Recursive approach to repeated collisions

ZHIWEI CHONG

1. *Introduction*

Mathematics and physics are fascinatingly connected with each other. Since [1] first pointed out the relationship between the number π and the total number of collisions between two blocks and a wall, it has generated wide interest among the general public [2, 3, 4, 5]. A carefully made video presents an elegant and crystal clear geometric explanation [6], which is followed by a slightly more advanced explanation based on linear algebra, matrix formalism in particular [7]. References [2, 3, 6] attribute the geometric approach to different individuals, but the relationship actually appeared much earlier in a physics journal which focused on the number of collisions rather than its connection with the number π [8].

Reference [1] contains another elegant geometric approach. An excellent video based on it was presented in [9], which unfortunately appears to be no longer available. In addition, solving this problem is connected with a quantum search algorithm [10, 11]. More references include [12, 13, 14]. In this Article, we aim to present a recursive approach that is accessible to high school students. It may be used as a student project or a mechanics problem.

2. *Repeated collisions and recursion*

Elastic collisions between two objects are discussed in almost all introductory physics textbooks [15, 16]. Let the masses for the two blocks be m_1 and m_2, and the their velocities before they collide are u and v, respectively. Then their velocities after the collision are

$$u' = \frac{m_1 - m_2}{m_1 + m_2}u + \frac{2m_2}{m_1 + m_2}v, \quad v' = \frac{2m_1}{m_1 + m_2}u + \frac{m_2 - m_1}{m_1 + m_2}v. \tag{1}$$

What really matters in an elastic collision is the mass ratio, and we rewrite the above two equations as

$$u' = \frac{k - 1}{k + 1}u + \frac{2}{1 + k}v, \; v' = \frac{2k}{k + 1}u + \frac{1 - k}{1 + k}v, \; k \equiv \frac{m_1}{m_2}. \tag{2}$$

For the collisions between two blocks and a wall, Figure 1 illustrates the situation before and after the first block-block collision. The mass of the left block is m_1 which is greater than m_2 for the right. Initially, the right block is at rest and the left block moves to the right at velocity u. After the collision, both move to the right. Then the right block bounces elastically from the wall and collides with the left block for the second time.

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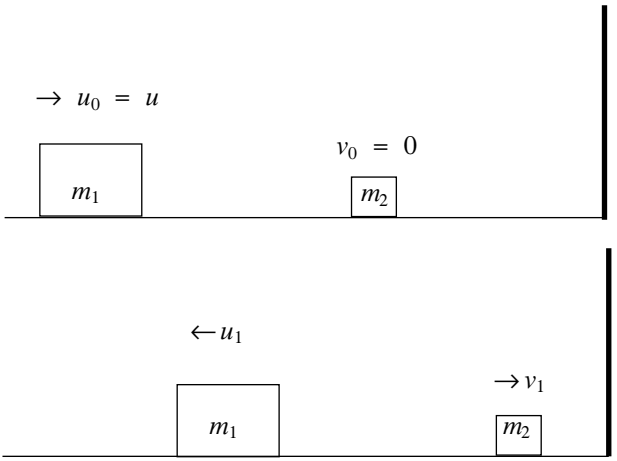


FIGURE 1: Situation before and after the first collision

For the multiple collisions under consideration, we denote the velocities for the left and the right block immediately after the nth block-block collision as u_n and v_n, respectively. We choose the right as our positive direction. Then their velocities before the $(n + 1)$th block-block collision are u_n and $-v_n$ because the right block moves in the opposite direction after it bounces elastically from the wall. Therefore the velocities immediately after the $(n + 1)$th block-block collision are

$$u_{n+1} = \frac{k - 1}{1 + k}u_n - \frac{2}{1 + k}v_n, \tag{3}$$

$$v_{n+1} = \frac{2k}{1 + k}u_n - \frac{1 - k}{1 + k}v_n. \tag{4}$$

From a mathematical point view of, they are recursive relations for u_n and v_n.

3. *Solution to the recursion*

It is natural to solve this problem by writing (3) and (4) in matrix form. However, since we are aiming to solve the equations in a way that is accessible to high school students, we want to reduce them to a single recursive relation for u_n or v_n. Here we choose to work with the recursive relation for u_n obtained in the following way.

