4

Hadwiger's conjecture

KEN-ICHI KAWARABAYASHI

1. Introduction
2. Complete graph minors: early results
3. Contraction-critical graphs
4. Algorithmic aspects of the weak conjecture
5. Algorithmic aspects of the strong conjecture
6. The odd conjecture
7. Independent sets and Hadwiger's conjecture
8. Other variants of the conjecture
9. Open problems
References

Hadwiger's conjecture states that any graph that does not have the complete graph K_k as a minor is $(k-1)$-colourable. It is well known that the case $k = 5$ is equivalent to the four-colour theorem. In 1993 Robertson, Seymour and Thomas proved that the case $k = 6$ is also equivalent to the four-colour theorem. For $k \geq 7$, the conjecture is still open. Our main focus in this chapter is to present recent results related to minimal counter-examples to Hadwiger's conjecture and some variations. We also consider algorithmic aspects of the conjecture and some of its variants, including the list-colouring version (which is false), the odd case of the conjecture, Hajós's conjecture and totally odd subdivisions, and Hadwiger's conjecture for some special classes of graphs.

1. Introduction

This chapter is motivated by Hadwiger's conjecture from 1943, which suggests a far-reaching generalization of the four-colour theorem. It is among the most challenging open problems in all of graph theory.

Hadwiger's conjecture (strong version) For all k, every k-colourable graph contains the complete graph K_k as a minor.

https://doi.org/10.1017/CBO9781139519793.007 Published online by Cambridge University Press

For $k \leq 3$, Hadwiger's conjecture is easy to prove, and for $k = 4$, it was proved independently by Hadwiger himself [31] and Dirac [21]. For $k = 5$, however, it becomes extremely difficult. In 1937 Wagner [74] proved that this case is equivalent to the four-colour theorem, so, given that result in [2], [3], [56], it follows that the case $k = 5$ also holds. In the deepest theorem in this area so far, Robertson, Seymour and Thomas [62] proved in 1993 that a minimal counter-example to the case $k = 6$ must have a vertex whose removal leaves a planar graph, so this case too follows from the four-colour theorem. For $k \geq 7$, the conjecture is still open. For $k = 7$, Kawarabayashi and Toft [46] proved that every 7-chromatic graph has K_7 or $K_{4,4}$ as a minor, and recently Kawarabayashi [38] proved that every 7-chromatic graph has K_7 or $K_{3,5}$ as a minor.

Because of the current state of affairs, there is considerable interest in a weaker version of Hadwiger's conjecture.

Hadwiger's conjecture (weak version) There exists a constant c such that, for $k \geq 1$, every ck-chromatic graph contains the complete graph K_k as a minor.

The best result of this type currently known is that there exists a constant c for which every $ck\sqrt{\log k}$-chromatic graph has K_k as a minor. This follows from the results in [66], [67], [50] and [49] (see Section 2). It was proved thirty years ago, and no one has been able to improve on the superlinear order $k\sqrt{\log k}$ of the bound on the chromatic number. Hence, it would be of great interest to establish that a linear function of the chromatic number k is sufficient to force K_k as a minor.

In this chapter, we survey recent progress concerning the two conjectures, paying special attention to the following.

Properties of minimal counter-examples, including the question of finiteness (Section 3).
Algorithmic aspects (Sections 4 and 5).
Other variants of Hadwiger's conjecture, including the odd Hadwiger conjecture (Section 6), immersion and Hajós's conjecture (Sections 6 and 8).
Hadwiger's conjecture for special families of graphs, including claw-free graphs and graphs with independence number 2 (Section 7).
The list-colouring version of Hadwiger's conjecture (Section 8).

The first step in attacking Hadwiger's conjecture is to find conditions on the minimum degree in a graph that force the existence of a large complete minor, an approach that has been taken since the 1960s. We survey these classical results in Section 2; for earlier results, see [73].

2. Complete graph minors: early results

Wagner and Mader studied extremal problems concerning the maximum number of edges in K_k-minor-free graphs. Wagner [75] proved that a sufficiently large chromatic

https://doi.org/10.1017/CBO9781139519793.007 Published online by Cambridge University Press

number (depending only on k) guarantees K_k as a minor, and Mader [52] showed that a sufficiently large average degree does the same. Later, Kostochka [50], [49] and Thomason [66] independently proved that the correct order of the average degree that forces K_k as a minor is $\Theta(k\sqrt{\log k})$.

Theorem 2.1 *There exist positive constants c_1 and c_2 such that every graph with average degree at least $c_1 k\sqrt{\ln k}$ contains K_k as a minor, while for $k \geq 3$ there exist graphs with average degree at least $c_2 k\sqrt{\ln k}$ that do not.*

More recently, Thomason [67] found the asymptotically sharp value of this 'extremal' function: it is $(\alpha + o(1))k\sqrt{\ln k}$, where $\alpha = 0.319...$ is an explicit constant determined by a particular equation.

These results show that if the minimum degree of a graph G is $o(k\sqrt{\ln k})$, then G does not necessarily contain a K_k-minor. This cannot be improved upon even if we add a connectivity condition; only connectivity of order $\Theta(k\sqrt{\ln k})$ forces the presence of a K_k-minor. However, as Thomason pointed out, the extremal graphs are close to disjoint unions of suitable dense random graphs. Such graphs cannot have too many vertices, a point we return to in Section 4.

Precise extremal numbers of edges for K_k-minors are known only for $k \leq 9$. For $k \leq 7$, these were found by Dirac [23] and Mader [52], while the result for $k = 8$ is due to Jørgensen [34], and the case $k = 9$ was recently settled by Song and Thomas [64].

There is an analogue to Theorem 2.1 for topological minors (recall that a graph H is a topological minor of a graph G if a subgraph of G is homeomorphically reducible to H) [11], [48]. Here the extremal function is of order $\Theta(k^2)$.

Theorem 2.2 *There exist positive constants c_1 and c_2 such that every graph with average degree at least $c_1 k^2$ contains K_k as a topological minor, while for $k \geq 3$ there exist graphs with average degree at least $c_2 k^2$ that do not contain K_k as a topological minor.*

3. Contraction-critical graphs

In attacking Hadwiger's conjecture, it is natural to consider minimal counter-examples, and in this section, we give several properties of such graphs.

Let G be a graph that is k-chromatic, is minimal with respect to the minor-relation in the class of all k-chromatic graphs, and does not contain K_k as a minor. Such a graph is called *k-contraction-critical* and cannot be a complete graph. These graphs were introduced by Dirac [22], [24], and one of his results was that the minimum degree in such a graph is at least k. The following result is shown in [36]; its proof is given below (following Theorem 3.5). Kawarabayashi and Yu [47] have recently shown that the fraction $\frac{2}{27}$ can be replaced by $\frac{1}{9}$.

Theorem 3.1 *Every k-contraction-critical graph is $\frac{2}{27}k$-connected.*

https://doi.org/10.1017/CBO9781139519793.007 Published online by Cambridge University Press

In order to prove Theorem 3.1, we need to define 'highly linked' subgraphs, which will be discussed in the next subsection.

The next result is a variation of an old theorem of Mader [53]; its non-algorithmic counterpart appeared in [9]. For completeness, we include its full proof, needed to establish the algorithmic part. The following concept will be useful in the proof. Given a graph G, a pair (A, B) of induced subgraphs is called a *separation* of G if $G = A \cup B$. Its *order* is $|V(A \cap B)|$. It is also useful to have a name for a $2k$-connected graph H that has at least $5k|V(H)|$ edges; we call it *extra-$2k$-connected*.

