RAMANUJAN'S CONTINUED FRACTION

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I. Introduction

The best of the theorems on continued fractions to be found in Ramanujan's manuscript note-book may be stated as follows:

Let
$$P=\Pi\Gamma\left(\frac{\alpha\pm\beta\pm\gamma\pm\delta\pm\epsilon+1}{2}\right) \quad (0,\ 2 \text{ or } 4 \text{ minus signs}),$$
$$Q=\Pi\Gamma\left(\frac{\alpha\pm\beta\pm\gamma\pm\delta\pm\epsilon+1}{2}\right) \quad (1 \text{ or } 3 \text{ minus signs}),$$

where there are eight gamma-functions in each product and the ambiguous signs are so chosen that the argument of each gamma-function contains one of the specified numbers of minus signs. Then

$$\frac{P-Q}{P+Q}=\frac{8\alpha\beta\gamma\delta\epsilon}{1\{2(\Sigma\alpha^4+1)-(\Sigma\alpha^2-1)^2-2^2\}}+\frac{64\Pi(\alpha^2-1^2)}{3\{2(\Sigma\alpha^4+1)-(\Sigma\alpha^2-5)^2-6^2\}}$$
$$+\frac{\Pi(\alpha^2-2^2)}{5\{2(\Sigma\alpha^4+1)-(\Sigma\alpha^2-13)^2-14^2\}}+\ldots,$$

provided that one of the numbers $\beta, \gamma, \delta, \epsilon$ is equal to $\pm n$, where n is a positive integer; and the products and sums on the right range over the numbers $\alpha, \beta, \gamma, \delta, \epsilon$.

If n were zero, the theorem would, of course, be trivial. When the continued fraction is written for brevity in the form

$$\frac{A_0}{b_0}+\frac{a_1}{b_1}+\frac{a_2}{b_2}+\ldots,$$

the denominator of the $(m+1)$th element is given by the formula*

$$b_m=(2m+1)[2(\Sigma\alpha^4+1)-(2m^2+2m+2)^2-\{\Sigma\alpha^2-(2m^2+2m+1)\}^2].$$

It may be mentioned that Ramanujan also asserted that a sufficient condition for the validity of his formula is that $\alpha=\pm n$, but this is incorrect; for, when $\alpha=\pm n$, the continued fraction is an elementary function of $\beta, \gamma, \delta, \epsilon$, while, in general, the expression on the left is not.

The reader who is familiar with the collection of theorems on continued fractions which is contained in Perron's treatise† will realize that the theorem dis-

* Through an error in copying which occurred when I previously published an enunciation of the theorem, *Journal London Math. Soc.* 6 (1931), 146, the numbers in b_2 were inadvertently given as 9 and 10 instead of 13 and 14.

† O. Perron, *Die Lehre von den Kettenbrüchen* (2. Aufl. 1929).

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covered by Ramanujan has to deal with a continued fraction much more elaborate than any continued fraction hitherto expressed in terms of the ordinary functions of analysis; the discovery seems an ample justification of Hardy's assertion that "his mastery of continued fractions was, on the formal side at any rate, beyond that of any mathematician in the world".

For some years I have been in possession of a proof of Ramanujan's theorem which I constructed on the lines of Dougall's proof by induction of his theorem on generalized hypergeometric functions. Since Hardy's announcement of Ramanujan's discovery of Dougall's theorem appeared in this periodical,* it seems not unfitting that my contribution towards the inauguration of the new series should consist in the communication of my proof of the theorem on continued fractions.

Just as it is apparently unknown whether Ramanujan's proof of Dougall's theorem was the same as Dougall's proof, so also I do not know whether my proof of the theorem on continued fractions is the same as Ramanujan's proof; but I have no doubt that, in the event of the two proofs being different, the structures of the two proofs are of the same degree of complexity

In § V I give two limiting forms of Ramanujan's theorem (one explicitly obtained by him, the other not), and I conclude the paper by giving in § VI a generalization of Ramanujan's theorem. Parallel results concerning generalized hypergeometric series make it fairly evident that a generalization on the lines which I give must exist, but I found that the task of constructing the exact form of the generalized formula involved rather more tedious analysis than would be realized from an inspection of the proof of the result.

II. Statement and proof of two lemmas

In order to effect the proof by induction of Ramanujan's theorem, I find it necessary to use two lemmas. The first is trivial, though I do not remember having seen it explicitly stated in print. The second, which involves a special case of a continued fraction discussed by Laguerre,† is a little more interesting; it is a special case of the continued fraction which represents the quotient of two hypergeometric series. I give an investigation by which the numerator and denominator of the last convergent are obtained in a compact form with a minimum of analysis.

