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q-SERIES IN MARKOV CHAINS WITH BINOMIAL TRANSITIONS

STUDYING A QUEUE WITH SYNCHRONIZATION

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We consider a single-server Markovian queue with synchronized services and setup times. The customers arrive according to a Poisson process and are served simultaneously. The service times are independent and exponentially distributed. At a service completion epoch, every customer remains satisfied with probability p (independently of the others) and departs from the system; otherwise, he stays for a new service. Moreover, the server takes multiple vacations whenever the system is empty.

Some of the transition rates of the underlying two-dimensional Markov chain involve binomial coefficients dependent on the number of customers. Indeed, at each service completion epoch, the number of customers n is reduced according to a binomial (n, p) distribution. We show that the model can be efficiently studied using the framework of q-hypergeometric series and we carry out an extensive analysis including the stationary, the busy period, and the sojourn time distributions. Exact formulas and numerical results show the effect of the level of synchronization to the performance of such systems.

1. INTRODUCTION

In the queuing literature, there exists a significant number of articles dealing with two-dimensional Markovian models. In this kind of queuing models, the state of the system is represented by a vector (n, i), where n records the number of customers and i gives information about some special feature of the system (e.g., state of the server(s), number of priority customers, phase of the arrival or the service process). These models have received considerable attention because, on the one hand, they

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can easily represent various queuing characteristics and, on the other hand, many of them are analytically tractable, thus providing effective computations of performance measures.

However, most of the analytically tractable models have two limitations: First, only one of the two variables is unlimited (usually the number of customers) while the other is assumed to take only a finite number of values. Second, the underlying Markov chain exhibits—at least to a certain degree—spatial homogeneity. A variety of methods has been developed for such models. Matrix analytic methods (see, e.g., Neuts [27,28], Latouche and Ramaswami [25], and Bini, Latouche, and Meini [9]) and generating function techniques (see, e.g., Mitrani and Chakka [26], Gail, Hantler, and Taylor [13], and Grassmann [15]) have been proved very efficient.

In systems that do not satisfy the above constraints, the analysis is much more complicated. When only the first constraint is violated (i.e., both variables are unlimited but the Markov chain is space-homogeneous), the matrix analytic methods still apply, although there exist serious difficulties (see, e.g., Kroese, Scheinhardt, and Taylor [24]). An alternative is to use analytic tools. One widely used approach is the reduction to a Riemann–Hilbert boundary value problem (see, e.g., Cohen and Boxma [11]). The compensation method (Adan [2]) has also been used for the efficient solution of a number of models.

The second constraint (i.e., the spatial homogeneity) is usually violated due to features such as retrials, reneging, or infinite number of servers. Indeed, in these situations, there are transitions out of a state (n, i) to a state $(n-1, i')$ with rates proportional to n. This is also the rule in stochastic models in Mathematical Biology, since every individual is associated with births and deaths. There are only few works trying to extend matrix analytic methods within this framework. In most cases, the authors use truncation or generalized truncation ideas to study the systems (see, e.g., Artalejo and Pozo [7] and the references therein) or they apply generating function methods. However, we have to note that in this case the partial generating functions of the number of individuals in system satisfy a system of linear differential equations in contrast with the state homogeneous case, in which they satisfy linear algebraic equations. Such systems are usually intractable or can be solved in terms of hypergeometric series (see, e.g., Altman and Yechiali [4], Artalejo and Gomez-Corral [6], Baykal-Gursoy and Xiao [8], Krishnamoorthy, Deepek, and Joshua [23], and Keilson and Servi [17]).

The primary aim of the present article is the study of a synchronization characteristic that also leads to spatially inhomogeneous Markov chains. Indeed, suppose that all of the n present customers of a given system are served concurrently, according to an exponential distribution with parameter μ. If at the service completion epoch every customer is satisfied and departs with probability p or repeats his service with probability q, independently of the others, then there are binomial transition rates of the form $\mu\binom{n}{n'}p^{n-n'}q^{n'}$, from a state (n, i) to states (n', i), for $0 \leq n' \leq n$. Similar Markov chains occur in Mathematical Biology in the study of population processes subject to binomial catastrophes (see, e.g., Brockwell, Gani, and Resnick [10], Economou [12],

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and Artalejo, Economou, and Lopez–Herrero [5]). Moreover, Neuts [29] studied a one-dimensional discrete-time model with similar dynamics.

The model of the article is a queuing system with synchronized services and setup times (also known as multiple server's vacations). The literature on queuing systems with vacations is vast (see, e.g., Takagi [30] and Tian and Zhang [31] and the references therein). However, to the best of our knowledge, there are no articles dealing with vacation queuing systems with some kind of synchronization. A potential application of this type of models is the representation of distance-learning service systems (e.g., webseminars), where all customers are served concurrently and then decide independently whether to repeat their service or not. The purpose of this article is threefold:

- To introduce and study a simple model combining synchronized services and vacations
- To illustrate the use of the framework of q-hypergeometric series in computations regarding systems with this kind of binomial transitions arising from synchronization
- To point out the effect of synchronization in the performance of systems with the same mean parameters (arrival and departure rates).

The rest of the article is organized as follows. In Section 2 we describe the model and introduce the appropriate notation. In Section 3 we carry out the equilibrium analysis of the system state and we derive several exact formulas and iterative algorithmic schemes. Some limiting regimes are discussed in more detail. The busy period and sojourn time distributions are studied in Section 4. Finally, in Section 5 we provide several numerical studies and we discuss the effect of the level of synchronization to the system performance.

2. MODEL DESCRIPTION AND NOTATION

We consider a queuing system in which customers arrive according to a Poisson process at rate λ. The service is provided by a single server, who serves simultaneously all present customers. The successive service times of the server are assumed to be exponential random variables with rate μ. At a service completion epoch, each customer is satisfied and departs with probability p or repeats his service with probability $q = 1 - p$, independently of the others. Whenever the system becomes empty, the server is deactivated immediately. As soon as an arrival enters an empty system, the server begins a setup time to reactivate. The setup times are exponentially distributed at rate ϑ. An alternative interpretation is that the server takes exponentially distributed multiple vacations with parameter ϑ as long as the system remains empty.

The system is represented by a continuous-time Markov chain $\{(N(t), I(t)) : t \geq 0\}$, where $N(t)$ is the number of customers in the system at time t and $I(t)$ denotes

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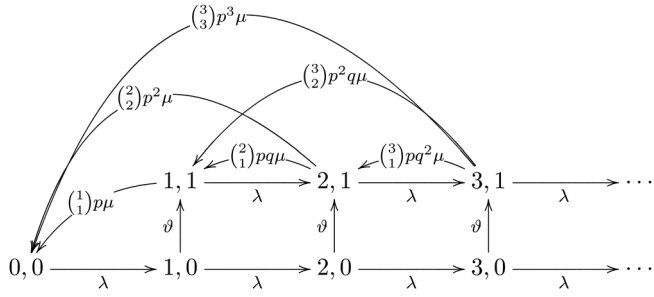


FIGURE 1. Transition rate diagram of $\{(N(t), I(t))\}$.

the state of the server at time t ($0 =$ off and $1 =$ on), $t \geq 0$. The corresponding transition diagram is given in Figure 1.

Let $\{\pi(n,i) : \ i = 0,1 \text{ and } n \geq i\}$ denote the equilibrium distribution of $\{(N(t), I(t))\}$. We also define the partial probability generating functions $\Pi_0(z)$ and $\Pi_1(z)$ by

$$\Pi_0(z) = \sum_{n=0}^{\infty} \pi(n,0) z^n \quad \text{and} \quad \Pi_1(z) = \sum_{n=1}^{\infty} \pi(n,1) z^n.$$

In Section 3 we will determine $\Pi_0(z)$ and $\Pi_1(z)$ in terms of q-hypergeometric series (also known as basic hypergeometric series). Moreover, in Section 4 we will see that the Laplace–Stieltjes transform of the busy period of this system is also expressed in terms of q-hypergeometric series.

There exists a rich theory for the class of q-hypergeometric series and their q-calculus, which enables fast calculations and simplifications. In the queuing theory literature there exist only a few articles in which this theory has been applied (see, e.g., Ismail [16], Kemp [18–20], and Kemp and Newton [21]). For this reason and for the sake of self-completeness, we will briefly summarize the basic definitions and results of this theory. The interested reader can find more details on the definitions and the results below (with proofs and extensions) in Gasper and Rahman [14, Chaps. 1–3, Appendexes I–III].