Theorem 3.2 *Let G be a graph with n vertices and m edges, and let k be such that $n \geq \frac{5}{2}k$ and $m \geq \frac{25}{4}kn - \frac{25}{2}k^2$. Then $n \geq 10k + 2$ and G contains an extra-$2k$-connected subgraph H. Furthermore, H can be found in time $O(n^3)$.*

Proof Clearly, if G is a graph of order n with at least $\frac{25}{4}kn - \frac{25}{2}k^2$ edges, then $\frac{25}{4}kn - \frac{25}{2}k^2 \leq \binom{n}{2}$. Hence, either

$$n \leq \tfrac{25}{4}k + \tfrac{1}{2} - \tfrac{1}{4}\sqrt{(25k+2)^2 - 400k^2} < \tfrac{5}{2}k$$

or

$$n \geq \tfrac{25}{4}k + \tfrac{1}{2} + \tfrac{1}{4}\sqrt{(25k+2)^2 - 400k^2} > 10k + 1.$$

Since $n \geq \frac{5}{2}k$, it follows that $n \geq 10k + 2$.

Suppose now that the theorem is false. Let G be a graph with n vertices and m edges, and let k satisfy the conditions of the theorem. Assume, moreover, that n is the least order of such a graph with the property that no subgraph is extra-$2k$-connected.

We now show that the minimum degree of G satisfies $\delta(G) > \frac{25}{4}k$. Suppose that G has a vertex v with degree at most $\frac{25}{4}k$, let $G' = G - v$, and let n' and m' be the numbers of vertices and edges in G'. By our choice of G, G' has no extra-$2k$-connected subgraphs. It follows from the fact that $n \geq 10k + 2$ that $n' = n - 1 \geq \frac{5}{2}k$. Also, $m' \geq m - \frac{25}{4}k \geq \frac{25}{4}kn' - \frac{25}{2}k^2$. Since $n' < n$, this contradicts the minimality of n and thus establishes that $\delta(G) > \frac{25}{4}k$.

Since $m \geq \frac{25}{4}kn - \frac{25}{2}k^2$ and $n \geq 10k + 2$, we have $m \geq 5kn$. But no subgraph of G is extra-$2k$-connected, so G cannot be $2k$-connected. Since $n > 2k$, this implies that G has a separation (A_1, A_2) for which both $A_1 - A_2$ and $A_2 - A_1$ are non-empty and $|A_1 \cap A_2| \leq 2k - 1$. By the condition on δ, $|A_i| \geq \frac{25}{4}k + 1$. For $i = 1, 2$, let G_i be a subgraph of G with vertex-set A_i for which $G = G_1 \cup G_2$ and $E(G_1 \cap G_2) = \varnothing$. Further, let n_i be the number of vertices and m_i the number of edges in G_i, and suppose that $m_i < \frac{25}{4}kn_i - \frac{25}{2}k^2$ for $i = 1, 2$. Then

$$\frac{25}{4}kn - \frac{25}{2}k^2 \leq m = m_1 + m_2 < \frac{25}{4}k(n + |A_1 \cap A_2|) - 25k^2 \leq \frac{25}{4}kn - \frac{25}{2}k^2,$$

a contradiction. Hence, we may assume that $m_1 \geq \frac{25}{4}kn_1 - \frac{25}{2}k^2$. Since $n > n_1 \geq \frac{25}{4}k + 1$ and G_1 has no extra-$2k$-connected subgraph H, this contradicts the minimality of n, and the existence of H is established.

https://doi.org/10.1017/CBO9781139519793.007 Published online by Cambridge University Press

The above proof yields an $O(n^3)$ algorithm for finding H. First, we remove vertices of degree at most $\frac{25}{4}k$ as long as the minimum degree remains less than $\frac{25}{4}k$; this can be done in linear time – first we list those vertices whose degree is at most $\frac{25}{4}k$, and then we remove them one by one. At each step, degrees change only at the neighbours of the removed vertex, and it takes constant time to lower their degrees by 1 and to move those whose degree drops to $\lfloor \frac{25}{4}k \rfloor$ into the list of vertices to be removed.

Next, we check whether G is $2k$-connected. By an algorithm of Henzinger, Rao and Gabow [32], this can be done in $O(n^2)$ time. At the same time we find a separation (A_1, A_2) of order less than $2k$ if one exists. As shown above, one of the corresponding subgraphs G_1 or G_2 can be used to continue the process. The recursion brings another factor of n to the time complexity, so the subgraph H can be found in $O(n^3)$ time. ■

A graph G is said to be *k-linked* if it has at least $2k$ vertices, and for each pair of k-tuples $(v_1, v_2, \ldots, v_k)$ and $(w_1, w_2, \ldots, w_k)$ of $2k$ distinct vertices, there exist k disjoint paths $P_1, P_2, \ldots, P_k$ with P_i a v_i–w_i path.

An important tool in what follows is the following theorem of Thomas and Wollan [65].

Theorem 3.3 *Every extra-$2k$-connected graph G is k-linked.*

This theorem implies that every $10k$-connected graph is k-linked. We note that Bollobás and Thomason [11] had proved earlier that every $22k$-connected graph is k-linked, and that Kawarabayashi, Kostochka and Yu [39] improved this to $12k$-connected.

Combining Theorems 3.2 and 3.3, we have the following result.

Corollary 3.4 *Let G be a graph with n vertices and m edges, and let k be such that $n \geq \frac{5}{2}k$ and $m \geq \frac{25}{4}kn - \frac{25}{2}k^2$. Then G contains a k-linked subgraph, and such a subgraph can be found in $O(n^3)$ time.*

Our next goal is to prove Theorem 3.1. To this end, it is useful to have a further definition and a lemma. Let (A, B) be a separation in a graph G and let $\mathbf{P} = \{S_1, S_2, \ldots, S_r\}$ be a partition of $S = V(A) \cap V(B)$. Then we say that A is *contractible* over $\{S_1, S_2, \ldots, S_r\}$ if there is a family $\mathbf{H} = \{H_1, H_2, \ldots, H_r\}$ of disjoint connected subgraphs of A such that $S_i \subseteq V(H_i)$ for each i. The following result is the lemma that we need.

Theorem 3.5 *Let G be a graph with n vertices and minimum degree d. Let (A, B) be a separation in G of minimum order, and let $S = V(A) \cap V(B)$ and $s = |S|$. If $s \leq \lfloor \frac{2}{27}d \rfloor$ and $\mathbf{P}$ is any partition of S, then A is contractible over $\mathbf{P}$, and the corresponding family $\mathbf{H}$ of subgraphs can be found in $O(n^3)$ time.*

Proof We may assume that the subgraph $A_1 = A - S$ is connected. Note that each vertex in A_1 has degree at least $d - s \geq \frac{25}{27}d$. Therefore, A_1 has at least $\frac{25}{27}d$ vertices and at least $\frac{25}{54}d|V(A_1)|$ edges. Let $k = \lfloor \frac{2}{27}d \rfloor$. By Theorem 3.2, A_1 contains an extra-$2k$-connected subgraph H_1, which (by Corollary 3.4) is k-linked. By the minimality of S and Menger's theorem, there are s disjoint paths $P_1, P_2, \ldots, P_s$ from S to $V(H_1)$, with

https://doi.org/10.1017/CBO9781139519793.007 Published online by Cambridge University Press

P_i joining $v_i \in S$ and $w_i \in H_i$. Since H_1 is $2k$-connected, it contains s independent edges $e_1, e_2, \ldots, e_s$ with $e_i = w_i w_i'$ for $i = 1, 2, \ldots, s$.

We now assume that the vertices $v_1, v_2, \ldots, v_s$ of S are listed so that the first $|S_1|$ are in S_1, the next $|S_2|$ are in S_2, and so on. Since H_1 is k-linked and $s \leq k$, there are $s-1$ disjoint paths $P_1', P_2', \ldots, P_{s-1}'$ in H_1 such that P_i' joins w_i and w_{i+1}'. The subgraph of A consisting of the s paths P_i, the s edges e_i, and the $s-1$ paths P_i' is a tree whose contraction gives rise to identification of all vertices of S_1 into a single vertex. Similarly, there are such trees for $S_2, S_3, \ldots, S_r$, and they are all disjoint. Hence A is contractible for $\{S_1, S_2, \ldots, S_r\}$.