From (3), v_n is expressed as

$$2v_n = (k - 1)u_n - (1 + k)u_{n+1}. \tag{5}$$

Recursively, v_{n+1} is expressed as

$$2v_{n+1} = (k - 1)u_{n+1} - (1 + k)u_{n+2}. \tag{6}$$

Substituting (5) and (6) into (4) gives the recursive relation for u_n, i.e.,

$$(1 + k)u_{k+2} + 2(1 - k)u_{n+1} + (1 + k)u_n = 0. \tag{7}$$

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We need initial conditions to solve this recursion. For this purpose, substituting the initial conditions $u_0 = u$ and $v_0 = 0$ into (3) gives u_1, and the initial conditions are

$$u_0 = u, \qquad u_1 = \frac{k-1}{1+k}u. \tag{8}$$

The following is a rather standard way to solve this kind of recursion [17]. The characteristic equation for (7) is

$$(1+k)r^2 + 2(1-k)r + (1+k) = 0. \tag{9}$$

The roots are

$$r_\pm = \frac{k-1}{1+k} \pm i\frac{2\sqrt{k}}{1+k} = \cos 2\beta \pm i \sin\beta = e^{\pm 2i\beta}, \cot\beta = \sqrt{k}, \tag{10}$$

where we have used the trigonometric identities

$$\cos 2x = \frac{\cot^2 x - 1}{\cot^2 x + 1}, \qquad \sin 2x = \frac{2\cot x}{\cot^2 x + 1}. \tag{11}$$

The solution to recursion (7) is therefore

$$u_n = \lambda_+ r_+^n + \lambda_- r_-^n, \tag{12}$$

where $\lambda_\pm$ are determined by the initial conditions in (8), i.e.,

$$\lambda_+ + \lambda_- = u, \qquad \lambda_+ r_+ + \lambda_- r_- = \frac{k-1}{1+k}u. \tag{13}$$

The solutions are

$$\lambda_+ = \lambda_- = \tfrac{1}{2}u, \tag{14}$$

Then (12) becomes

$$u_n = \tfrac{1}{2}\left(r_+^n + r_-^n\right)u = \tfrac{1}{2}\left(e^{2in\beta} + e^{-2in\beta}\right)u = u\cos 2n\beta. \tag{15}$$

As for v_n, it can be obtained from (5) as

$$\begin{aligned} v_n &= \tfrac{1}{2}\left[(k-1)u_n - (k+1)u_{n+1}\right] = \tfrac{1}{2}(k+1)\left(\frac{k-1}{k+1}u_n - u_{n+1}\right) \\ &= \tfrac{1}{2}(1+k)\left(\cos 2\beta \cos 2n\beta - \cos 2(n+1)\beta\right)u \\ &= \tfrac{1}{2}(1+k)u \sin 2\beta \sin 2n\beta = \sqrt{k}u \sin 2n\beta, \end{aligned} \tag{16}$$

where the third equality follows from (11) and (15). We will use (15) and (16) to calculate the total number of collisions in the following section.

4. *The total number of collisions and* π

We first focus on the last collision. For the left block, its last collision must be with the right block. Moreover, it will move to the left, that is, $u_n < 0$ if its nth collision is the last one. For the right block, however, its last collision may be either with the left block or with the wall. The upper part of Figure 2 illustrates the former. The right block also moves to the left but at a smaller speed than the left block. The lower figure illustrates the latter. The right block moves to the right and bounces from the wall. Thereafter it moves to the left again at a smaller speed than the left block, but it will not be able to catch up with it, which means that an additional block-block collision will not happen.

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A. *The last collision is block-block*

In this case, both blocks move to the left and the right block follows the left one at a smaller speed, which means

$$u_n < 0,\ v_n < 0,\ u_n < v_n. \tag{17}$$

To be specific, from (15), (16) and (17) we have

$$\cos 2n\beta < 0, \quad \sin 2n\beta < 0, \quad u\cos 2n\beta < \sqrt{k}u\sin 2n\beta. \tag{18}$$