The q-hypergeometric series are series of the form $\sum_{n=0}^{\infty} c_n$, where $c_0 = 1$ and c_{n+1}/c_n is a rational function of q^n for a deformation parameter q, which is usually taken to satisfy $|q| < 1$. They were initially introduced by Heine, who developed their basic theory, following Gauss' fundamental work on hypergeometric series. Observing that the ratio c_{n+1}/c_n, being rational in q^n, can be written in the form

$$\frac{c_{n+1}}{c_n} = \frac{(1-a_1q^n)(1-a_2q^n)\cdots(1-a_rq^n)}{(1-q^{n+1})(1-b_1q^n)\cdots(1-b_sq^n)}(-q^n)^{1+s-r}z, \tag{2.1}$$

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we have that every such series assumes the form

$$
{}_r\phi_s\left(\begin{matrix}a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_s\end{matrix}; q, z\right) \equiv {}_r\phi_s(a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_s; q, z)
$$

$$
= \sum_{n=0}^{\infty} \frac{(a_1; q)_n (a_2; q)_n \cdots (a_r; q)_n}{(q; q)_n (b_1; q)_n \cdots (b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n, \tag{2.2}
$$

where $(a; q)_0 = 1$ and $(a; q)_n = (1 - a)(1 - aq)(1 - aq^2) \cdots (1 - aq^{n-1}),\ n \geq 1$. In the definition of a q-series through (2.2), it is assumed that $b_i \neq q^{-m}$ for $m = 0, 1, \ldots$ and $i = 1, 2, \ldots, s$. This is the standard ${}_r\phi_s$ notation for q-series that is currently used instead of the old one without the factor $\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}$. If $0 < |q| < 1$, the ${}_r\phi_s$ series converge absolutely for all z when $r \leq s$ and for $|z| < 1$ when $r = s + 1$. We use the abbreviation $(a_1, a_2, \ldots, a_r; q)_n$ to denote the product $(a_1; q)_n (a_2; q)_n \cdots (a_r; q)_n$. The quantity $(a; q)_n$ is referred to as the q-shifted factorial. We also define $(a; q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k)$ and use the abbreviation $(a_1, a_2, \ldots, a_r; q)_\infty$ to denote the product $(a_1; q)_\infty (a_2; q)_\infty \cdots (a_r; q)_\infty$. A q-calculus has been developed that parallels the theory of hypergeometric functions. The most important summation formula for the q-hypergeometric series is given by the q-binomial theorem (see, e.g., Gasper and Rahman [14, Sect. 1.3])

$$
{}_1\phi_0(a; -; q, z) = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} z^n = \frac{(az; q)_\infty}{(z; q)_\infty}, \qquad |z| < 1. \tag{2.3}
$$

Of particular importance are the two q-analogues of the exponential function e^z. They are the functions $e_q(z) = 1/(z; q)_\infty$ and $E_q(z) = (-z; q)_\infty$. The q-binomial theorem enables one to express the q-exponential functions in the form of a q-series. If we set $a = 0$ in (2.3), we get

$$
e_q(z) = {}_1\phi_0(0; -; q, z) = \sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n} = \frac{1}{(z; q)_\infty}, \qquad |z| < 1. \tag{2.4}
$$

If we replace z by $-z/a$ in (2.3) and let $a \to \infty$, we get

$$
E_q(z) = {}_0\phi_0(-; -; q, -z) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q; q)_n} z^n = (-z; q)_\infty, \quad |z| < 1.
$$

The definite q-integral on an interval $[0, a]$ is defined by

$$
\int_0^a f(t)\, d_q t = a(1 - q) \sum_{n=0}^{\infty} f(aq^n) q^n.
$$

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As $q \to 1^-$, the q-analogues reduce to their standard counterparts. In particular, we have the relationships

$$\lim_{q\to 1^-} e_q(z(1-q)) = \lim_{q\to 1^-} E_q(z(1-q)) = e^z, \tag{2.5}$$

$$\lim_{q\to 1^-} \int_0^a f(t)\, d_q t = \int_0^a f(t)\, dt. \tag{2.6}$$

As $q \to 0^+$, we can easily see that

$$\lim_{q\to 0^+} e_q(z) = \frac{1}{1-z}, \tag{2.7}$$

$$\lim_{q\to 0^+} E_q(z) = 1 + z. \tag{2.8}$$

Of particular importance are the transformation formulas of q-hypergeometric series (see Gasper and Rahman [14, Appendix III]). In Section 3 we will use the transformation formulas

$$ {}_{r+1}\phi_r\left(\begin{matrix} a_1, \cdots, a_{r+1} \\ b_1, \cdots, b_r \end{matrix}; q, q^z\right) = \frac{(a_1, \ldots, a_{r+1}; q)_\infty}{(1-q)(q, b_1, \ldots, b_r; q)_\infty} \times \int_0^1 t^{z-1} \frac{(qt, b_1 t, \ldots, b_r t; q)_\infty}{(a_1 t, \ldots, a_{r+1} t; q)_\infty}\, d_q t \tag{2.9}$$

(see Gasper and Rahman [14, p. 26, 1.4(iii)]) and

$$ {}_3\phi_2\left(\begin{matrix} a, q, 0 \\ aq, bq \end{matrix}; q; q\right) = (1-a)(1-b) \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{a^k b^{n-k-1} (q;q)_k}{(q;q)_n}; \tag{2.10}$$

for $a = 0$, we obtain

$$ {}_2\phi_1(q, 0; bq; q, q) = \frac{1-b}{b}(e_q(b) - 1). \tag{2.11}$$

We will also use the limiting relation

$$\lim_{q\to 0^+} {}_{r+1}\phi_r\left(\begin{matrix} a_1, \cdots, a_{r+1} \\ b_1, \cdots, b_r \end{matrix}; q, q\right) = 1. \tag{2.12}$$

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3. THE EQUILIBRIUM STATE DISTRIBUTION

The balance equations of the model are given as follows:

$$\lambda\pi(0,0) = \mu\sum_{n=1}^{\infty} p^n\pi(n,1) = \mu\Pi_1(p),$$

$$(\lambda+\vartheta)\pi(n,0) = \lambda\pi(n-1,0), \qquad n \geq 1, \tag{3.1}$$

$$(\lambda+\mu)\pi(1,1) = \vartheta\pi(1,0) + \mu\sum_{j=1}^{\infty}\binom{j}{1}p^{j-1}q\pi(j,1), \tag{3.2}$$

$$(\lambda+\mu)\pi(n,1) = \vartheta\pi(n,0) + \lambda\pi(n-1,1) + \mu\sum_{j=n}^{\infty}\binom{j}{n}p^{j-n}q^n\pi(j,1), \qquad n \geq 2. \tag{3.3}$$

These equations can be solved efficiently by employing generating function methods and we obtain the following.

THEOREM 3.1: *The equilibrium state probability of an empty system* $\pi(0,0)$ *is given by*

$$\pi(0,0) = \left[\frac{\lambda+\vartheta}{\vartheta} + \frac{\lambda}{\lambda+\mu}E_q\left(\frac{\lambda}{\mu}\right)\,{}_3\phi_2\left(-\frac{\lambda}{\vartheta}, q, 0; -\frac{\lambda}{\vartheta}q, -\frac{\lambda}{\mu}q; q, q\right)\right]^{-1}. \tag{3.4}$$

The partial probability generating functions $\Pi_0(z)$ *and* $\Pi_1(z)$ *are given by*

$$\Pi_0(z) = \frac{\lambda+\vartheta}{\lambda+\vartheta-\lambda z}\pi(0,0), \tag{3.5}$$

$$\Pi_1(z) = \left[1 - \frac{\lambda+\vartheta}{\vartheta}\pi(0,0)\right]e_q\left(-\frac{\lambda}{\mu}(1-z)\right) - \frac{\lambda(\lambda+\vartheta)(1-z)}{(\lambda+\vartheta-\lambda z)(\lambda+\mu-\lambda z)} \times \pi(0,0)\,{}_3\phi_2\left(\begin{matrix} -\frac{\lambda}{\vartheta}(1-z), q, 0 \\ -\frac{\lambda}{\vartheta}q(1-z), -\frac{\lambda}{\mu}q(1-z)\end{matrix}; q, q\right). \tag{3.6}$$

The convergence of the series is absolute in $\{z \in \mathbb{C} : |z| \leq 1\}$ *and uniform in every compact subset of* $\{z \in \mathbb{C} : |z| < 1\}$.

PROOF: Using (3.1) we have immediately that

$$\pi(n,0) = \left(\frac{\lambda}{\lambda+\vartheta}\right)^n\pi(0,0), \qquad n \geq 0. \tag{3.7}$$

We have now that (3.5) follows easily from (3.7).