As far as the algorithm is concerned, by Theorem 3.2, we can find H_1 in $O(n^3)$ time. To find paths $P_1, P_1, \ldots, P_s$, we can apply an augmenting path algorithm $s(< 2k)$ times, which can be done in $O(n^2)$ time. Finally, the disjoint connecting paths $P_1', P_2', \ldots, P_{s-1}'$ can be found by the graph minors algorithm of Robertson and Seymour [57] in $O(n^3)$ time. This completes the proof of the lemma. ■

We now show that this result implies Theorem 3.1.

Proof of Theorem 3.1 Take a minimal cutset S of G. Then $|S| < \lceil \frac{2}{27}k \rceil$. Let A_1 be a component of $G - S$ and let $A_2 = G - A_1 - S$. Then both $A_1 \cup S$ and $A_2 \cup S$ have chromatic number at most $k-1$. Let S_1 be a maximum independent set in $\langle S \rangle$, let S_2 be a maximum independent set in $\langle S - S_1 \rangle$, and so on, thereby obtaining a partition $\mathbf{P} = \{S_1, S_2, \cdots, S_r\}$ of S.

By Theorem 3.5, we can contract both A_1 and A_2 over $\mathbf{P}$ so that each of the resulting graphs on S is complete on S. Now let G_1 be the resulting graph together with A_2 and let G_2 be the resulting graph together with A_1. Then by the minimality of G, $\chi(G_1)$ and $\chi(G_2) \leq k-1$. But clearly we can combine the colourings of G_1 and G_2 to give a colouring of the whole graph G with at most $k-1$ colours. This contradiction completes the proof. ■

Returning to the algorithmic portion of the result, we note that we can use Robertson and Seymour's algorithm from [57] to find the disjoint connecting paths $P_1', P_2', \ldots, P_{s-1}'$. However, it turns out that when we apply Theorem 3.5 in the algorithm for Theorem 4.1 in the next section, we may assume that the k-linked subgraph H_1 is of bounded size; that is, if $N(k)$ is the constant from Theorem 3.6 below and $|V(H_1)| \geq N(k)$, then we know that H_1 contains K_k as a minor. Since this is one of the possible outcomes of the algorithm, we may assume that $|V(H_1)| < N(k)$, and hence the paths $P_1', P_2', \ldots, P_{s-1}'$ can be found in constant time.

The following result is proved in [9].

Theorem 3.6 *For each integer l, there exists a constant $N(l)$ such that every $2l$-connected graph with at least $N(l)$ vertices and minimum degree at least $\frac{31}{2}l$ contains K_l as a minor.*

Actually, the main result proved in [9] is stronger: For any integers s, k and l, there is a constant $N(s,k,l)$ such that every $(3l+2)$-connected graph of order at least

https://doi.org/10.1017/CBO9781139519793.007 Published online by Cambridge University Press

$N(s,k,l)$ and minimum degree at least d contains either $K_{sk,l}$ as a topological minor or a minor isomorphic to s copies of $K_{k,l}$. Since G could be the complete bipartite graph $K_{d,m}$ with m arbitrarily large, it is necessary to include the possibility of having a subdivision of $K_{sk,l}$.

The next result follows from Theorems 3.1 and 3.6.

Theorem 3.7 *If G is a $\frac{31}{2}k$-contraction-critical graph of order n, then there is a function $f(k)$ for which $n \leq f(k)$.*

This theorem tells us that there are only finitely many minimal counter-examples to the weak conjecture.

4. Algorithmic aspects of the weak conjecture

Although the weaker version of Hadwiger's conjecture is still open, we can show that from an algorithmic point of view, this problem can be 'decided' in polynomial time (with complexity $O(n^3)$).

Theorem 4.1 *For each $k \in \mathbb{N}$, there is a polynomial-time algorithm for deciding that*

(1) *a given graph G is $27k$-colorable*
(2) *G contains a K_k-minor*
or
(3) *there is a function $f(k)$ for which G contains a minor H of order at most $f(k)$ that does not contain a K_k-minor and is not $27k$-colourable.*

Note that if (3) holds, then H is a counter-example to Hadwiger's conjecture – in fact, to the weaker conjecture that every $27k$-chromatic graph has K_k as a minor. If (1) holds, then the graph can be coloured with $27k$ colours, while if (c) holds, then we can exhibit the minor H.

To prove this theorem, we need Theorem 3.6, but we do not need any result in the graph minor series. However, the proof of Theorem 3.6 given in [9] does depend on Robertson and Seymour's deep results (see [9], [59], [58]).

Before we give an algorithm for Theorem 4.1, we need some additional definitions. A *tree decomposition* of a graph G is a pair (T,Y), where T is a tree and Y is a family $\{Y_t : t \in V(T)\}$ of sets of vertices of G for which the following two properties hold.

(W1) $\bigcup_{t\in V(T)} Y_t = V(G)$, and each edge of G has both ends in some Y_t.

(W2) If $t,t',t'' \in V(T)$ and if t' lies on the t–t'' path in T, then $Y_t \cap Y_{t''} \subseteq Y_{t'}$.

The *width* of a tree decomposition (T,Y) is $\max_{t\in V(T)}(|Y_t| - 1)$.

We are now ready to describe our algorithm.

https://doi.org/10.1017/CBO9781139519793.007 Published online by Cambridge University Press

Algorithm for Theorem 4.1
Input: A graph G
Output: As described in Theorem 4.1.

Description:

Step 1 If G has a vertex of degree less than $27k$, we delete it. We continue this procedure until there are no vertices of degree less than $27k$. (This can be done in linear time.) Let G' be the resulting graph. Proceed to Step 2.

Step 2 Test whether the tree-width of G' is small or not, say smaller than some value $g(k)$. For simplicity in later steps, we assume that $g(k) \geq N(k)$, where $N(k)$ is as in Theorem 3.6: this can be done in linear time by an algorithm of Bodlaender [8]. If the tree-width is at least $g(k)$, go to Step 3. Otherwise, use the linear-time algorithm of Arnborg and Proskurowski [4] to colour G'. If G' can be coloured with $27k$ colours, then colour $G-G'$ greedily, and output the colouring of G. If G' is not $27k$-colourable, check whether it contains a K_k-minor; again, this can be done using the algorithm of Arnborg and Proskurowski (or that of Robertson and Seymour [57]). If G' contains K_k as a minor, output that G contains K_k as a minor. If G' does not contain K_k as a minor, then proceed as argued below. (Up to this point, the whole process can be done in linear time.)

Let (T, Y) be the tree-decomposition found above. The dynamic programming approach of Arnborg and Proskurowski assumes that T is a rooted tree whose edges are directed away from the root. For $tt' \in E(T)$ (where the root is closer to t than to t'), define $S(t,t') = Y_t \cap Y_{t'}$ and let $G'(t,t')$ be the induced subgraph $G'[\bigcup Y_s]$, where the union runs over those vertices s in the component of $T - tt'$ that does not contain the root. The algorithm of Arnborg and Proskurowski starts at all leaves of T and for each edge tt' in T, computes the set $C(t,t')$ of all $27k$-colourings of $S(t,t')$ that can be extended to the whole of $G'(t,t')$. If T has a vertex t of very large degree, then for two neighbours t' and t'', $S(t,t') = S(t,t'')$ and $C(t,t') = C(t,t'')$. Then $G'(t,t')$ can be deleted, and we still have a graph of bounded tree-width without a K_k minor and with no $27k$-colourings.

If all vertices of T have bounded degree, then T has a long path and there are two edges t_1t_1' and t_2t_2' on this path (with the second further from the root) for which $|S(t,t')| = |S(t,t'')|$ and $C(t,t') = C(t,t'')$. In the same way as argued in [9], we may assume that there are $|S(t,t')|$ disjoint paths joining $S(t,t')$ and $S(t,t'')$. By contracting these paths and replacing $G'(t,t')$ by $G'(t,t'')$, we obtain a minor of G' that is still of bounded tree-width, without a K_k minor and without $27k$-colourings. Repeating this, we eventually end up with the desired minor of G' of bounded size. This bound is actually a doubly exponential value expressed in terms of k.

