contained. We note that the centroid G and the symmedian point K are isogonal conjugates and conjecture that for any interior point P with isogonal conjugate P^*, the incentre I is within the disc with diameter PP^*.

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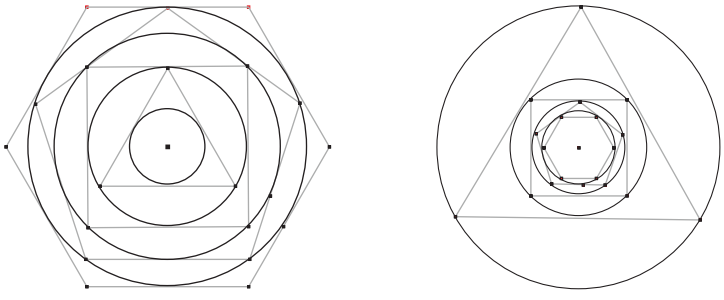
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106.26 The nested polygons problem revisited

The title refers to the following puzzle popularised recently by Ian Stewart in [1]; his account is based on [2].

A circle of unit radius is circumscribed by an equilateral triangle which is then circumscribed by another circle. Repeat, but on successive stages use a square, regular pentagon, and so on. Does the figure (shown in Figure 1(a)) become arbitrarily large or does it remain bounded in size?



(a) (b)

FIGURE 1: (a) Nested outward polygons; (b) nested inward polygons

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If R_n denotes the radius of the circle circumscribing the regular n-gon of the construction then (readily) $R_n = \sec\frac{\pi}{3}\sec\frac{\pi}{4}\ldots\sec\frac{\pi}{n}$. The version of the problem above focuses on whether or not the increasing sequence (R_n) is bounded above. Other versions, including [3, 4], ask for an estimation of the value of $\lim_{n\to\infty} R_n$, including the correction of a numerical slip in [3]. This requires relatively sophisticated manipulation of infinite products/sums, [1, 2, 4, 5, 6]. Singmaster in [7] traces the history of this nested polygons configuration back to Kepler and also quotes a 1980s 55 decimal place evaluation of $\lim_{n\to\infty} R_n$, which is close to 8.7.

My aim here is more modest: to give a short elementary proof that (R_n) is bounded above in order to make the problem accessible to as wide (and as young) an audience as possible. Showing that (R_n) is bounded above is equivalent to showing that the decreasing sequence $r_n = \cos\frac{\pi}{3}\cos\frac{\pi}{4}\ldots\cos\frac{\pi}{n}$ is bounded below away from zero; the r_n may be interpreted as the radii of the circles involved in carrying out the iterative construction inwards rather than outwards from the initial unit circle, as in Figure 1(b), with the puzzle then being whether the process converges to a point or to a circular hole.

By taking blocks of 4, 8, 16, … terms, we see that

$$\begin{aligned} r_{2^{n+1}-1} &= \cos\frac{\pi}{3}\left(\cos\frac{\pi}{4}\ldots\cos\frac{\pi}{7}\right)\left(\cos\frac{\pi}{8}\ldots\cos\frac{\pi}{15}\right)\ldots\left(\cos\frac{\pi}{2^n}\ldots\cos\frac{\pi}{2^{n+1}-1}\right) \\ &> \cos\frac{\pi}{3}\cos^4\frac{\pi}{4}\cos^8\frac{\pi}{8}\ldots\cos^{2^n}\frac{\pi}{2^n} \\ &= \frac{1}{8}\cos^8\frac{\pi}{8}\ldots\cos^{2^n}\frac{\pi}{2^n}. \end{aligned} \tag{1}$$

Let $p_n = \cos^8\frac{\pi}{8}\ldots\cos^{2^n}\frac{\pi}{2^n}$ and apply the inequality $\cos^4\theta \geqslant \cos 2\theta$ (see below) to each term to obtain

$$\begin{aligned} p_n &\geqslant \cos^2\frac{\pi}{4}\cos^4\frac{\pi}{8}\ldots\cos^{2^{n-2}}\frac{\pi}{2^{n-1}} \\ &> \frac{1}{2}\cos^4\frac{\pi}{8}\ldots\cos^{2^{n-2}}\frac{\pi}{2^{n-1}}\cos^{2^{n-1}}\frac{\pi}{2^n}, \text{ since the added term is less than 1,} \\ &= \frac{1}{2}\sqrt{p_n}. \end{aligned}$$

Thus $p_n > \frac{1}{4}$ so, from (1), $r_{2^{n+1}-1} > \frac{1}{32}$ which, since (r_n) is decreasing, suffices to guarantee that $r_n > \frac{1}{32}$ for all $n \geqslant 3$.