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Multiplying both sides of (3.2) and (3.3) by z and z^n, respectively, and summing for all $n = 1, 2, \ldots$ taking into account (3.7), we obtain

$$\sum_{n=1}^{\infty}(\lambda+\mu)\pi(n,1)z^n = \vartheta\sum_{n=1}^{\infty}\left(\frac{\lambda}{\lambda+\vartheta}\right)^n\pi(0,0)z^n + \sum_{n=2}^{\infty}\lambda\pi(n-1,1)z^n + \sum_{n=1}^{\infty}\mu\sum_{j=n}^{\infty}\binom{j}{n}p^{j-n}q^n\pi(j,1)z^n. \tag{3.8}$$

After some algebraic manipulation in (3.8), we obtain

$$(\lambda+\mu-\lambda z)\Pi_1(z) = \mu\Pi_1(1-q+qz) + \frac{\lambda(\lambda+\vartheta)(z-1)}{\lambda+\vartheta-\lambda z}\pi(0,0). \tag{3.9}$$

Defining

$$A(z) = \frac{\lambda(\lambda+\vartheta)(z-1)}{\mu(\lambda+\vartheta-\lambda z)},$$

we arrive at

$$\Pi_1(z) = \frac{\mu}{\lambda+\mu-\lambda z}\Pi_1(1-q+qz) + \frac{\mu}{\lambda+\mu-\lambda z}A(z)\pi(0,0). \tag{3.10}$$

From the normalization equation $\Pi_0(1) + \Pi_1(1) = 1$ and (3.5), we obtain that

$$\Pi_1(1) = 1 - \frac{\lambda+\vartheta}{\vartheta}\pi(0,0). \tag{3.11}$$

By iterating (3.10) we can prove inductively that

$$\Pi_1(z) = \Pi_1(1-q^{n+1}+q^{n+1}z)\prod_{k=0}^{n}\frac{\mu}{\mu+\lambda(1-z)q^k} + \sum_{j=0}^{n}A(1-q^j+q^jz)\prod_{k=0}^{j}\frac{\mu}{\mu+\lambda(1-z)q^k}\pi(0,0), \qquad n \geq 0. \tag{3.12}$$

Taking the limit as $n \to \infty$ in (3.12) and using (3.11) results in

$$\Pi_1(z) = \left(1-\frac{\lambda+\vartheta}{\vartheta}\pi(0,0)\right)\prod_{k=0}^{\infty}\frac{\mu}{\mu+\lambda(1-z)q^k} + \sum_{j=0}^{\infty}A(1-q^j+q^jz)\prod_{k=0}^{j}\frac{\mu}{\mu+\lambda(1-z)q^k}\pi(0,0), \tag{3.13}$$

provided that the series and the infinite product converge. To prove the convergence of the series and of the infinite product, we will express (3.13) in terms of q-hypergeometric series.

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Let

$$a_j(z) = A(1 - q^j + q^j z) \prod_{k=0}^{j} \frac{\mu}{\mu + \lambda(1-z)q^k}, \qquad j \geq 0.$$

It is easy to see that the $a_j(z)$ are well defined for $|z| \leq 1$. We have that the ratio of two successive terms assumes the form

$$\begin{aligned} \frac{a_{j+1}(z)}{a_j(z)} &= \frac{\vartheta + \lambda q^j(1-z)}{\vartheta + \lambda q^{j+1}(1-z)} \frac{\mu}{\mu + \lambda q^{j+1}(1-z)} q \\ &= \frac{1 + (\lambda/\vartheta)q^j(1-z)}{1 + (\lambda/\vartheta)q^{j+1}(1-z)} \frac{1}{1 + (\lambda/\mu)q^{j+1}(1-z)} q \\ &= \frac{(1-qq^j)(1-(-(\lambda/\vartheta)(1-z))q^j)(1-0q^j)}{(1-q^{j+1})(1-(-(\lambda/\vartheta)q(1-z))q^j)(1-(-(\lambda/\mu)q(1-z))q^j)} q, \end{aligned} \tag{3.14}$$

and, hence, is a rational function of q^j, of the form stated in (2.1) for $r = s + 1 = 3$. Therefore, the series converges for all $|z| \leq 1$.

We define $b_k(z) = \frac{\lambda}{\mu}(1-z)q^k$, and it is clear that in $\{z \in \mathbb{C} : |z| \leq 1\}$, we have $1 + b_k(z) \neq 0$ for $k = 0, 1, 2, \ldots$. So we obtain that

$$\prod_{k=0}^{\infty} \frac{1}{(1 + b_k(z))} = \prod_{k=0}^{\infty} \frac{\mu}{\mu + \lambda(1-z)q^k} = \frac{1}{(-(\lambda/\mu)(1-z); q)_\infty} = e_q\left(-\frac{\lambda}{\mu}(1-z)\right) \tag{3.15}$$

is a nonvanishing analytic function. We conclude that (3.13) assumes the form (3.6).

The uniform and absolute convergence of the series in (3.13) can be also proved using the Weierstrass M-test (see, e.g., Ahlfors [3, Chap. 2, §2.2.3]). It remains to prove (3.4). To this end we set $z = 0$ in (3.6). Then we obtain

$$\begin{aligned} 0 = \Pi_1(0) = &\left[1 - \frac{\lambda + \vartheta}{\vartheta}\pi(0,0)\right] e_q\left(-\frac{\lambda}{\mu}\right) \\ &- \frac{\lambda}{\lambda + \mu}\pi(0,0)\, {}_3\phi_2\left(-\frac{\lambda}{\vartheta}, q, 0; -\frac{\lambda}{\vartheta}q, -\frac{\lambda}{\mu}q; q, q\right). \end{aligned} \tag{3.16}$$

We multiply (3.16) by $E_q(\lambda/\mu)$, take into account that $E_q(\lambda/\mu)e_q(-\lambda/\mu) = 1$, solve for $\pi(0,0)$, and we obtain (3.4). ■

We can now proceed to the calculation of the moments for the equilibrium distribution of the number of customers in the system. We have the following.

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THEOREM 3.2: *The factorial moments* $m_{(n)} = E[N(N-1)(N-2)\cdots(N-n+1)]$ *of the equilibrium number of customers in the system are given by*

$$m_{(n)} = \frac{(\lambda+\vartheta)\lambda^n n!}{\vartheta^{n+1}}\pi(0,0) + \sum_{k=1}^{n} \frac{(\lambda+\vartheta)\lambda^n n!}{\vartheta^k \mu^{n-k+1}} \frac{(q;q)_{k-1}}{(q;q)_n}\pi(0,0) + \frac{\lambda^n n!}{\mu^n (q;q)_n}\left[1 - \frac{\lambda+\vartheta}{\vartheta}\pi(0,0)\right], \qquad n \geq 1. \tag{3.17}$$

In particular,

$$E[N] = \frac{(\lambda+\vartheta)\lambda}{\vartheta^2}\pi(0,0) + \frac{\lambda}{\mu(1-q)}, \tag{3.18}$$

$$\mathrm{Var}[N] = \frac{2(\lambda+\vartheta)\lambda^2}{\vartheta^3}\pi(0,0) + \frac{2(\lambda+\vartheta)\lambda^2}{\vartheta^2\mu(1-q^2)}\pi(0,0) + \frac{2\lambda^2}{\mu^2(1-q)(1-q^2)} + \frac{(\lambda+\vartheta)\lambda}{\vartheta^2}\pi(0,0) + \frac{\lambda}{\mu(1-q)} - \left(\frac{(\lambda+\vartheta)\lambda}{\vartheta^2}\pi(0,0) + \frac{\lambda}{\mu(1-q)}\right)^2.$$

PROOF: The factorial moment generating function $P(z) = \sum_{n=0}^{\infty} m_{(n)}(z^n/n!)$ is given by

$$P(z) = \Pi_0(1+z) + \Pi_1(1+z). \tag{3.19}$$

We observe that

$$ {}_3\phi_2\left(\begin{matrix}\frac{\lambda}{\vartheta}z, q, 0 \\ \frac{\lambda}{\vartheta}qz, \frac{\lambda}{\mu}qz\end{matrix}; q, q\right) \overset{(2.10)}{=} \left(1-\frac{\lambda}{\vartheta}z\right)\left(1-\frac{\lambda}{\mu}z\right)\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\frac{(\lambda z)^{n-1}(q;q)_k}{\vartheta^k\mu^{n-k-1}(q;q)_n} = \frac{(\vartheta-\lambda z)(\mu-\lambda z)}{\vartheta\mu}\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{(\lambda z)^{n-1}(q;q)_{k-1}}{\vartheta^{k-1}\mu^{n-k}(q;q)_n}. \tag{3.20}$$

Moreover,

$$e_q\left(\frac{\lambda}{\mu}z\right) = \sum_{n=0}^{\infty}\frac{((\lambda/\mu)z)^n}{(q;q)_n}. \tag{3.21}$$

Plugging (3.20) and (3.21) into (3.6) and using (3.5), (3.19) yields, after some simplifications, (3.17). ■

The exact inversion of $\Pi_1(z)$ given by (3.6), although possible using (2.10), is practically useless, since the corresponding formulas for the stationary probabilities involve infinite sums. Therefore, we cannot obtain the equilibrium state distribution of the system in closed form, but we can exploit (3.6) using some numerical inversion algorithm (see, e.g., Abate, Choudhury, and Whitt [1]) to obtain the equilibrium

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distribution for given values of the parameters up to any desired degree of accuracy. Nevertheless, we can obtain closed–form expressions for some limiting regimes. We present them in Theorems 3.3, 3.4, and 3.5.