Step 3 Test whether G' is $2k$-connected. By the assumption in Step 2, if G' is $2k$-connected, then $|G'| \geq g(k) \geq N(k)$; it then follows from Theorem 3.6 that G' contains K_k as a minor. So output that G has K_k as a minor. If G' is not $2k$-connected, go to Step 4.

https://doi.org/10.1017/CBO9781139519793.007 Published online by Cambridge University Press

Step 4 If G' is not $2k$-connected, find a minimal separation (A, B); this can be done in polynomial time by standard methods. The best known algorithm for this is due to Henzinger, Rao and Gabow [32] and needs time $O(n^2)$.

Let $S = V(A) \cap V(B)$. Then $|S| < 2k$. As in the proof of Theorem 3.5, let A'_1 be a component of $G' - S$ and $A'_2 = G' - A'_1 - S$. Let S_1 be a maximal independent set in $G'[S]$. Let S_i be a maximal independent set in $G'(S) - \bigcup_{j=1}^{i-1} S_l$, for $i = 2, 3, \ldots$. These maximal independent sets can be found greedily (or by other means) in constant time (since $|S| < 2k$). Next, we collapse each non-empty set S_i to a single vertex s_i; then the resulting graph on $S' = \{s_1, s_2, \ldots, s_r\}$ is complete. Let A_1, A_2 be the corresponding graphs obtained from A'_1, A'_2 by adding the clique S' and the corresponding edges between A'_i and S'.

Finally, we test A_1 and A_2 (recursively), starting from Step 1. If both A_1 and A_2 are $27k$-colourable, they give rise to a $27k$-colouring of their union since S' is a clique. Since the sets S_i that were collapsed into single vertices in S' are independent in G', this colouring gives rise to a colouring of G' which can then be extended to all of G.

If one of the graphs, say A_1, contains a K_k-minor, then we obtain a K_k-minor in G' (after contracting A_2 onto S') by using Theorem 3.5 with $d = 27k$. Similarly, if outcome (c) occurs for A_1, we get the same outcome for G' by using a contraction of A_2 onto S'.

This algorithm stops when the order of the current graph is small or when the current graph is $2k$-connected with minimum degree at least $27k$. The correctness follows from Theorem 3.5.

Now we estimate the time-complexity of the algorithm. All steps except the application of Theorem 3.5 can be done in time proportional to n^2. Another factor of n arises because of the recursion in Step 4. Finally, Theorem 3.5 is applied only when we backtrack from the recursion. If we apply it on the graph A_1 of order n_1, then we spend $O(n_1^3)$ time, but we never use it again on the same vertices. Therefore applications of Theorem 3.5 use only $O(n^3)$ time in total. This completes the proof of the correctness and of the stated time-complexity of the algorithm.

5. Algorithmic aspects of the strong conjecture

Robertson and Seymour found the result stated in the next theorem; it gives rise to a polynomial-time algorithm for k-colouring K_k-minor-free graphs if Hadwiger's conjecture is true. They did not publish a proof, but one can be found in Kawarabayashi and Reed [44].

Theorem 5.1 *For each $k \in \mathbb{N}$, there is a polynomial-time algorithm for deciding that*

(1) *a given graph G is k-colourable.*
(2) *G contains a K_{k+1}-minor.*

https://doi.org/10.1017/CBO9781139519793.007 Published online by Cambridge University Press

or

(3) *G contains a minor H of order at most $N(k)$ that is K_{k+1}-minor-free and is not k-colourable.*

Robertson has pointed out that in order to prove this theorem, he and Seymour used the following result. It is not only of independent interest, but is arguably the strongest known result in this direction.

Theorem 5.2 *For $k \geq 4$ and any graph G with no K_{k+1}-minor, one of the following statements holds:*

(1) *There exists an integer $f(k)$ such that G has tree-width at most $f(k)$.*
(2) *G has a vertex of degree at most k.*
(3) *G has a vertex v of degree $k+1$ with three mutually non-adjacent neighbours.*
(4) *G has a separation (A, B) of order at most k and with $A \neq G$ such that A can be contracted to a clique K on $A \cap B$ with each vertex of $A \cap B$ in a different vertex of K.*
(5) *G has a set S of at most $k-4$ vertices for which $G - S$ is planar.*

Note that (2), (3) and (4) cannot happen in a minimal counter-example to Hadwiger's conjecture, and that a graph satisfying (5) is also not a counter-example, by the four-colour theorem. The proof is complicated and uses the graph minor structure theory heavily, but does not use the well-quasi-ordering result (see [59] or [58]). The proof in [44] is much simpler, but still depends on the graph minor structure theory. Seymour has also pointed out that property (3) can be eliminated at the expense of a considerably longer proof.

6. The odd conjecture

Recently, the concept of an 'odd minor' has received considerable attention from researchers. We say that G has an *odd complete minor* of size k if it contains k disjoint trees with any two having adjacent vertices and which can be properly 2-coloured in such a way that the edges between trees have both vertices the same colour (hence the vertices of all trivial trees must receive the same colour). We say that G has an *odd K_k minor* if it has an odd complete minor of size k. It is easy to see that any graph with an odd K_k minor contains K_k as a minor.

Gerards and Seymour (see [33]) conjectured an analogue of Hadwiger's conjecture for odd complete graphs.

The odd conjecture For each $k \in \mathbb{N}$, every graph with no odd K_{k+1} minor is k-colorable.

This conjecture is trivially true for $k = 1$ and 2. In fact, for $k = 2$, it says simply that every graph with no odd cycles is 2-colourable. The case $k = 3$ was proved by Catlin [13]. Recently, Bertrand Guenin announced a solution of the $k = 4$ case,

https://doi.org/10.1017/CBO9781139519793.007 Published online by Cambridge University Press

which would imply the four-colour theorem since a graph that has an odd K_5-minor certainly has a K_5-minor. The conjecture remains open for $k \geq 5$.

The following theorem, first proved by Geelen *et al*. [29], is clearly a generalization of Theorem 2.1.

Theorem 6.1 *Any graph with no odd K_k-minor is $O(k\sqrt{\log k})$-colourable.*

Our proof [37] uses induction on the number of vertices. To this end, we prove the stronger statement below. We let c_1 be as in Theorem 2.1; thus, any graph with minimum degree at least $c_1 k\sqrt{\log k}$ contains a K_k-minor.

Theorem 6.2 *If G is a graph of order n and if $k \leq \frac{1}{4}n$, then either G has an odd K_k-minor or every precolouring of an induced subgraph of order $4k$ can be extended to a $\lfloor 64c_1 k\sqrt{\log k} + 4k \rfloor$-colouring of G.*

Proof Suppose that the result does not hold. We assume that $4k < n$ since otherwise there is nothing to prove. Let G be a counter-example of minimum order, and let Z be a set of $4k$ vertices of G. If $G - Z$ has a vertex v of degree at most $64c_1 k\sqrt{\log k} + 4k - 1$, then, by the minimality of G, $G - v$ has a $\lfloor 64c_1 k\sqrt{\log k} + 4k \rfloor$-colouring. This colouring can clearly be extended to a colouring of G, which is a contradiction. Hence, we assume that $\delta(G - Z) \geq 64c_1 k\sqrt{\log k} + 4k$.

We prove the theorem by establishing five claims.

(1) *G does not have a separation (A, B) of order $2k$ or less for which both $A - B - Z$ and $B - A - Z$ are non-empty.*

Suppose that there is such a separation (A, B) of order at most $2k$, and let $S = A \cap B$. Since $|S| \leq 2k$ and $|Z| \leq 4k$, it follows that either $|S \cup (A \cap Z)| \leq 4k$ or $|S \cup (B \cap Z)| \leq 4k$, say the former. By the minimality of G, there is a colouring of $G[B \cup Z]$ with Z precoloured. Note that the subgraph of G induced on $B \cup Z$ is smaller than G.