The two lemmas are as follows.

LEMMA 1. *In the continued fraction*

$$1\,(x^2-1)+\frac{-1^2\,(x^2-1^2)^2}{3\,(x^2-3)}+\frac{-2^2\,(x^2-2^2)^2}{5\,(x^2-7)}+\frac{-3^2\,(x^2-3^2)^2}{7\,(x^2-13)}+\ldots,$$

* G. H. Hardy, "A chapter from Ramanujan's note-book", *Proc. Camb. Phil. Soc.* 21 (1923), 492–503.

† E. Laguerre, "Sur la fonction $\left(\frac{x+1}{x-1}\right)^{\omega}$", *Bull. Soc. Math. France*, 8 (1879), 36–52.

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in which the denominator of the $(m+1)$*th element is*

$$(2m+1)\{x^2-(m^2+m+1)\},$$

the numerator of the m*th convergent* is*

$$m!\,(x^2-1^2)(x^2-2^2)(x^2-3^2)\dots(x^2-m^2).$$

The truth of the lemma is evident for $m=1$, and for $m=2$, since

$$1.3\,(x^2-1)(x^2-3)-1^2(x^2-1^2)^2\equiv 2!\,(x^2-1^2)(x^2-2^2);$$

it is then immediately deducible by induction from the recurrence formula

$$p_{m+1}=(2m+1)\{x^2-(m^2+m+1)\}p_m-m^2(x^2-m^2)^2p_{m-1},$$

which is satisfied by taking

$$p_m=m!\,(x^2-1^2)(x^2-2^2)(x^2-3^2)\dots(x^2-m^2),$$

in view of the obvious identity

$$(m+1)\{x^2-(m+1)^2\}\equiv(2m+1)\{x^2-(m^2+m+1)\}-m(x^2-m^2).$$

LEMMA 2. *The numerator and denominator of the last convergent of the continued fraction*

$$(-1)+\frac{n^2-1^2}{(-3)}+\frac{n^2-2^2}{(-5)}+\frac{n^2-3^2}{(-7)}+\dots+\frac{n^2-(n-1)^2}{(1-2n)}$$

are equal to $(-)^n(2n-1)(2n-2)\dots(n+1)n$

and $(-)^{n-1}(2n-1)(2n-2)\dots(n+2)(n+1)$

respectively.

This is most easily proved by induction, the continued fraction being built up from the lower end. Write

$$k_m=(-)^{n-m}(2n-1)(2n-2)\dots(n+m+1)(n+m)$$

for brevity, and assume that

$$\frac{n^2-m^2}{(-2m-1)}+\frac{n^2-(m+1)^2}{(-2m-3)}+\dots+\frac{n^2-(n-1)^2}{(1-2n)}=\frac{-(n-m)k_m}{k_m};$$

then

$$\frac{n^2-(m-1)^2}{(1-2m)}+\frac{n^2-m^2}{(-2m-1)}+\dots+\frac{n^2-(n-1)^2}{(1-2n)}$$

$$=\frac{n^2-(m-1)^2}{(1-2m)}+\frac{-(n-m)k_m}{k_m}$$

$$=\frac{(n-m+1)(n+m-1)k_m}{-(n+m-1)k_m}=\frac{-(n-m+1)k_{m-1}}{k_{m-1}}.$$

* Here and elsewhere it is supposed that the "numerators" and "denominators" of the convergents of a continued fraction are calculated without cancelling common factors, if such exist. Thus the numerators of the convergents of the continued fraction

$$b_0+\frac{a_1}{b_1}+\frac{a_2}{b_2}+\frac{a_3}{b_3}+\dots$$

are $b_0,\quad b_0b_1+a_1,\quad b_0b_1b_2+a_1b_2+a_2b_0,\quad \dots$

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This establishes the induction; the lemma then follows by evaluating

$$-1+\frac{-(n-1)k_1}{k_1}.$$

III. A RESTATEMENT OF RAMANUJAN'S THEOREM

We can now being the attack on the main theorem. The first step is to evolve a compact notation for the expressions which Ramanujan asserts to be equal. Consider the continued fraction

$$b_0+\frac{a_1}{b_1}+\frac{a_2}{b_2}+\frac{a_3}{b_3}+\dots$$

in which

$$a_m=64\,(\alpha^2-m^2)\,(\beta^2-m^2)\,(\gamma^2-m^2)\,(\delta^2-m^2)\,(n^2-m^2),$$
$$b_m=(2m+1)\,[2\,(\alpha^4+\beta^4+\gamma^4+\delta^4+n^4+1)-(2m^2+2m+2)^2$$
$$-\{\alpha^2+\beta^2+\gamma^2+\delta^2+n^2-(2m^2+2m+1)\}^2].$$

The numerator and the denominator of the nth convergent of this continued fraction will be called

$$N_n(\alpha,\beta,\gamma,\delta),\quad D_n(\alpha,\beta,\gamma,\delta)$$

respectively, so that $N_n(\alpha,\beta,\gamma,\delta)$ and $D_n(\alpha,\beta,\gamma,\delta)$ are polynomials in the four variables α, β, γ, δ, of degrees $4n$ and $4n-4$ respectively.