To emphasize the dependence on the parameters of the model in the rest of this section, we will denote $\pi(n,i)$, $\Pi_0(z)$, and $\Pi_1(z)$ by $\pi(n,i;\lambda,\mu,p,\vartheta)$, $\Pi_0(z;\lambda,\mu,p,\vartheta)$, and $\Pi_1(z;\lambda,\mu,p,\vartheta)$, respectively. Note that μp can be thought of as the effective service rate per customer. Indeed, the overall service time of a customer is a geometric sum of exponentially distributed random variables with rate μ and so we can easily see that it is also exponentially distributed with parameter μp. Under this perspective, if we have two models with the same parameters λ and ϑ that differ only in μ and p, but with $\mu p = \mu^*$ fixed, we can think that the models have identical arrival rates, effective service rates per customer and setup rates λ, μ^* and ϑ and differ only in the "level of synchronization" p. Indeed, the case $p \to 0^+$ corresponds to no synchronization, since the customers depart almost singly at the service completion epochs. On the contrary, the case $p \to 1^-$ corresponds to full synchronization, since almost all present customers depart simultaneously.

We are interested in studying the equilibrium behavior of the system for the case where λ, μ^*, and ϑ are kept fixed in the two limiting cases $p \to 0^+$ ($q \to 1^-$) and $p \to 1^-$ ($q \to 0^+$). The corresponding results are presented in Theorems 3.3 and 3.4.

THEOREM 3.3: *For a system with arrival rate λ, effective service rate per customer μ^*, and setup rate ϑ, the equilibrium state distribution $\pi^{(1)}(n,i) = \lim_{q\to 1^-} \pi(n,i;\lambda,\mu^*/(1-q),1-q,\vartheta)$ in the limiting case of no synchronization is given by*

$$\pi^{(1)}(0,0) = \left[\frac{\lambda+\vartheta}{\vartheta} + \frac{(\lambda+\vartheta)\lambda}{\mu^*}\int_0^1 \frac{\exp((\lambda/\mu^*)s)}{\vartheta+\lambda s}\,ds\right]^{-1}, \tag{3.22}$$

$$\pi^{(1)}(n,0) = \left(\frac{\lambda}{\lambda+\vartheta}\right)^n \pi^{(1)}(0,0), \qquad n \geq 1, \tag{3.23}$$

$$\pi^{(1)}(n,1) = \left(\frac{\lambda}{\lambda+\vartheta}\right)^n \frac{1}{n!}\sum_{k=1}^{n}\left(\frac{\lambda+\vartheta}{\mu^*}\right)^k (n-k)!\,\pi^{(1)}(0,0), \qquad n \geq 1. \tag{3.24}$$

PROOF: We use (2.9) and we obtain that

$$\begin{aligned}(1-q)\,{}_3\phi_2\left(-\frac{\lambda}{\vartheta}(1-z),q,0;-\frac{\lambda}{\vartheta}q(1-z),-\frac{\lambda}{\mu}q(1-z);q,q\right)\\ = \frac{(-(\lambda/\vartheta)(1-z),q,0;q)_\infty}{(q,-(\lambda/\vartheta)q(1-z),-(\lambda/\mu)q(1-z);q)_\infty}\\ \times\int_0^1 \frac{(qt,-(\lambda/\vartheta)q(1-z)t,-(\lambda/\mu)q(1-z)t;q)_\infty}{(-(\lambda/\vartheta)(1-z)t,qt,0;q)_\infty}\,d_qt.\end{aligned} \tag{3.25}$$

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By simplifying several terms and taking into account the relations $(0;q)_\infty = 1$ and $(a;q)_\infty = (1-a)(aq;q)_\infty$, we have that (3.25) assumes the form

$$\begin{aligned}(1-q)\,{}_3\phi_2\left(-\frac{\lambda}{\vartheta}(1-z),q,0;-\frac{\lambda}{\vartheta}q(1-z),-\frac{\lambda}{\mu}q(1-z);q,q\right)\\ = \frac{1+(\lambda/\vartheta)(1-z)}{(-(\lambda/\mu)q(1-z);q)_\infty}\int_0^1 \frac{(-(\lambda/\mu)q(1-z)t;q)_\infty}{1+(\lambda/\vartheta)(1-z)t}\,d_qt\\ = (\vartheta+\lambda(1-z))e_q\left(-\frac{\lambda}{\mu}q(1-z)\right)\int_0^1 \frac{E_q((\lambda/\mu)q(1-z)t)}{\vartheta+\lambda(1-z)t}\,d_qt. \qquad \textbf{(3.26)}\end{aligned}$$

Replacing μ by $\mu^*/(1-q)$ and using (2.5), we have

$$\lim_{q\to1^-} e_q\left(-\frac{\lambda}{\mu}q(1-z)\right) = \lim_{q\to1^-} e_q\left(-\frac{\lambda}{\mu^*}q(1-z)(1-q)\right) = e^{-(\lambda/\mu^*)(1-z)}, \qquad \textbf{(3.27)}$$

$$\lim_{q\to1^-} E_q\left(\frac{\lambda}{\mu}q(1-z)t\right) = \lim_{q\to1^-} E_q\left(\frac{\lambda}{\mu^*}q(1-z)t(1-q)\right) = e^{(\lambda/\mu^*)(1-z)t}. \qquad \textbf{(3.28)}$$

We can then take the limit as $q\to1^-$ in (3.26), using (3.27), (3.28), and (2.6). We obtain

$$\begin{aligned}\lim_{q\to1^-}(1-q)\,{}_3\phi_2\left(-\frac{\lambda}{\vartheta}(1-z),q,0;-\frac{\lambda}{\vartheta}q(1-z),-\frac{\lambda}{\mu}q(1-z);q,q\right)\\ = (\vartheta+\lambda(1-z))e^{-(\lambda/\mu^*)(1-z)}\int_0^1 \frac{e^{(\lambda/\mu^*)(1-z)t}}{\vartheta+\lambda(1-z)t}\,dt\\ = (\vartheta+\lambda(1-z))e^{-(\lambda/\mu^*)(1-z)}\int_0^{1-z} \frac{e^{(\lambda/\mu^*)s}}{\vartheta+\lambda s}\frac{1}{1-z}\,ds. \qquad \textbf{(3.29)}\end{aligned}$$

For $z=0$, we have, in particular,

$$\lim_{q\to1^-}(1-q)\,{}_3\phi_2\left(-\frac{\lambda}{\vartheta},q,0;-\frac{\lambda}{\vartheta}q,-\frac{\lambda}{\mu}q;q,q\right) = (\vartheta+\lambda)e^{-(\lambda/\mu^*)}\int_0^1 \frac{e^{(\lambda/\mu^*)s}}{\vartheta+\lambda s}\,ds. \qquad \textbf{(3.30)}$$

Taking the limit as $q\to1^-$ in (3.4), taking into account (3.30) yields easily (3.22). Equation (3.23) is obvious in light of (3.7). To obtain (3.24), we begin by taking limit as $q\to1^-$ in (3.6). Using (3.27) and (3.29), we have

$$\begin{aligned}\Pi_1^{(1)}(z) &= \lim_{q\to1^-}\Pi_1(z)\\ &= \left[1-\frac{\lambda+\vartheta}{\vartheta}\pi^{(1)}(0,0)\right]e^{-(\lambda/\mu^*)(1-z)} - \frac{\lambda(\lambda+\vartheta)}{\mu^*}e^{-(\lambda/\mu^*)(1-z)}\\ &\quad\times\int_0^{1-z}\frac{e^{(\lambda/\mu^*)s}}{\vartheta+\lambda s}\,ds\,\pi^{(1)}(0,0). \qquad \textbf{(3.31)}\end{aligned}$$

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We differentiate (3.31) with respect to z and we obtain

$$\frac{d}{dz}\Pi_1^{(1)}(z) = \frac{\lambda}{\mu^*}\Pi_1^{(1)}(z) + \frac{\lambda(\lambda+\vartheta)}{\mu^*(\vartheta+\lambda(1-z))}\pi^{(1)}(0,0).$$

We expand $d/dz\Pi_1^{(1)}(z)$, $\Pi_1^{(1)}(z)$, and $\lambda(\lambda+\vartheta)/\mu^*(\vartheta+\lambda(1-z))$ in a power series and we equate the coefficients of z^n. We obtain the stable recursive scheme

$$\pi^{(1)}(1,1) = \frac{\lambda}{\mu^*}\pi^{(1)}(0,0), \tag{3.32}$$

$$\pi^{(1)}(n,1) = \frac{\lambda}{n\mu^*}\pi^{(1)}(n-1,1) + \frac{\lambda}{n\mu^*}\left(\frac{\lambda}{\lambda+\vartheta}\right)^{n-1}\pi^{(1)}(0,0), \qquad n \geq 2. \tag{3.33}$$