Again, by the minimality of G, we can colour $G[A]$ with $Z' = S \cup (A \cap Z)$ precoloured, where the precolouring of vertices in S comes from the colouring of $G[B \cup Z]$. Recall that $|Z'| \leq 4k$ and that the subgraph induced on A is smaller than G. Hence, there is a colouring of $G[A]$ with Z' precoloured. A combination of the obtained colourings of $G[B]$ and $G[A]$ yields a desired colouring of G, giving the required contradiction and proving the claim.

We choose a spanning bipartite graph H of $G - Z$ whose minimum degree is as large as possible. An old theorem of Erdős says that every graph of minimum degree at least $2l$ has a spanning bipartite graph with minimum degree at least l. Since $G - Z$ has minimum degree at least $64c_1 k\sqrt{\log k}$, H has minimum degree at least $32c_1 k\sqrt{\log k}$. The following result now follows by Mader's theorem [53].

(2) *H has an $8c_1 k\sqrt{\log k}$-connected subgraph L.*

We say that P is a *parity-breaking path* for L if P is disjoint from L except for its endpoints, and L together with P has an odd cycle. We need a generalization of

https://doi.org/10.1017/CBO9781139519793.007 Published online by Cambridge University Press

Mader's well-known S-path theorem for paths of odd length, and for this we use the following result of Geelen *et al.* [29].

Theorem 6.3 *For any set S of vertices in a graph G and for any integer k, either there are k disjoint paths of odd length and both ends in S, or there is a set X of at most $2k-2$ vertices for which $G-X$ has no such set of paths.*

Since L is $8c_1k\sqrt{\log k}$-connected, each of the partite sets of L has at least $8c_1k\sqrt{\log k}$ vertices.

(3) *There are at least k disjoint parity-breaking paths for L.*

Take one partite set S of L and apply Theorem 6.3 to G and S. If there are at least k disjoint odd S paths in G, then we can clearly find k disjoint parity-breaking paths for L, since otherwise, by Theorem 6.3, there is a separation (A,B) of order at most $2k-2$ such that A contains S and $A-B$ has no such paths. Since $A\cap B$ has at most $2k-2$ vertices and L is $8c_1k\sqrt{\log k}$-connected, it follows that A contains L. By Claim **1**, $B-A-Z=\varnothing$. It follows from Theorem 6.3 that $A-B$ can be expressed as $L'\cup W_1\cup W_2\cup\cdots\cup W_m$, for some integer m with L' a bipartite graph containing L and each W_i a block. In addition, each W_i contains a cut-vertex v_i to L' so that W_i-v_i consists of vertices in Z, since otherwise there would be a separation (A',B') of order at most $2k$ and both $A'-B'-Z$ and $B'-A'-Z$ non-empty, contradicting Claim **1**. Since $|Z|\leq 4k$, we have $m\leq 4k$. Because $B-A-Z=\varnothing$, we need at most $4k$ colours for Z, at most $4k$ colours for the vertices $v_1,v_2,\ldots,v_m$, two colours for L', and at most $2k-2$ colours for $A\cap B$ (since $A\cap B$ has order at most $2k-2$). These colours are enough to colour G, and we need only at most $10k\leq 8c_1k\sqrt{\log k}+4k$ colours for the colouring of G. This completes the proof of Claim **3**.

The next claim is proved in [35], although the proof there is slightly different; its proof is inspired by proofs of Robertson and Seymour [57] and Bollobás and Thomason [11].

(4) *Let G be a graph, let $S=\{s_1,s_2,\ldots,s_k\}$ be a set of k vertices, and suppose that G has a K_{2k}-minor and is k-connected. Then G has k vertex-disjoint non-empty connected subgraphs $C_1,C_2,\ldots,C_k$ such that, for each $1\leq i\leq k$, the subgraph C_i contains s_i and is adjacent to all of the other subgraphs $C_1,C_2,\ldots,C_{i-1},C_{i+1},\ldots,C_k$.*

Let $P_1,P_2,\ldots,P_k$ be the parity-breaking paths for L with ends p_i and p_i', for $1\leq i\leq k$. By our choice of c, L has a K_{4k}-minor. So claims **3** and **4** imply the following result.

(5) *G has a K_{2k}-minor M consisting of $2k$ disjoint trees $N_1,N_2,\ldots,N_{2k}$, each of which is contained in the bipartite graph L, and in which there are k disjoint parity-breaking paths $P_1,P_2,\ldots,P_k$ in which P_i has one end p_i in N_{2i-1} and the other end p_i' in N_{2i}, but is otherwise disjoint from M.*

The induced subgraph consisting of trees $N_1,N_2,\ldots,N_{2k}$ in L is clearly bipartite. We now use Claim **5** to construct an odd-K_k-minor. Suppose that we can 2-colour

https://doi.org/10.1017/CBO9781139519793.007 Published online by Cambridge University Press

$N_j \cup P_j \cup N_{j+k}$ so that the trees $N_i \cup P_i \cup N_{i+k}$ consist of an odd K_j-minor for $1 \leq i \leq j$. We claim that we can 2-colour $N_{j+1} \cup P_{j+1} \cup N_{j+1+k}$ so that the trees $N_i \cup P_i \cup N_{i+k}$ consist of an odd K_{j+1}-minor for $1 \leq i \leq j+1$. This is certainly possible since the 2-colouring of N_{j+1+k} can be interchanged with the 2-colouring of N_{j+1} in the bipartition of L. So, for any $i \leq j$, either the edge between N_i and N_{j+1} or the edge between N_{i+k} and N_{j+1+k} has both vertices the same colour. In this way, we can construct the desired odd K_k-minor, and this completes the proof of the theorem. ■

7. Independent sets and Hadwiger's conjecture

There is another way to think about Hadwiger's conjecture, based on the relation between the independence number and the chromatic number of a graph. Recall that in a coloring of a graph, each color-class is a set of independent vertices, so that if G has independence number α, then $\chi(G) \geq n/\alpha$. Hadwiger's conjecture asserts that G has $K_{\chi(G)}$ as a minor, and so, if true, implies the following conjecture, which was first explicitly stated and put into context by Woodall [77]. Throughout this section, n denotes the order of the graph G, and α denotes its independence number.

Conjecture A Every graph G has a $K_{\lceil n/\alpha \rceil}$ minor.

This conjecture seems weaker than Hadwiger's conjecture; however, for $\alpha = 2$, the two are equivalent (see [54]). In 1982 Duchet and Meyniel [25] proved the following remarkable result.

Theorem 7.1 *Every graph G has a $K_{\lceil n/(2\alpha-1) \rceil}$ minor.*

This result was improved first by Plummer, Toft and Kawarabayashi (see [41]) and subsequently in [6], [27] and [48]. Currently, the strongest result of this type is due to Balogh and Kostochka [5], who showed that $\lceil n/(2\alpha-1) \rceil$ can be replaced by $\lceil n/(2-c)\alpha \rceil$, for some constant $c > \frac{1}{19.2} = 0.052\,0833....$

We note that it seems quite hard to improve the constant $\frac{1}{3}$ in Theorem 7.1 for $\alpha = 2$ (a question asked independently by Seymour and Mader). As far as we know, no progress has yet been made on this question, although some related results appear in [16].

Analogous questions for the odd Hadwiger conjecture have also been studied; one is the following, due to Komlós and Szemerédi [48].

Theorem 7.2 *Every graph G has an odd $K_{\lceil n/(2\alpha-1) \rceil}$ minor.*

A slightly stronger result is known when $\alpha(G) = 3$. This is an analogue of a result in [48].

Theorem 7.3 *Every graph G with $\alpha = 3$ has an odd $K_{\lceil n/4 \rceil}$ minor.*

A graph is called *claw-free* if it does not contain $K_{1,3}$ as an induced subgraph; clearly every graph G with $\alpha = 2$ has this property. It is therefore likely to be very

https://doi.org/10.1017/CBO9781139519793.007 Published online by Cambridge University Press

hard to show Hadwiger's conjecture for claw-free graphs. One result on minors and claw-free graphs was proved by Fradkin [28].