It is to be observed that, when any one of the variables, say δ, is equal to $\pm m$, where m is a positive integer less than n, then

$$\frac{N_n(\alpha,\beta,\gamma,\pm m)}{N_m(\alpha,\beta,\gamma,n)}\quad\text{and}\quad\frac{D_n(\alpha,\beta,\gamma,\pm m)}{D_m(\alpha,\beta,\gamma,n)}$$

are both equal to the polynomial in α, β, γ, of degree $4(n-m)$, which is the numerator of the last convergent of the continued fraction

$$b_m+\frac{a_{m+1}}{b_{m+1}}+\dots+\frac{a_{n-1}}{b_{n-1}}$$

(in which δ has been replaced by $\pm m$); this polynomial will be called $E_{m,n}(\alpha,\beta,\gamma)$.

Next consider

$$8\alpha\beta\gamma\delta n\frac{P+Q}{P-Q},$$

where P and Q are defined as in the enunciation due to Ramanujan, save for the substitution of n for ϵ.

It is evident that P and Q are proportional to the expressions

$$\prod_{r=0}^{n-1}\{(\alpha+\beta+\gamma+\delta+n-2r-1)\,(\alpha+\beta-\gamma-\delta+n-2r-1)$$
$$\times(\alpha-\beta-\gamma+\delta+n-2r-1)\,(\alpha-\beta+\gamma-\delta+n-2r-1)\}$$

and

$$\prod_{r=0}^{n-1}\{(\alpha+\beta+\gamma-\delta+n-2r-1)\,(\alpha+\beta-\gamma+\delta+n-2r-1)$$
$$\times(\alpha-\beta+\gamma+\delta+n-2r-1)\,(\alpha-\beta-\gamma-\delta+n-2r-1)\}$$

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respectively. These expressions are polynomials of degree $4n$ in the four variables $\alpha, \beta, \gamma, \delta$. The expressions are obviously interchanged by the substitution of any one of $-\beta, -\gamma, -\delta$ for β, γ, δ respectively; they are also interchanged by the substitution of $-\alpha$ for α, as may be seen by making this change in the second expression and then (by writing $n-1-r$ for r) reversing the order of the factors in each of the four products composing the expression; the effect of the change and the reversal is to give the first expression.

Moreover, the expressions are symmetrical in $\alpha, \beta, \gamma, \delta$; the symmetry in β, γ, δ is evident, and symmetry in the four variables is established by reversing the order of the factors in appropriate products.

We now write

$$M_n(\alpha, \beta, \gamma, \delta), \quad C_n(\alpha, \beta, \gamma, \delta)$$

for the sum of the two expressions and for their difference divided by $8n\alpha\beta\gamma\delta$. Then $M_n(\alpha, \beta, \gamma, \delta)$ is a symmetrical polynomial of degree $4n$ in the variables $\alpha, \beta, \gamma, \delta$, and it is an even function of each variable; while $C_n(\alpha, \beta, \gamma, \delta)$ is a symmetrical polynomial of degree $4n-4$ in the same variables, and it is also an even function of each variable.*

If m is a positive integer less than n, it is easy to see that

$$\frac{M_n(\alpha, \beta, \gamma, \pm m)}{M_m(\alpha, \beta, \gamma, n)} \quad \text{and} \quad \frac{C_n(\alpha, \beta, \gamma, \pm m)}{C_m(\alpha, \beta, \gamma, n)}$$

are both equal to

$$\prod_{r=0}^{n-m-1} \{(\alpha+\beta+\gamma-m+n-2r-1)(\alpha-\beta-\gamma-m+n-2r-1) \times (\alpha+\beta-\gamma-m+n-2r-1)(\alpha-\beta+\gamma-m+n-2r-1)\}.$$

This product, which will be called $F_{m,n}(\alpha, \beta, \gamma)$, is a polynomial of degree $4(n-m)$ in α, β, γ.