Iterating (3.33) and using (3.32) yields (3.24). ■

THEOREM 3.4: *For a system with arrival rate λ, effective service rate per customer μ^*, and setup rate ϑ, the equilibrium state distribution $\pi^{(2)}(n,i) = \lim_{q\to 0^+}\pi(n,i;\lambda,\mu^*/(1-q),1-q,\vartheta)$ in the limiting case of full synchronization is given by*

$$\pi^{(2)}(0,0) = \frac{\vartheta\mu^*}{\vartheta\mu^*+\lambda\mu^*+\lambda\vartheta}, \tag{3.34}$$

$$\pi^{(2)}(n,0) = \left(\frac{\lambda}{\lambda+\vartheta}\right)^n \pi^{(2)}(0,0), \qquad n \geq 1, \tag{3.35}$$

$$\pi^{(2)}(n,1) = \begin{cases} \dfrac{\mu^*}{\lambda+\mu^*} n\left(\dfrac{\lambda}{\lambda+\mu^*}\right)^n \pi^{(2)}(0,0), & n \geq 1, \text{if } \mu^* = \vartheta \\ \dfrac{\vartheta}{\mu^*-\vartheta}\left(\left(\dfrac{\lambda}{\lambda+\vartheta}\right)^n - \left(\dfrac{\lambda}{\lambda+\mu^*}\right)^n\right)\pi^{(2)}(0,0), & n \geq 1, \text{if } \mu^* \neq \vartheta. \end{cases} \tag{3.36}$$

PROOF: We take the limit as $q \to 0^+$ in (3.4), using (2.8) and (2.12). This yields

$$\pi^{(2)}(0,0) = \left[\frac{\lambda+\vartheta}{\vartheta} + \frac{\lambda}{\lambda+\mu^*}\left(1+\frac{\lambda}{\mu^*}\right)\right]^{-1}, \tag{3.37}$$

which is easily reduced to (3.34). Equation (3.35) is immediate from (3.7). Taking $q \to 0^+$ in (3.6), taking into account (2.7), (2.12), and (3.34) implies, after some

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simplifications, that

$$\begin{aligned}\Pi_1^{(2)}(z) &= \lim_{q\to 0^+} \Pi_1(z)\\ &= \left[1-\frac{\lambda+\vartheta}{\vartheta}\pi^{(2)}(0,0)\right]\frac{1}{1+(\lambda/\mu^*)(1-z)}\\ &\quad -\frac{\lambda(\lambda+\vartheta)(1-z)}{(\lambda+\vartheta-\lambda z)(\lambda+\mu^*-\lambda z)}\pi^{(2)}(0,0)\\ &= \frac{\vartheta\lambda z}{(\lambda+\mu^*-\lambda z)(\lambda+\vartheta-\lambda z)}\pi^{(2)}(0,0). \qquad \textbf{(3.38)}\end{aligned}$$

Expanding $\Pi_1^{(2)}(z)$, $(\lambda+\mu^*-\lambda z)^{-1}$ and $(\lambda+\vartheta-\lambda z)^{-1}$ in a power series and equating the coefficients of z^n yields

$$\pi^{(2)}(n,1) = \vartheta\lambda^n \sum_{k=1}^{n}\left(\frac{1}{\lambda+\mu^*}\right)^k\left(\frac{1}{\lambda+\vartheta}\right)^{n-k+1}\pi^{(2)}(0,0), \qquad n\geq 1. \qquad \textbf{(3.39)}$$

By computing the geometric sum in (3.39) for the two cases $\mu=\vartheta$ and $\mu\neq\vartheta$, we obtain the two branches of (3.36). ■

The last limiting regime that we consider is for fixed λ, μ and q, when $\vartheta\to\infty$. In that case, the server is not deactivated. This system can be seen as the Poisson arrival process subject to binomial catastrophes and it reduces to a special case of Economou [12]. We then have the following.

THEOREM 3.5: *For a system with arrival rate λ, service rate μ and service repeat probability q, the equilibrium state distribution $\pi^{(3)}(n,i)=\lim_{\vartheta\to\infty}\pi(n,i;\lambda,\mu,1-q,\vartheta)$ in the limiting case of no deactivation of the server is given by*

$$\pi^{(3)}(0,0) = e_q\left(-\frac{\lambda}{\mu}\right), \qquad \textbf{(3.40)}$$

$$\pi^{(3)}(n,0) = 0, \qquad n\geq 1 \qquad \textbf{(3.41)}$$

$$\pi^{(3)}(n,1) = \sum_{k=n}^{\infty}\frac{(-1)^n(-\lambda/\mu)^k\binom{k}{n}}{(q;q)_k}, \qquad n\geq 1. \qquad \textbf{(3.42)}$$

PROOF: Taking the limit as $\vartheta\to\infty$ in (3.4) implies that

$$\pi^{(3)}(0,0) = \left[1+\frac{\lambda}{\lambda+\mu}E_q\left(\frac{\lambda}{\mu}\right){}_3\phi_2\left(0,q,0;0,-\frac{\lambda}{\mu}q;q,q\right)\right]^{-1}. \qquad \textbf{(3.43)}$$

However, ${}_3\phi_2(0,q,0;0,-(\lambda/\mu)q;q,q)$ is simplified to ${}_2\phi_1(q,0;-(\lambda/\mu)q;q,q)$. Then, using (2.11), we have that (3.43) yields

$$\pi^{(3)}(0,0) = \left[1+\frac{\lambda}{\lambda+\mu}E_q\left(\frac{\lambda}{\mu}\right)\frac{1+\lambda/\mu}{-\lambda/\mu}\left(e_q\left(-\frac{\lambda}{\mu}\right)-1\right)\right]^{-1}, \qquad \textbf{(3.44)}$$

which gives (3.40) after some simplifications.

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Taking $\vartheta \to \infty$ in (3.5) and using (3.44) yields

$$\Pi_0^{(3)}(z) = \pi^{(3)}(0,0) = e_q\left(-\frac{\lambda}{\mu}\right), \tag{3.45}$$

so we immediately have (3.41). We now take $\vartheta \to \infty$ in (3.6) and we obtain

$$\begin{aligned}\Pi_1^{(3)}(z) = &\left[1 - \pi^{(3)}(0,0)\right] e_q\left(-\frac{\lambda}{\mu}(1-z)\right) \\ &- \frac{\lambda(1-z)}{\lambda+\mu-\lambda z}\pi^{(3)}(0,0)\ {}_3\phi_2\left(0,q,0;0,-\frac{\lambda}{\mu}q(1-z);q,q\right).\end{aligned} \tag{3.46}$$

However, by (2.11), we have

$$\begin{aligned}{}_3\phi_2\left(0,q,0;0,-\frac{\lambda}{\mu}q(1-z);q,q\right) &= {}_2\phi_1\left(q,0;-\frac{\lambda}{\mu}q(1-z);q,q\right) \\ &= \frac{1+(\lambda/\mu)(1-z)}{-(\lambda/\mu)(1-z)}\left(e_q\left(-\frac{\lambda}{\mu}(1-z)\right)-1\right).\end{aligned} \tag{3.47}$$

Plugging (3.47) in (3.46) results, after some simplification, in

$$\Pi_1^{(3)}(z) = e_q\left(-\frac{\lambda}{\mu}(1-z)\right) - e_q\left(-\frac{\lambda}{\mu}\right). \tag{3.48}$$

Expanding (3.48) in a power series of $(1-z)$ using (2.4) yields

$$\begin{aligned}\Pi_1^{(3)}(z) &= \sum_{k=0}^{\infty}\frac{(-\lambda/\mu)^k}{(q;q)_k}\sum_{n=0}^{k}\binom{k}{n}(-z)^n - e_q\left(-\frac{\lambda}{\mu}\right) \\ &= \sum_{n=1}^{\infty}\sum_{k=n}^{\infty}\frac{(-\lambda/\mu)^k}{(q;q)_k}\binom{k}{n}(-z)^n,\end{aligned}$$

which proves (3.42). ■

4. BUSY PERIOD AND SOJOURN TIME DISTRIBUTIONS

We now study the busy period of the model, i.e., the time from the arrival of a customer at an empty system until the next epoch that the system is empty again. The busy period L is a first passage time starting from the state $(1,0)$ to reach $(0,0)$. Let $L_{(n,i)}$, $i = 0, 1$ and $n \geq i$, be a generic random variable representing a first passage time to state $(0,0)$ starting from (n,i) and denote by $\varphi_{(n,i)}(s) = E[e^{-sL_{(n,i)}}]$ its Laplace–Stieltjes