Theorem 7.4 *Every claw-free graph with no K_{2k}-minor is $3k$-colourable.*

8. Other variants of the conjecture

In this section we consider some other concepts for which Hadwiger-like conjectures are quite natural; in particular, we consider subdivisions and immersions of graphs, and then move on to list-colourings.

Subdivisions

Although Kuratowski's well-known theorem can be stated in terms of both subdivisions and minors, similarities between the two concepts do not seem to carry much beyond that. For example, graphs without K_5-minors were already characterized by Wagner [74] in 1937, but graphs without K_5-subdivisions are still mysterious and a characterization is perhaps out of reach. Another difference between the two is that, as shown by Robertson and Seymour [60] in proving Wagner's famous conjecture, graphs are well-quasi-ordered by the minor relation, but not by the subdivision relation.

In connection with the four-colour problem, Hajós conjectured in the 1940s that, for all k, every graph without a subdivision of K_k is $(k-1)$-colourable. For $k \leq 3$ the result is obvious, while for $k = 4$ it was proved independently by Hadwiger [31] and Dirac [21]. However, it was disproved by Catlin [13] for $k \geq 7$. In fact, Erdős and Fajtlowicz [26] proved that the conjecture is false for almost all graphs (see also Bollobás and Catlin [10]). Recently, Thomassen [71] has given many families of graphs that are counter-examples to Hajós's conjecture. In fact, it is known that the order of magnitude for the chromatic number of graphs without a K_k-subdivision is $\Theta(k^2)$ (see [10], [11] and [45]). Thus, we have this modified version (which some might consider more of a question) of Hajós's original conjecture.

Conjecture B Every graph without a subdivision of K_4 is 3-colourable, and every graph without a subdivision of K_5 is 4-colourable.

The story does not end there however, as a restricted form of subdivision has led to some interesting questions. In a *totally odd subdivision* of a graph G, each edge of G has been replaced by a path of odd length. Totally odd subdivisions of K_4 play an important role in both graph theory and combinatorial optimization. In graph theory, their importance arises in graph colouring. Toft [72] conjectured that any graph with no totally odd K_4-subdivision is 3-colourable, a generalization of the $k = 4$ case of Hajós's original conjecture. This was finally settled by Zang [78] and Thomassen [70] independently. (This conjecture also relates to the so-called 't-perfect graphs'

https://doi.org/10.1017/CBO9781139519793.007 Published online by Cambridge University Press

and 'strongly t-perfect graphs' from the theory of perfect graphs; for more details, see the book by Schrijver [63].)

A general result by Thomassen [68] asserts that there is a function $f : \mathbb{N} \to \mathbb{N}$ with the property that every $f(k)$-chromatic graph has a subdivision of K_k in which each edge corresponds to a path of prescribed parity in the subdivision. As pointed out by Thomassen [69], totally odd K_k-subdivisions are also interesting in this connection. Unfortunately, the bound on the chromatic number given by Thomassen [68] is far from best possible, even in the totally odd case. Here we give an essentially sharp bound.

Theorem 8.1 *For any set S of at most k^2 vertices in a graph G, either G has a totally odd K_k-subdivision or every precolouring of the induced subgraph $\langle S \rangle$ can be extended to a $\frac{79}{4}k^2$-colouring of G.*

A result in [11] and [45] asserts that there exists a constant c for which every graph with minimum degree ck^2 contains a subdivision of K_k. This implies that the chromatic number of any graph without a subdivision of K_k is at most $ck^2 - 1$; furthermore, this result is sharp (up to constants). Hence, our result is best possible in a sense, and generalizes the result in [11] and [45].

Because the proof of Theorem 8.1 is quite similar to that of Theorem 6.2, we do not include it here. We note, however, that the proof implies that the same conclusion holds if 'totally odd' is replaced by 'prescribed parity'.

Immersions

An *immersion* of one graph H into another graph G is a mapping η that is one-to-one from $V(H)$ into $V(G)$ and maps the edges in H to edge-disjoint paths in G in such a way that if $e = vw$ is an edge of H, then $\eta(e)$ is a path joining $\eta(v)$ and $\eta(w)$ in G.

Previous investigations on immersions were mainly conducted from an algorithmic standpoint (see [12]), but it would be interesting to consider structural issues, since the notions of an immersion and a minor seem to be quite similar, and a structural approach concerning graph minors has been extremely successful. In fact, Robertson and Seymour [61] extended their proof of Wagner's conjecture [60] to prove that graphs are well-quasi-ordered by the immersion relation, and this proves a conjecture of C. Nash-Williams. The proof is based on the entire series of graph minors papers, and thus we may expect that a structural approach to immersions is very difficult – perhaps as hard as structural results on minors.

Immersion containment is not quite the same as either subdivision or minor containment. Clearly, a subdivision of H in G implies the existence of an immersion of H, but the converse is apparently not true. Also, a subdivision of H implies a minor of H, but the converse is not true. So, the existence of a subdivision is stronger than either a minor or an immersion. On the other hand, a minor and an immersion do not seem to be comparable. It would therefore be interesting to investigate

https://doi.org/10.1017/CBO9781139519793.007 Published online by Cambridge University Press

relations between the chromatic number of a graph G and the largest order of a complete graph immersed in G. In fact, Abu-Khzam and Langston [1] conjectured the following.

Conjecture C Every graph with no immersions of K_k is $(k-1)$-colourable.

This conjecture, like Hadwiger's conjecture and Hajós's conjecture, is true for $k \leq 4$. In fact, this follows from the truth of Hajós's conjecture for $k \leq 4$. Abu-Khzam and Langston [1] proved the weaker result that K_5 with one edge removed has an immersion in every 5-chromatic graph. They also pointed out that structural investigations have been extensively carried out for graphs without a K_4-immersion (see [12]).

Independently, DeVos *et al*. [19] and Lescure and Meyniel [51] have proved cases $k = 5, 6$ and 7 of the conjecture. Recently, DeVos *et al*. [18] proved a very general result that says that Conjecture **C** is true if k is replaced by $200k$ – in fact, they proved that the average degree $200k$ is enough to force a K_k-immersion. This actually differentiates between Hadwiger's conjecture and Conjecture **C** since the average degree $\Theta(k\sqrt{\log k})$ forces K_k as a minor (see Kostochka [50] and Thomason [66]), and this bound is sharp. On the other hand, only an average degree of $200k$ is needed to force K_k as an immersion.

List-colourings

The last variation that we consider in this section involves a different type of colouring rather than a different type of graph. Motivated by the weak Hadwiger conjecture, Kawarabayashi and Mohar [40] proved the following result. (A constant $f(k)$ is *computable* means that $f(k)$ can be expressed as a specific value, depending on k (see [9]).)

Theorem 8.2 *For each positive integer k, there exists a computable constant $f(k)$ such that every graph G with no K_k-minor admits a vertex-partition $\{V_1, V_2, \ldots, V_{\lceil 31k/2 \rceil}\}$ for which every component of each $G[V_i]$ has at most $f(k)$ vertices.*

When $f(k) = 1$, we get a colouring of G. Thus, the theorem gives a lessening of the requirements of a colouring, and hence a relaxation of Hadwiger's conjecture, and is the first result in this direction. In fact, since it is still not known whether there exists a constant c for which every ck-chromatic graph has a K_k-minor, this may be viewed as a first step in attacking that question.

Actually, the main result in [9] is the list-colouring version of Theorem 8.2 (see Chapter 6). When any of a set of ck colours is allowed at each vertex (called *ck-choosability*), we obtain this list-colouring version of Hadwiger's conjecture.

Conjecture D *There is a constant c for which every graph without a K_k-minor is ck-choosable.*

https://doi.org/10.1017/CBO9781139519793.007 Published online by Cambridge University Press

The list-colouring version of Hadwiger's conjecture itself is false; for example, there are planar graphs, which of course have no K_5 minors, that are not 4-choosable. Furthermore, Conjecture **D** is not true for $c = 1$; an example of Barat, Joret and Wood [7] shows that there is a K_{3k+2}-minor-free graph that is not $4k$-choosable.