In the notation which has now been introduced, Ramanujan's theorem assumes the form

$$\frac{M_n(\alpha, \beta, \gamma, \delta)}{C_n(\alpha, \beta, \gamma, \delta)} = \frac{N_n(\alpha, \beta, \gamma, \delta)}{D_n(\alpha, \beta, \gamma, \delta)}.$$

IV. Proof of Ramanujan's theorem by induction

We proceed to prove that the polynomial

$$D_n(\alpha, \beta, \gamma, \delta) M_n(\alpha, \beta, \gamma, \delta) - C_n(\alpha, \beta, \gamma, \delta) N_n(\alpha, \beta, \gamma, \delta)$$

which is symmetrical and of degree $8n-4$ in $\alpha, \beta, \gamma, \delta$ is identically zero. We assume that this theorem has been proved for general $\alpha, \beta, \gamma, \delta$ when n is replaced by each of the integers $1, 2, 3, \ldots, n-1$.

* The properties of the original expressions show that their difference vanishes when $\alpha = 0$, and so the difference contains α as a factor; similarly for β, γ, δ. Hence $C_n(\alpha, \beta, \gamma, \delta)$ is an integral function.

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We then have, for $m = 1, 2, 3, \ldots, n-1$,

$$\begin{aligned} & D_n(\alpha, \beta, \gamma, \pm m) M_n(\alpha, \beta, \gamma, \pm m) - C_n(\alpha, \beta, \gamma, \pm m) N_n(\alpha, \beta, \gamma, \pm m) \\ & \quad = E_{m,n}(\alpha, \beta, \gamma) F_{m,n}(\alpha, \beta, \gamma) \\ & \quad \times [D_m(\alpha, \beta, \gamma, n) M_m(\alpha, \beta, \gamma, n) - C_m(\alpha, \beta, \gamma, n) N_m(\alpha, \beta, \gamma, n)] \\ & \quad = 0, \end{aligned}$$

by hypothesis. Hence the general polynomial under consideration of degree $8n-4$ in $\alpha, \beta, \gamma, \delta$ contains $\delta \mp m$ as factors ($m = 1, 2, 3, \ldots, n-1$), and by symmetry it also contains $\alpha \mp m$, $\beta \mp m$, $\gamma \mp m$ as factors. These factors are all distinct for general $\alpha, \beta, \gamma, \delta$, and so we have

$$\begin{aligned} & D_n(\alpha, \beta, \gamma, \delta) M_n(\alpha, \beta, \gamma, \delta) - C_n(\alpha, \beta, \gamma, \delta) N_n(\alpha, \beta, \gamma, \delta) \\ & \qquad = f_n(\alpha, \beta, \gamma, \delta) \prod_{m=1}^{n-1} \{(\alpha^2 - m^2)(\beta^2 - m^2)(\gamma^2 - m^2)(\delta^2 - m^2)\}, \end{aligned}$$

where $f_n(\alpha, \beta, \gamma, \delta)$ is a polynomial in $\alpha, \beta, \gamma, \delta$ which evidently is a symmetrical function of degree four in these variables and is an even function of each of them.

Hence $f_n(\alpha, \beta, \gamma, \delta)$ is expressible in the form

$$\begin{aligned} f_n(\alpha, \beta, \gamma, \delta) \equiv & \lambda_n(\alpha^4 + \beta^4 + \gamma^4 + \delta^4) - \mu_n(\alpha^2 + \beta^2 + \gamma^2 + \delta^2)^2 \\ & + \rho_n(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) + \sigma_n, \end{aligned}$$

where $\lambda_n, \mu_n, \rho_n, \sigma_n$ are independent of $\alpha, \beta, \gamma, \delta$

Now consider the special set of values

$$\alpha = \xi + \tfrac{1}{2}, \qquad \beta = \xi - \tfrac{1}{2},$$
$$\gamma = n, \qquad \delta = 0,$$

with ξ arbitrary; we shall show that, for this set of values, both of the functions $M_n(\alpha, \beta, \gamma, \delta)$, $N_n(\alpha, \beta, \gamma, \delta)$ vanish.