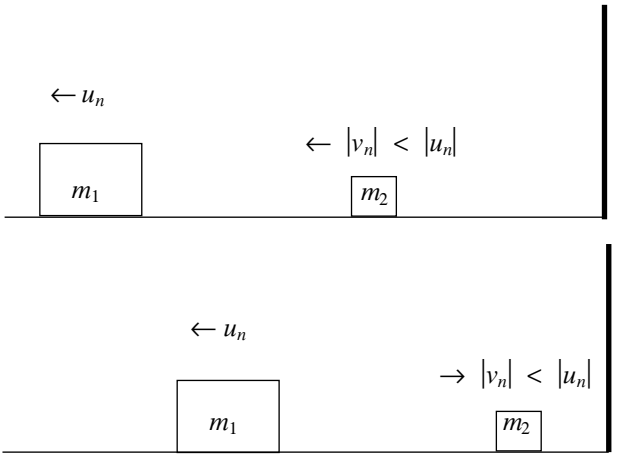


FIGURE 2: Two cases for the right block after the last block-block collision

Therefore

$$\pi < 2n\beta < \tfrac{3}{2}\pi, \qquad \tan 2n\beta < \frac{1}{\sqrt{k}} = \tan\beta, \tag{19}$$

and the last inequality in (19) implies

$$2n\beta - \pi < \beta. \tag{20}$$

Combining the first inequality in (19) with (20) gives

$$\pi < 2n\beta < \pi + \beta. \tag{21}$$

The total number N of both block-block and block-wall collisions is $N = 2n - 1$. Then from (21) we have

$$\frac{\pi}{\beta} - 1 < N < \frac{\pi}{\beta}. \tag{22}$$

Moreover, if $k = \dfrac{m_1}{m_2}$ is very large, then $\tan\beta = \dfrac{1}{\sqrt{k}}$ is very small, and we have

$$\beta \approx \tan\beta = \frac{1}{\sqrt{k}}. \tag{23}$$

Then (22) becomes

$$\pi\sqrt{k} - 1 < N < \pi\sqrt{k}. \tag{24}$$

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What makes people fascinated is the observation that an approximate value for π can be calculated from (24) by counting the total number of collisions, since,

$$\frac{N}{\sqrt{k}} < \pi < \frac{N+1}{\sqrt{k}}. \tag{25}$$

Equations (24) and (25) are the central results in this paper; they relate the important number π in mathematics with the total number of collisions in physics. For example, if $k = 10^2$, then $N = 31$; if $k = 10^6$, then $N = 3141$. It is worth pointing out that (13) in [8] is very close to (24). However, the focus in that paper is on an expression for the number of block-block collisions, which involves π. What is really interesting is to look at the problem the other way around: calculate an approximate value for π by counting the total number of collisions!

B. *The last collision is block-wall*

In this case the right block moves to the right at a smaller speed than the left block after the last block-block collision, which means

$$u_n < 0, \qquad v_n > 0, \qquad u_n < -v_n. \tag{26}$$

To be specific, from (15), (16) and (26) we have

$$\cos 2n\beta < 0, \qquad \sin 2n\beta > 0, \qquad u\cos 2n\beta < -\sqrt{k}u\sin 2n\beta. \tag{27}$$

Then,

$$\frac{\pi}{2} < 2n\beta < \pi, \qquad \tan 2n\beta > -\frac{1}{\sqrt{k}} = -\tan\beta, \tag{28}$$

and the last inequality in (28) implies

$$\pi - 2n\beta < \beta. \tag{29}$$

Combining the first inequality in (28) with (29) gives

$$\pi - \beta < 2n\beta < \pi. \tag{30}$$

The total number N of both block-block and block-wall collisions is not $2n - 1$ but $2n$, since we have an additional collision between the right block and the wall. Then from (30) we have

$$\frac{\pi}{\beta} - 1 < N < \frac{\pi}{\beta}, \tag{31}$$

which is the same as (22). Then (24) follows naturally.

5. *Summary and discussion*

The geometric approaches in [1, 6, 8] are truly elegant but rather challenging in transforming the physics problem into a geometric one. Once it becomes a geometric problem, it is easy to obtain the result. The natural algebraic approach is the matrix formalism in [7]. It is a bit demanding for most high school students, but it is a good exercise for those who have taken further mathematics. The approach in this paper is also algebraic but much less demanding; the method to solve the recursion is standard for students.

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ZHIWEI CHONG
Guangdong Country Garden School, Shunde District, Foshan, Guangdong, China
e-mail: *zhiwei.chong@qq.com*

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