Fractional colourings

Fractional colourings are closely related to the independent sets in a graph; in fact, the definition involves them, so we let $\mathcal{I}$ denote the family of independent sets in a graph G. For k a non-negative rational number, a *fractional k-colouring* of graph G is a map $q : \mathcal{I} \to \mathbb{Q}^+$ (the set of non-negative rationals) such that

- for each vertex v, $\sum q(S) = 1$, where the sum is over all $S \in \mathcal{I}$ that contain v
- $\sum q(S) \leq k$, where the sum is over all $S \in \mathcal{I}$.

Thus, when k is an integer, a graph G is k-colourable if and only if it has a fractional k-colouring q that has its values in $(0, 1)$. Consequently, if true, Hadwiger's conjecture would imply that every graph with no K_{t+1}-minor has a fractional t-colouring. In this direction, Reed and Seymour [55] proved the following result.

Theorem 8.3 *For each integer $t \geq 1$, every graph with no K_t-minor has a fractional $2t$-colouring.*

Motivated by the odd conjecture and Theorem 8.3, Kawarabayashi and Reed [43] deduced the same conclusion when only odd minors are excluded.

Theorem 8.4 *For each integer $t \geq 1$, every graph with no odd K_t-minor has a fractional $2t$-colouring.*

9. Open problems

We conclude the chapter with some open questions related to Hadwiger's conjecture. The first unsettled case of the conjecture is the following.

Conjecture E Every 7-chromatic graph has a K_7 minor.

Mader [52] proved that a minimal counter-example to Hadwiger's conjecture is 7-connected. Hence, Conjecture **E** is implied by the next one.

Conjecture F Every 7-connected graph with no K_7 minor has two vertices whose removal leaves a planar graph.

Motivated by the fact that a complete characterization for graphs without K_6 minors is not known, Jørgensen [34] made the following beautiful conjecture.

Conjecture G Every 6-connected graph G with no K_6 minors contains a vertex whose removal leaves a planar graph.

https://doi.org/10.1017/CBO9781139519793.007 Published online by Cambridge University Press

This conjecture was solved for large graphs in [41], but their method cannot give a complete solution. The next conjecture actually seems much harder than Conjecture **G**, and even out of reach for large graphs.

Conjecture H Every 7-connected graph has K_6 as a minor.

In fact, for a complete characterization of graphs with no K_6 minors, we may need to find a characterization of 5-connected graphs with no K_6 minor. The following weakening of Conjecture **H** would be a step in the right direction.

Conjecture I Every sufficiently large 7-connected graph G with no K_7-minors has two vertices v, w such that $G - v - w$ is planar.

As we have seen in this chapter, there are many weaker versions of Hadwiger's conjecture, all of which currently appear to be intractable. Let us state again the weak version of Hadwiger's conjecture.

Hadwiger's conjecture (weak version) There exists a constant c such that for $k \geq 1$, every ck-chromatic graph contains the complete graph K_k as a minor.

A stronger conjecture has also been made (see [40]).

Conjecture J There is a constant c for which every graph without a K_k-minor is ck-choosable.

Finally, we mention two other related conjectures. The first was made independently by Chartrand, Geller and Hedetniemi [14] and Woodall [77], while the second is due to Bojan Mohar.

Conjecture K For integers $1 \leq h \leq k$, every graph G without K_{k+1} or a $K_{\lfloor (k+2)/2 \rfloor, \lceil (k+2)/2 \rceil}$-minor, has a partition of G into $k-h+1$ parts, each part inducing a graph without K_{h+1} or a $K_{\lfloor (h+2)/2 \rfloor, \lceil (h+2)/2 \rceil}$-minor.

Conjecture L There are only finitely many 3-connected k-colour-critical graphs without K_k as a minor.

References

1. F. N. Abu-Khzam and M. A. Langston, Graph coloring and the immersion order, to appear.
2. K. Appel and W. Haken, Every planar map is four colorable, Part I. Discharging, *Illinois J. Math.* **21** (1977), 429–490.
3. K. Appel, W. Haken and J. Koch, Every planar map is four colorable, Part II. Reducibility, *Illinois J. Math.* **21** (1977), 491–567.
4. S. Arnborg and A. Proskurowski, Linear time algorithms for NP-hard problems restricted to partial k-trees, *Discrete Appl. Math.* **23** (1989), 11–24.
5. J. Balogh and A. Kostochka, Large minors in graphs with given independent number, *Discrete Math.* **311** (2011), 2204–2215.
6. J. Balogh, J. Lenz and H. Wu, Complete minors, independent sets, and chordal graphs, *Discuss. Math. Graph Theory*, to appear.

https://doi.org/10.1017/CBO9781139519793.007 Published online by Cambridge University Press

7. J. Barat, G. Joret and D. Wood, Disproof of the list Hadwiger conjecture, *Electron. J. Combin.* **18** (2011), P232.
8. H. L. Bodlaender, A linear-time algorithm for finding tree-decompositions of small treewidth, *SIAM J. Comput.* **25** (1996), 1305–1317.
9. T. Böhme, K. Kawarabayashi, J. Maharry and B. Mohar, Linear connectivity forces large complete bipartite minors, *J. Combin. Theory (B)* **99** (2009), 557–582.
10. B. Bollobás and P. A. Catlin, Topological cliques of random graphs, *J. Combin. Theory (B)* **30** (1981), 224–227.
11. B. Bollobás and A. Thomason, Highly linked graphs, *Combinatorica* **16** (1996), 313–320.
12. H. D. Booth, R. Govindan, M. A. Langston and S. Ramachandramurthis, Sequential and parallel algorithms for K_4-immersion testing, *J. Algorithms* **30** (1999), 344–378.
13. P. A. Catlin, A bound on the chromatic number of a graph, *Discrete Math.* **22** (1978), 81–83.
14. G. Chartrand, D. Geller and S. Hedetniemi, Graphs with forbidden subgraphs, *J. Combin. Theory (B)* **10** (1971), 12–41.
15. M. Chudnovsky and A. O. Fradkin, Hadwiger's conjecture for quasi-line graphs, *J. Graph Theory* **59** (2008), 17–33.
16. M. Chudnovsky and P. Seymour, Packing seagulls, *Combinatorica*, to appear.
17. V. Chvátal, On certain polytopes associated with graphs, *J. Combin. Theory (B)* **18** (1975), 138–154.
18. M. DeVos, Z. Dvořák, J. Fox, J. McDonald, B. Mohar and D. Scheide, A minimum degree condition forcing complete graph immersion, *Combinatorica* **34** (2014), 279–298.
19. M. DeVos, K. Kawarabayashi, B. Mohar and H. Okamura, Immersing small complete graphs, *Ars Math. Contemp.* **3** (2010), 139–146.
20. R. Diestel, *Graph Theory* (2nd edn.), Springer, 2000.
21. G. A. Dirac, A property of 4-chromatic graphs and some remarks on critical graphs, *J. London Math. Soc.* **27** (1952), 85–92.
22. G. A. Dirac, Trennende Knotenpunktmengen und Reduzibilität abstrakter Graphen mit Anwendung auf das Vierfarbenproblem, *J. Reine Angew. Math.* **204** (1960), 116–131.
23. G. A. Dirac, Homomorphism theorems for graphs, *J. Reine Angew. Math.* **153** (1964), 69–80.
24. G. A. Dirac, On the structure of 5- and 6-chromatic abstract graphs, *Math. Ann.* **214/215** (1964), 43–52.
25. P. Duchet and H. Meyniel, On Hadwiger's number and the stability number, *Graph Theory* (Cambridge, 1981), North-Holland (1982), 71–73.
26. P. Erdős and S. Fajtlowicz, On the conjecture of Hajós, *Combinatorica* **1** (1981), 141–143.
27. J. Fox, Complete minors and independence number, *SIAM J. Discrete Math.* **24** (2010) 1313–1321.
28. A. O. Fradkin, Clique minors in claw-free graphs, *J. Combin. Theory (B)* **102** (2012), 71–85.
29. J. Geelen, B. Gerards, B. Reed, P. Seymour and A. Vetta, On the odd variant of Hadwiger's conjecture, *J. Combin. Theory (B)* **99** (2009), 20–29.
30. B. Gerards, An extension of König's theorem to graphs with no odd-K_4, *J. Combin. Theory (B)* **47** (1989), 330–348.
31. H. Hadwiger, Über eine Klassifikation der Streckenkomplexe, *Vierteljschr. Naturforsch. Ges. Zürich* **88** (1943), 133–142.
32. M. R. Henzinger, S. Rao and H. N. Gabow, Computing vertex connectivity: new bounds from old techniques, *J. Algorithms* **34** (2000), 222–250.
33. T. R. Jensen and B. Toft, *Graph Coloring Problems*, Wiley–Interscience, 1995.
34. L. Jørgensen, Contraction to K_8, *J. Graph Theory* **18** (1994), 431–448.