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transform. By conditioning on the time of the next event (first-step analysis), we obtain that the $\varphi_{(n,i)}(s)$ satisfy the system

$$\varphi_{(0,0)}(s) = 1, \tag{4.1}$$

$$\varphi_{(n,0)}(s) = \frac{\lambda}{\lambda+\vartheta+s}\varphi_{(n+1,0)}(s) + \frac{\vartheta}{\lambda+\vartheta+s}\varphi_{(n,1)}(s), \qquad n \geq 1, \tag{4.2}$$

$$\varphi_{(n,1)}(s) = \frac{\lambda}{\lambda+\mu+s}\varphi_{(n+1,1)}(s) + \frac{\mu p^n}{\lambda+\mu+s} + \frac{\mu}{\lambda+\mu+s}\sum_{j=1}^{n}\binom{n}{n-j}p^{n-j}q^j\varphi_{(j,1)}(s), \qquad n \geq 1. \tag{4.3}$$

We define the mixed transforms $\Phi_i(s,z)$, $i = 0, 1$, by

$$\Phi_i(s,z) = \sum_{n=i}^{\infty}\varphi_{(n,i)}(s)z^n, \qquad i = 0, 1. \tag{4.4}$$

These mixed transforms carry information for all first passage times distributions from an arbitrary state to $(0,0)$ and, in particular, for the busy period of the system. They can be expressed in terms of the q-hypergeometric series as follows.

THEOREM 4.1: *The Laplace–Stieltjes transforms $\varphi_{(1,1)}(s)$ and $\varphi_{(1,0)}(s)$ and the mixed transforms $\Phi_1(s,z)$ and $\Phi_0(s,z)$ are given by the equations*

$$\varphi_{(1,1)}(s) = \mu(1-q)\,{}_1\phi_1\left(-\frac{\lambda}{\mu+s};-\frac{\lambda q^2}{\mu+s};q,\frac{\mu q^2}{\mu+s}\right)(\lambda q+\mu+s)^{-1} \,{}_1\phi_1\left(-\frac{\lambda}{\mu+s};-\frac{\lambda q}{\mu+s};q,\frac{\mu q}{\mu+s}\right)^{-1}, \tag{4.5}$$

$$\begin{aligned}\Phi_1(s,z) = {}&\mu(1-q)z^2\,{}_2\phi_2\left(q,-\frac{z}{1-z};\frac{(\mu+s)zq}{\lambda(1-z)},-\frac{zq^2}{1-z};q,\frac{\mu zq^2}{\lambda(1-z)}\right)\\ &\times\left(((\lambda+\mu+s)z-\lambda)(1-(1-q)z)\right)^{-1}\\ &-\lambda z\varphi_{(1,1)}(s)\,{}_2\phi_2\left(q,-\frac{z}{1-z};\frac{(\mu+s)zq}{\lambda(1-z)},-\frac{zq}{1-z};q,\frac{\mu zq}{\lambda(1-z)}\right)\\ &\times\left((\lambda+\mu+s)z-\lambda\right)^{-1},\end{aligned} \tag{4.6}$$

$$\varphi_{(1,0)}(s) = \frac{\vartheta}{\lambda}\Phi_1\left(s,\frac{\lambda}{\lambda+\vartheta+s}\right), \tag{4.7}$$

$$\Phi_0(s,z) = 1 + \frac{\vartheta z}{(\lambda+\vartheta+s)z-\lambda}\left(\Phi_1(s,z) - \Phi_1\left(s,\frac{\lambda}{\lambda+\vartheta+s}\right)\right). \tag{4.8}$$

The involved q-hypergeometric series converge absolutely in $\{(s,z)\in\mathbb{C}^2 : |z|\leq 1, \mathrm{Re}(s)\geq 0\}$.

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PROOF: For the sake of notational convenience we introduce an operator notation. Let $T(z) = qz/(1-(1-q)z)$. This is a linear fractional transformation and we define its kth compositions by $T_0(z) = z$ and $T_k(z) = T(T_{k-1}(z))$ for $k \geq 1$. Then we can prove inductively that

$$T_k(z) = \frac{q^k z}{1-(1-q^k)z}, \qquad k \geq 0. \tag{4.9}$$

We also define the quantities

$$a_0(z) = 1, \tag{4.10}$$

$$a_k(z) = (-1)^k \left(\frac{\mu z}{\lambda(1-z)}\right)^k \frac{q^{\binom{k}{2}}}{\prod_{i=1}^k (1-[(\mu+s)z/\lambda(1-z)]q^i)}. \tag{4.11}$$

Multiplying (4.3) by $(\lambda+\mu+s)z^n$ and adding for all $n \geq 1$ results, after some manipulations, in

$$[(\lambda+\mu+s)z-\lambda]\,\Phi_1(s,z) = \frac{\mu p z^2}{1-pz} - \lambda z \varphi_{(1,1)}(s) + \frac{\mu z}{1-pz}\Phi_1(s,T(z)). \tag{4.12}$$

By iterating (4.12), we obtain

$$\begin{aligned}
&[(\lambda+\mu+s)z-\lambda]\Phi_1(s,z) \\
&\quad = \frac{\mu p z^2}{1-pz} - \lambda z \varphi_{(1,1)}(s) + \sum_{k=1}^{n} \left[\prod_{i=1}^{k} \frac{\mu T_i(z)}{q[(\lambda+\mu+s)T_i(z)-\lambda]}\right]\left[\frac{\mu p T_k^2(z)}{1-pT_k(z)}\right] \\
&\qquad - \sum_{k=1}^{n} \left[\prod_{i=1}^{k} \frac{\mu T_i(z)}{q[(\lambda+\mu+s)T_i(z)-\lambda]}\right] \lambda T_k(z)\varphi_{(1,1)}(s) \\
&\qquad + \prod_{i=1}^{n+1} \frac{\mu T_i(z)}{q[(\lambda+\mu+s)T_i(z)-\lambda]} [(\lambda+\mu+s)z-\lambda]\Phi_1(s,T_{n+1}(z)),
\end{aligned} \tag{4.13}$$

and taking limits as the number of iterations n goes to infinity results similarly with the derivation of (3.13) from (3.12) in

$$\begin{aligned}
&[(\lambda+\mu+s)z-\lambda]\Phi_1(s,z) \\
&\quad = \frac{\mu p z^2}{1-pz} - \lambda z \varphi_{(1,1)}(s) + \sum_{k=1}^{\infty} \left[\prod_{i=1}^{k} \frac{\mu T_i(z)}{q[(\lambda+\mu+s)T_i(z)-\lambda]}\right]\left[\frac{\mu p T_k^2(z)}{1-pT_k(z)}\right] \\
&\qquad - \sum_{k=1}^{\infty} \left[\prod_{i=1}^{k} \frac{\mu T_i(z)}{q[(\lambda+\mu+s)T_i(z)-\lambda]}\right] \lambda T_k(z)\varphi_{(1,1)}(s).
\end{aligned} \tag{4.14}$$

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The absolute convergence of the series can be proved straightforwardly by applying the ratio test. We can now easily verify that

$$\frac{T_i(z)}{(\lambda+\mu+s)T_i(s)-\lambda}=\frac{(-1)q^i z}{\lambda(1-z)(1-\frac{(\mu+s)z}{\lambda(1-z)}q^i)},\qquad i\geq 1,$$

so

$$\prod_{i=1}^{k}\frac{\mu T_i(z)}{q[(\lambda+\mu+s)T_i(s)-\lambda]}=a_k(z). \tag{4.15}$$

We plug (4.15) into (4.14) and we obtain

$$\begin{aligned}[(\lambda+\mu+s)z-\lambda]\Phi_1(s,z)=&\sum_{k=0}^{\infty}a_k(z)\left[\frac{\mu p T_k^2(z)}{1-pT_k(z)}\right]\\&-\sum_{k=0}^{\infty}a_k(z)\lambda T_k(z)\varphi_{(1,1)}(s).\end{aligned} \tag{4.16}$$

By setting $z=\lambda/(\lambda+\mu+s)$ in (4.16), we obtain

$$\begin{aligned}\phi_{(1,1)}(s)=&\left[\sum_{k=0}^{\infty}a_k\frac{\lambda}{\lambda+\mu+s}\left(\frac{\mu p T_k^2(\lambda/(\lambda+\mu+s))}{1-pT_k\left(\lambda/(\lambda+\mu+s)\right)}\right)\right]\\&\times\left[\sum_{k=0}^{\infty}a_k\left(\frac{\lambda}{\lambda+\mu+s}\right)\lambda T_k\left(\frac{\lambda}{\lambda+\mu+s}\right)\right]^{-1}.\end{aligned} \tag{4.17}$$

By writing the ratio of two consecutive terms of the sums in (4.17) as in (2.1), we can express the sums in ϕ-notation. Then (4.17) yields (4.5).