As a referee helpfully observed, this argument is reminiscent of the familiar proof that the harmonic series diverges by under-estimating blocks of 2, 4, 8, 16, … terms; a more general version of this occurs in Cauchy's condensation test for the convergence of a series of positive, decreasing terms.

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The unfinished business is the inequality $\cos^4\theta \geqslant \cos 2\theta$. We can see this by using a double-angle formula:

$$\cos^4\theta - \cos 2\theta = \cos^4\theta - 2\cos^2\theta + 1 = \left(1 - \cos^2\theta\right)^2 \geqslant 0.$$

(This actually proves more, namely the trigonometrical identity $\cos^4\theta - \sin^4\theta = \cos 2\theta$, which gives another route to a proof.)

But there is an amusing geometrical proof, valid for $0 < \theta < \frac{\pi}{4}$, which avoids the need to invoke the double-angle formula. In Figure 2, triangle ABC is isosceles with $AB = AC = 1$, $\angle BAC = 2\theta$, altitude AD bisects $\angle BAC$ and M is the midpoint of altitude BE; also DF is perpendicular to AC. Since F is the midpoint of EC, FM is parallel to BC with $\angle FPD = 90°$; PG is perpendicular to AC.

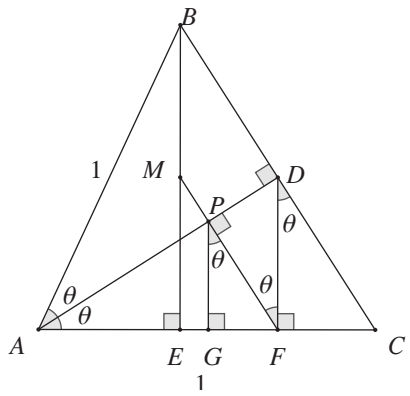


FIGURE 2

The angle-bisector theorem in triangle BAE shows that the bisector of $\angle BAE$ divides EB in the ratio $AE : 1$, so M is above the line AD; MF then intersects AD at P with P to the right of line BE. It follows that $AE < AG$. But $AE = \cos 2\theta$ and $AG = AP\cos\theta = AF\cos^2\theta = AD\cos^3\theta = 1.\cos^4\theta$!

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106.27 An interesting application of Ptolemy's inequality

Martin Lukarevski proposed the following inequalities in the *Monthly* [1] and the *Gazette* [2]

$$\frac{a}{l_a} + \frac{b}{l_b} + \frac{c}{l_c} \geqslant 2\sqrt{3}, \tag{1}$$

$$\frac{a^2}{l_a^2} + \frac{b^2}{l_b^2} + \frac{c^2}{l_c^2} \geqslant 4, \tag{2}$$

where a, b, c are the side lengths and l_a, l_b, l_c are the lengths of the corresponding angle bisectors of a given triangle. The goal of this note is to give a geometric proof of the following inequality of which (1) and (2) are consequences:

$$\frac{ab}{m_a m_b} + \frac{bc}{m_b m_c} + \frac{ca}{m_c m_a} \geqslant 4. \tag{3}$$

I found it interesting that I can only prove (3) via Ptolemy's inequality [3]. Consider triangle ABC. Let M, N, P be midpoints of BC, CA, AB respectively. Let $BC = a$, $CA = b$, $AB = c$, $AM = m_a$, $BN = m_b$, $CP = m_c$. Let G be the centroid of triangle ABC.

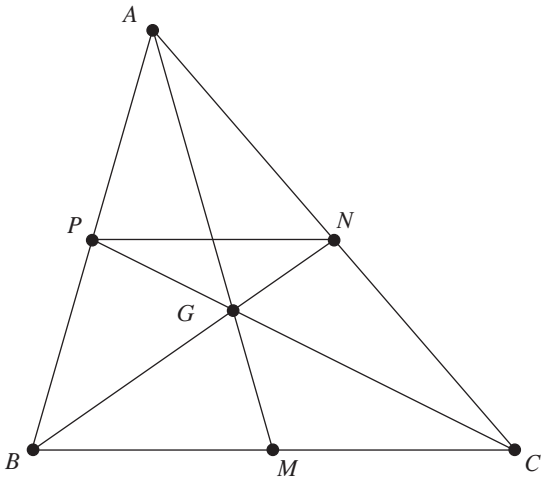


FIGURE 1

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