j=1}^{m+1} \frac{\sin\frac{\pi(\nu_j+1)}{q+2}}{(\nu_j+1)\sin\frac{\pi j}{q+2}} \det\left(\cos^{j-1} x_i\right)_{i,j=1}^{m+1} \\ &= 2^{\frac{m(m+1)}{2}} \prod_{j=1}^{m+1} \frac{\sin\frac{\pi(\nu_j+1)}{q+2}}{(\nu_j+1)\sin\frac{\pi j}{q+2}} \prod_{0\le k<l\le m+1} (\cos x_l - \cos x_k). \end{aligned}$$

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(iv) We have

$$U_{q+1}^{(1,0)}(t) = P_{q+1}^{(\frac{1}{2},-\frac{1}{2})} = \frac{\sqrt{2}\sin((q+3/2)\arccos t)}{(2q+3)\sqrt{1-t}},$$

and $t_j = \cos\frac{\pi j}{q+3/2}$, $j = 1, \dots, q+1$, are the zeros of $U_{q+1}^{(1,0)}$. As in the previous case, for $x_i = \frac{\pi(\nu_i+1/2)}{q+3/2}$,

$$D(\nu) = \det\Big(\frac{\sqrt{2}\sin jx_i}{(2\nu_i+1)\sqrt{1-t_j}}\Big)_{i,j=1}^{m+1} = \frac{\det(\sin jx_i)_{i,j=1}^{m+1}}{\prod_{j=1}^{m+1}(2\nu_j+1)\sqrt{(1-t_j)/2}}$$
$$= 2^{\frac{m(m+1)}{2}}\prod_{j=1}^{m+1}\frac{\sin\frac{\pi(\nu_j+1)}{q+2}}{(2\nu_j+1)\sqrt{(1-t_j)/2}}\prod_{0\le k<l\le m+1}(\cos x_l - \cos x_k).$$

A.2. Corollaries of Theorem 1.14 for trigonometric polynomials

Finally, writing the Chebyshev polynomials $U_{q+1}^{(0,0)}$ and $U_{q+1}^{(1,1)}$ in the trigonometric form, the monotonicity property of coefficients in expansion (1.15) given in Theorem 1.14 implies the following results.

Corollary A.1. *Suppose* $0 \le m \le q$, *then*

$$\frac{\cos(q+1)x}{\big(\cos x - \cos\frac{\pi}{2q+2}\big)\cdots\big(\cos x - \cos\frac{\pi(2m+1)}{2q+2}\big)} = \frac{a_{0,m}}{2} + \sum_{\nu=1}^{q-m} a_{\nu,m}\cos\nu x, \tag{A.1}$$

then

$$a_{0,m} > a_{1,m} > \cdots > a_{q-m,m} > 0.$$

Corollary A.2. *Suppose* $1 \le m \le q$, *then*

$$\frac{\sin(q+1)x}{\big(\cos x - \cos\frac{\pi}{q+1}\big)\cdots\big(\cos x - \cos\frac{\pi m}{q+1}\big)} = \sum_{\nu=1}^{q-m+1}\nu b_{\nu,m}\sin\nu x,$$

then

$$b_{1,m} > \cdots > b_{q-m+1,m} > 0.$$

It is worth mentioning that polynomials similar to (A.1) were considered by Yudin in [40].

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