We then solve (4.16) for $\Phi_1(s,z)$ and express the sums involved in the standard manner to put them in ϕ-notation. This yields (4.6).

We now multiply (4.1) and (4.2) by z^0 and z^n, respectively, and add for all $n\geq 0$. After some algebraic manipulations, we obtain that

$$[(\lambda+\vartheta+s)z-\lambda]\,\Phi_0(s,z)=(\lambda+\vartheta+s)z-\lambda-z\lambda\varphi_{(1,0)}(s)+\vartheta z\Phi_1(s,z). \tag{4.18}$$

We set $z=\lambda/(\lambda+\vartheta+s)$ in (4.18) and solve for $\varphi_{(1,0)}(s)$, so we obtain (4.7). Solving (4.18) for $\Phi_0(s,z)$, taking into account (4.7) results in (4.8). ■

We now consider a tagged customer and let S denote its sojourn time in the system. The following theorem shows that the distribution of S is either a mixture of exponential distributions with parameters $\mu(1-q)$ and ϑ (in the case where $\mu(1-q)\neq\vartheta$) or a mixture of an exponential distribution and an Erlang-2 distribution with parameter ϑ (in the case where $\mu(1-q)=\vartheta$). More specifically we have the following.

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THEOREM 4.2: *The distribution of the sojourn time of a customer in the system is given by*

$$F_S(x) = \begin{cases} (1-p_1)(1-e^{-\mu(1-q)x}) + p_1(1-e^{-\vartheta x}), & x \geq 0 \text{ if } \vartheta \neq \mu(1-q) \\ (1-p_2)(1-e^{-\vartheta x}) + p_2(1-e^{-\vartheta x} - xe^{-\vartheta x}), & x \geq 0 \text{ if } \vartheta = \mu(1-q), \end{cases} \tag{4.19}$$

where

$$p_1 = \frac{(\lambda+\vartheta)\mu(1-q)}{\vartheta(\mu(1-q)-\vartheta)}\pi(0,0), \qquad p_2 = \frac{\lambda+\vartheta}{\vartheta}\pi(0,0). \tag{4.20}$$

PROOF: The sojourn time of a customer in the system depends on whether the server is active at the time of his arrival. Because of the PASTA property we have that the probability that the server is active at an arrival instant is $\Pi_1(1)$ given by (3.11), whereas the probability that the server is deactivated is $\Pi_0(1)$. Then the sojourn time S has the representation

$$S = \begin{cases} Y + \sum_{j=1}^{J} X_j & \text{with probability } \dfrac{\lambda+\vartheta}{\vartheta}\pi(0,0) \\ \sum_{j=1}^{J} X_j & \text{with probability } 1 - \dfrac{\lambda+\vartheta}{\vartheta}\pi(0,0), \end{cases} \tag{4.21}$$

where Y and $X_1, X_2, \ldots$ are independent exponentially distributed random variables with rates ϑ and μ, respectively, and J is independent of them, representing the number of services that will be required until the customer leaves the system. We have that J is geometrically distributed with $P(J=j) = (1-q)q^{j-1}$, $j = 1, 2, \ldots$. Because of (4.21), the Laplace–Stieltjes transform of S assumes the form

$$w(s) = \frac{\lambda+\vartheta}{\vartheta}\pi(0,0)\frac{\vartheta}{\vartheta+s}\frac{\mu(1-q)}{\mu(1-q)+s} + \left(1 - \frac{\lambda+\vartheta}{\vartheta}\pi(0,0)\right)\frac{\mu(1-q)}{\mu(1-q)+s}. \tag{4.22}$$

The partial fraction expansion of (4.22) yields

$$w(s) = \begin{cases} (1-p_1)\dfrac{\mu(1-q)}{\mu(1-q)+s} + p_1\dfrac{\vartheta}{\vartheta+s} & \text{if } \vartheta \neq \mu(1-q) \\ (1-p_2)\dfrac{\vartheta}{\vartheta+s} + p_2\left(\dfrac{\vartheta}{\vartheta+s}\right)^2 & \text{if } \vartheta = \mu(1-q), \end{cases}$$

where p_1 and p_2 are given by (4.20) and we invert to obtain (4.19). ■

5. NUMERICAL RESULTS

In this section we present some numerical results that shed further light in the behavior of the model. We begin by illustrating the effect of each single parameter p, ϑ, and

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λ in the mean number of customers in the system, $E[N]$, when the other parameters are kept fixed. In all numerical scenarios we assume that the time unit has been rescaled to agree with the mean service time, i.e., we set $\mu = 1$. Our results are based on the computation of the mean number of customers given in (3.18) of Theorem 3.2. In Figure 2 we present the graph of $E[N]$ with respect to p for $\lambda = 3.5$ and $\vartheta = 1.5$. The function is increasing convex as p varies from 0 to 1. Figure 3 shows the influence of the setup rate ϑ in the mean number of customers in the system when $\lambda = 3.5$ and $p = 0.5$. The function is decreasing convex with a horizontal asymptote as $\vartheta \to \infty$. This agrees with our intuitive expectation and the result of Theorem 3.5. Figure 4 shows the behavior of $E[N]$ as the arrival rate λ varies, when $\vartheta = 1.5$ and $p = 0.5$. As expected, the mean number of customers in the system increases when λ increases.

Figures 5 and 6 demonstrate the effect of the level of synchronization in the mean and the variance, respectively, of the number of customers in the system. In Figures 5 and 6, the arrival rate λ, the effective service rate per customer $\mu^\star = \mu p$, and the setup rate ϑ are kept fixed: $\lambda = 3.5$, $\vartheta = 1.5$, $\mu^\star = 1$. We observe that both $E[N]$ and $\mathrm{Var}[N]$ are increasing convex functions of p; that is, both the average number of customers and its variability increase as the level of synchronization increases.

Finally, we present the probability densities of the busy period for three numerical scenarios, by numerically invert the Laplace–Stieltjes transform of the busy period, $\varphi_{(1,0)}(s)$, given in Theorem 4.1. Although there is no symbolic inversion of $\varphi_{(1,0)}(s)$,

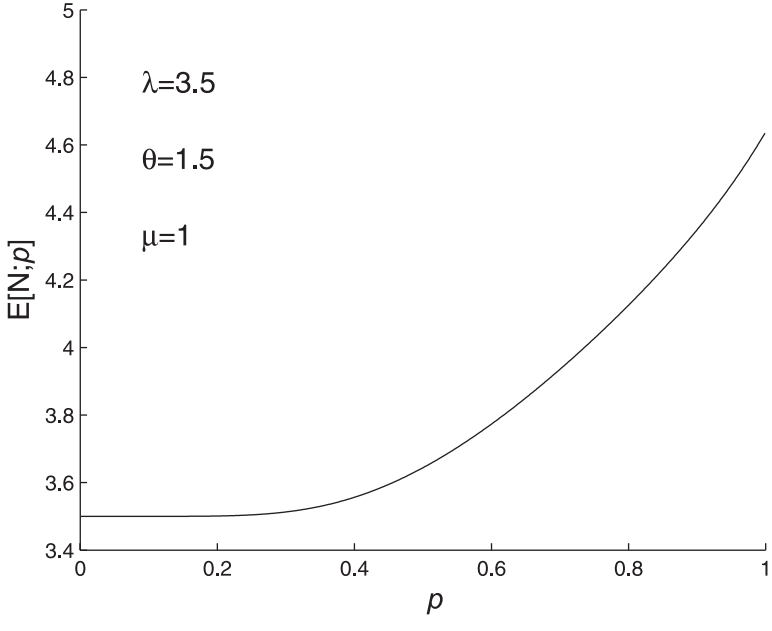


FIGURE 2. $E[N; \lambda, \mu, p, \vartheta]$ versus p.

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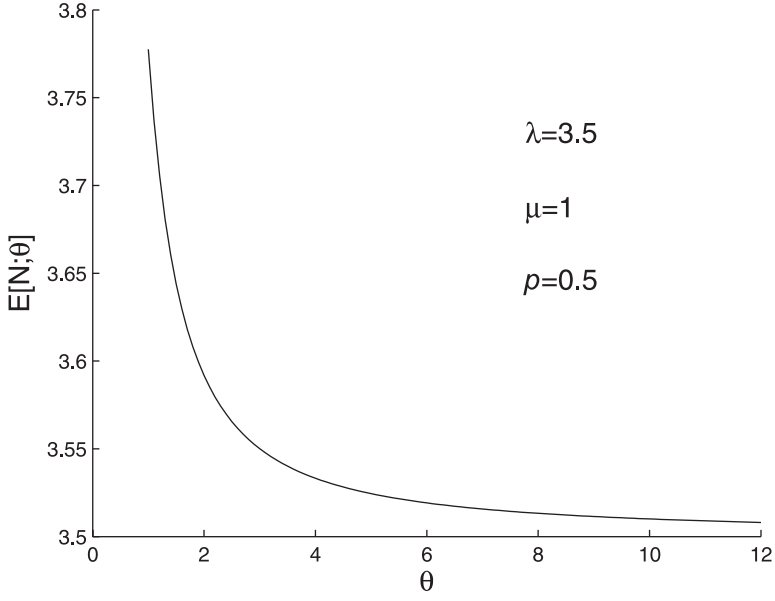


FIGURE 3. $E[N;\lambda,\mu,p,\vartheta]$ versus ϑ.