https://doi.org/10.1017/CBO9781139519793.007 Published online by Cambridge University Press

35. K. Kawarabayashi, Rooted minors problem in highly connected graphs, *Discrete Math.* **287** (2004), 121–123.
36. K. Kawarabayashi, On the connectivity of minimal counterexamples to Hadwiger's conjecture, *J. Combin. Theory (B)* **97** (2007), 144–150.
37. K. Kawarabayashi, Note on coloring graphs with no odd-K_k-minors, *J. Combin. Theory (B)* **99** (2009), 738–741.
38. K. Kawarabayashi, Minors in 7-chromatic graphs, preprint.
39. K. Kawarabayashi, A. Kostochka and G. Yu, On sufficient degree conditions for a graph to be k-linked, *Combin. Probab. Comput.* **15** (2006), 685–694.
40. K. Kawarabayashi and B. Mohar, A relaxed Hadwiger's conjecture for list-colorings, *J. Combin. Theory (B)* **97** (2007), 647–651.
41. K. Kawarabayashi, S. Norine, R. Thomas and P. Wollan, K_6 minors in large 6-connected graphs, submitted.
42. K. Kawarabayashi, M. Plummer and B. Toft, Improvements of the theorem of Duchet and Meyniel on Hadwiger's conjecture, *J. Combin. Theory (B)* **96** (2005), 152–167.
43. K. Kawarabayashi and B. Reed, Fractional coloring and the odd Hadwiger's conjecture, *Europ. J. Combin.* **29** (2008), 411–417.
44. K. Kawarabayashi and B. Reed, Hadwiger's conjecture is decidable, *STOC'09 Proc. 2009 ACM International Symposium on Theory of Computing*, ACM (2009), 445–454.
45. K. Kawarabayashi and Z. Song, Some remarks on the odd Hadwiger's conjecture, *Combinatorica* **27** (2007), 429–438.
46. K. Kawarabayashi and B. Toft, Any 7-chromatic graph has K_7 or $K_{4,4}$ as a minor, *Combinatorica* **25** (2005), 327–353.
47. K. Kawarabayashi and G. Yu, Connectivities for k-knitted graphs and for minimal counterexamples to Hadwiger's conjecture, *J. Combin. Theory (B)* **103** (2013), 320–326.
48. J. Komlós and E. Szemerédi, Topological cliques in graphs II, *Combin. Prob. Comput.* **5** (1996), 79–80.
49. A. Kostochka, The minimum Hadwiger number for graphs with a given mean degree of vertices, *Metody Diskret. Analiz* **38** (1982), 37–58.
50. A. Kostochka, Lower bound of the Hadwiger number of graphs by their average degree, *Combinatorica* **4** (1984), 307–316.
51. F. Lescure and H. Meyniel, On a problem upon configurations contained in graphs with given chromatic number, *Graph Theory in Memory of G. A. Dirac* (Sandbjerg, 1985), Ann. Discrete Math. **41**, North-Holland (1989), 325–331.
52. W. Mader, Homomorphiesätze für Graphen, *Math. Ann.* **178** (1968), 154–168.
53. W. Mader, Existenz n-fach zusammenhängender Teilgraphen in Graphen genügend grosser Kantendichte, *Abh. Math. Sem. Univ. Hamburg* **37** (1972), 86–97.
54. M. Plummer, M. Stiebitz and B. Toft, On a special case of Hadwiger's conjecture, *Discuss. Math. Graph Theory* **23** (2003), 333–363.
55. B. Reed and P. D. Seymour, Fractional chromatic number and Hadwiger's conjecture, *J. Combin. Theory (B)* **74** (1998), 147–152.
56. N. Robertson, D. P. Sanders, P. D. Seymour and R. Thomas, The four-color theorem, *J. Combin. Theory (B)* **70** (1997), 2–44.
57. N. Robertson and P. D. Seymour, Graph minors XIII. The disjoint paths problem, *J. Combin. Theory (B)* **63** (1995), 65–110.
58. N. Robertson and P. D. Seymour, Graph minors XVII. Taming a vortex, *J. Combin. Theory (B)* **77** (1999), 162–210.
59. N. Robertson and P. D. Seymour, Graph minors XVI. Excluding a non-planar graph, *J. Combin. Theory (B)* **89** (2003), 43–76.

https://doi.org/10.1017/CBO9781139519793.007 Published online by Cambridge University Press

60. N. Robertson and P. D. Seymour, Graph minors XX. Wagner's conjecture, *J. Combin. Theory (B)* **92** (2004), 325–357.
61. N. Robertson and P. D. Seymour, Graph minors XXIII, Nash-Williams' immersion conjecture, *J. Combin. Theory (B)* **100** (2010), 181–205.
62. N. Robertson, P. D. Seymour and R. Thomas, Hadwiger's conjecture for K_6-free graphs, *Combinatorica* **13** (1993), 279–361.
63. A. Schrijver, *Combinatorial Optimization: Polyhedra and Efficiency*, Springer, 2003.
64. Z. Song and R. Thomas, The extremal function for K_9 minors, *J. Combin. Theory (B)* **96** (2006), 240–252.
65. R. Thomas and P. Wollan, An improved linear edge bound for graph linkages, *Europ. J. Combin.* **26** (2005), 309–324.
66. A. Thomason, An extremal function for contractions of graphs, *Math. Proc. Cambridge Philos. Soc.* **95** (1984), 261–265.
67. A. Thomason, The extremal function for complete minors, *J. Combin. Theory (B)* **81** (2001), 318–338.
68. C. Thomassen, Graph decomposition with applications to subdivisions and path systems modulo k, *J. Graph Theory* **7** (1983) 261–271.
69. C. Thomassen, Parity, cycle space, and K_4-subdivision in graphs, *Surveys in Combinatorics, 1999*, London Math. Soc. Lecture Notes **267**, Cambridge University Press (1999), 223–238.
70. C. Thomassen, Totally odd K_4-subdivisions in 4-chromatic graphs, *Combinatorica* **21** (2001), 417–443.
71. C. Thomassen, Some remarks on Hajós' conjecture, *J. Combin. Theory (B)* **93** (2005), 95–105.
72. B. Toft, Problem 11, *Recent Advances in Graph Theory* (ed. M. Fiedler), Academia Praha (1975), 543–544.
73. B. Toft, A survey of Hadwiger's conjecture, *Congr. Numer.* **115** (1996), 249–283.
74. K. Wagner, Über eine Eigenschaft der ebenen Komplexe, *Math. Ann.* **114** (1937), 570–590.
75. K. Wagner, Beweis einer Abschwächung der Hadwiger-Vermutung, *Math. Ann.* **153** (1964), 139–141.
76. D. Wood, Independent sets in graphs with an excluded clique minor, *Discrete Math. Theor. Comput. Sci.* **9** (2007), 171–175.
77. D. R. Woodall, Subcontraction-equivalence and Hadwiger's conjecture, *J. Graph Theory* **11** (1987), 197–204.
78. W. Zang, Proof of Toft's conjecture: every graph containing no fully odd K_4 is 3-colorable, *J. Combin. Optim.* **2** (1998), 117–188.

https://doi.org/10.1017/CBO9781139519793.007 Published online by Cambridge University Press