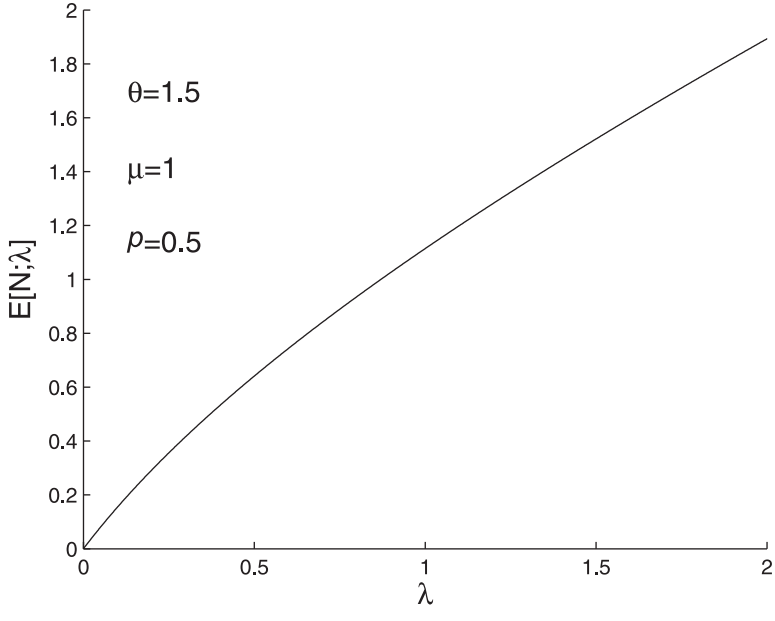


FIGURE 4. $E[N;\lambda,\mu,p,\vartheta]$ versus λ.

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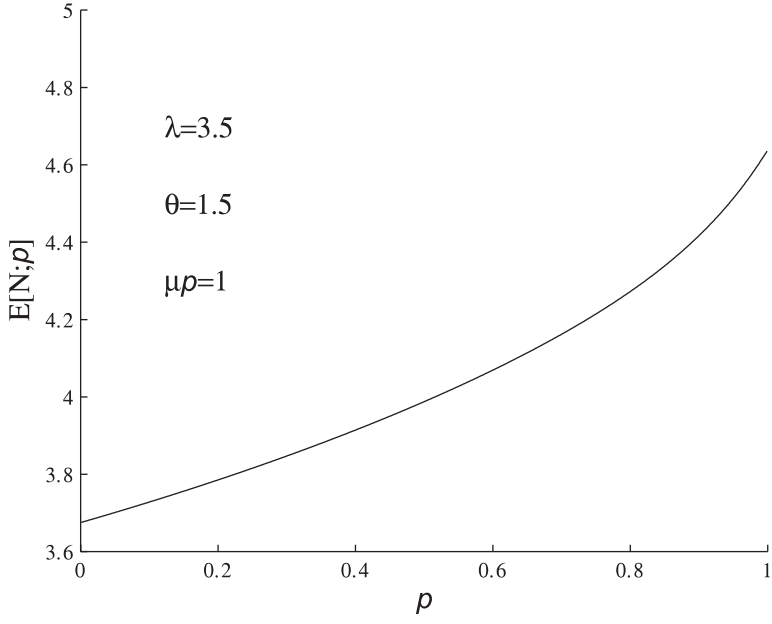


FIGURE 5. $E[N;\lambda,\mu^*/p,p,\vartheta]$ versus p.

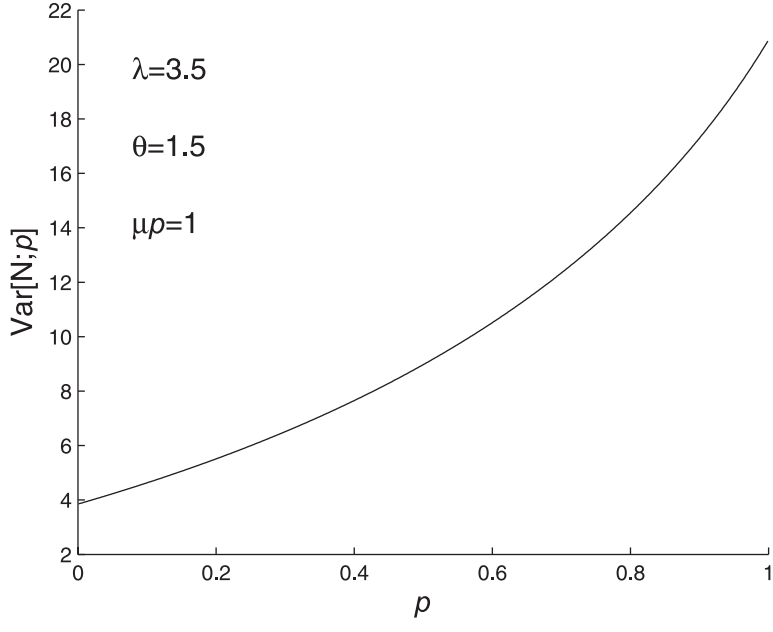


FIGURE 6. $\mathrm{Var}[N;\lambda,\mu^*/p,p,\vartheta]$ versus p.

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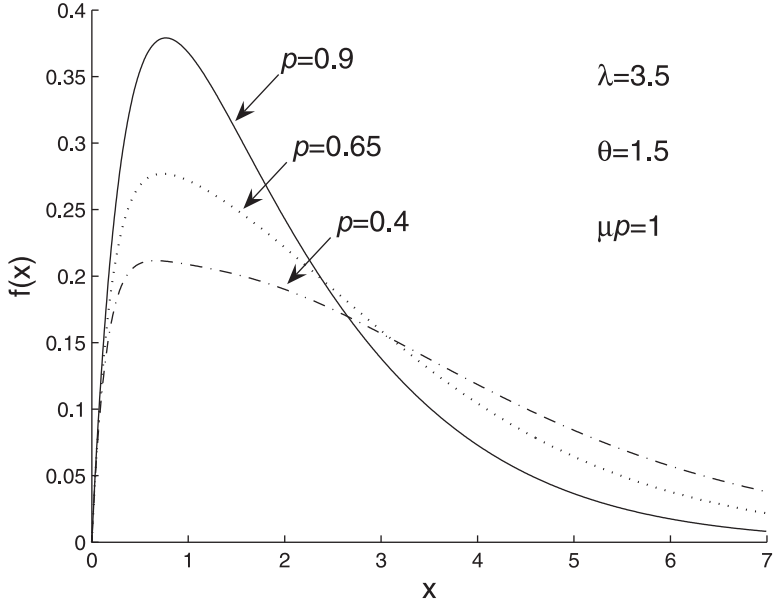


FIGURE 7. Busy period densities.

the use of the ${}_r\phi_s$ notation enables us to compute the numerical inversion very easily, by exploiting the available procedures and algorithms that support the computation of q-hypergeometric series in all standard mathematical software packages. In these scenarios we assume that $\lambda = 3.5$, $\vartheta = 1.5$, and $\mu p = 1$ and we consider three cases for p: 0.4, 0.65, and 0.9. Again, the numerical findings in Figure 7 are in absolute accordance with the fact that the mean number of customers increases as the level of synchronization increases.

6. GENERAL REMARKS–CONCLUSION

In the present article we have illustrated that the framework and the theory of a q-hypergeometric series provide efficient tools for the study of Markov chains with binomial transitions. We have seen that both the probability generating function for the equilibrium state distribution and the Laplace–Stieltjes transforms of first passage times are expressed in terms of q-series. Moreover, the q-calculus can facilitate the proofs of convergence results within this class of Markov chains.

In particular, the synchronization phenomenon arising in many queuing models, due to diverse characteristics, yields Markovian models with binomial transitions. We have studied such a model and we propose this methodology to be applied in similar situations such as synchronized retrials, synchronized reneging etc.

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In the framework of Mathematical Biology, there exists already a significant number of Markovian models with binomial transitions. Nevertheless, to the best of our knowledge, there are no applications of the q-calculus in the study of these models. A first step towards direction should be the study of any simple population growth model with binomial catastrophes.

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