Evidently these values make

$$\alpha - \beta - \gamma + \delta + n - 1 = 0, \quad \alpha - \beta - \gamma - \delta + n - 1 = 0,$$

and so

$$M_n(\xi + \tfrac{1}{2}, \xi - \tfrac{1}{2}, n, 0) = 0.$$

Moreover they make

$$\begin{aligned} a_m &= 64(\alpha^2 - m^2)(\beta^2 - m^2)(\gamma^2 - m^2)(\delta^2 - m^2)(n^2 - m^2) \\ &= -64m^2(n^2 - m^2)^2(m + \tfrac{1}{2} + \xi)(m - \tfrac{1}{2} + \xi)(m + \tfrac{1}{2} - \xi)(m - \tfrac{1}{2} - \xi), \\ b_m &= (2m+1)[2(\alpha^4 + \beta^4 + \gamma^4 + \delta^4 + n^4 + 1) - (2m^2 + 2m + 2)^2 \\ &\qquad - \{\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + n^2 - (2m^2 + 2m + 1)\}^2] \\ &= 8(2m+1)\{n^2 - (m^2 + m + 1)\}(m + \tfrac{1}{2} + \xi)(m + \tfrac{1}{2} - \xi), \end{aligned}$$

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whence it follows that the numerator of the mth convergent for this special form of the continued fraction

$$b_0+\frac{a_1}{b_1}+\frac{a_2}{b_2}+\dots$$

is equal to $$2^{3m}\prod_{r=0}^{m-1}\{(r+\tfrac{1}{2}+\xi)(r+\tfrac{1}{2}-\xi)\}$$

multiplied by the numerator of the mth convergent of the continued fraction

$$b_0'+\frac{a_1'}{b_1'}+\frac{a_2'}{b_2'}+\dots,$$

for which $$a_m'=-m^2(n^2-m^2)^2,$$
$$b_m'=(2m+1)\{n^2-(m^2+m+1)\},$$

the last continued fraction being the result of simplifying the preceding continued fraction by cancelling factors.

In particular $$N_n(\xi+\tfrac{1}{2},\xi-\tfrac{1}{2},n,0)$$

is equal to $$2^{3n}\prod_{r=0}^{n-1}\{(r+\tfrac{1}{2}+\xi)(r+\tfrac{1}{2}-\xi)\}$$

multiplied by the numerator of the nth convergent of the continued fraction

$$1(n^2-1)+\frac{-1^2(n^2-1^2)^2}{3(n^2-3)}+\frac{-2^2(n^2-2^2)^2}{5(n^2-7)}+\frac{-3^2(n^2-3^2)^2}{7(n^2-13)}+\dots.$$

From Lemma 1 it now follows that, for all ξ,

$$N_n(\xi+\tfrac{1}{2},\xi-\tfrac{1}{2},n,0)=0.$$

Combining this with the result

$$M_n(\xi+\tfrac{1}{2},\xi-\tfrac{1}{2},n,0)=0,$$

already proved, and observing that

$$\prod_{m=1}^{n-1}[\{(\xi+\tfrac{1}{2})^2-m^2\}\{(\xi-\tfrac{1}{2})^2-m^2\}(n^2-m^2)(-m^2)]\neq 0$$

for general values of ξ, we infer from the definition of $f_n(\alpha,\beta,\gamma,\delta)$ that

$$f_n(\xi+\tfrac{1}{2},\xi-\tfrac{1}{2},n,0)=0$$

for all ξ. That is to say,

$$\lambda_n(2\xi^4+3\xi^2+\tfrac{1}{8}+n^4)-\mu_n(2\xi^2+\tfrac{1}{2}+n^2)^2+\rho_n(2\xi^2+\tfrac{1}{2}+n^2)+\sigma_n\equiv 0$$

for all ξ, and therefore

$$\lambda_n=2\mu_n,$$
$$\rho_n=2(n^2-1)\mu_n,$$
$$\sigma_n=-(3n^4-2n^2-1)\mu_n.$$

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We have now proved that

$$D_n(\alpha,\beta,\gamma,\delta)\,M_n(\alpha,\beta,\gamma,\delta)-C_n(\alpha,\beta,\gamma,\delta)\,N_n(\alpha,\beta,\gamma,\delta),$$
$$=\mu_n\{2(\alpha^4+\beta^4+\gamma^4+\delta^4)-(\alpha^2+\beta^2+\gamma^2+\delta^2)^2$$
$$+2(n^2-1)(\alpha^2+\beta^2+\gamma^2+\delta^2)-(3n^4-2n^2-1)\}$$
$$\times\prod_{m=1}^{n-1}\{(\alpha^2-m^2)(\beta^2-m^2)(\gamma^2-m^2)(\delta^2-m^2)\}.$$

The final step is to put $\beta=\alpha$, $\delta=-\gamma$, and then to compare the terms of degree $8n-4$ in α and γ on both sides of the last result. The only such term on the right is

$$-8\mu_n(\alpha\gamma)^{4n-2}.$$

The terms of highest degree in α and γ in

$$N_n(\alpha,\alpha,\gamma,-\gamma) \quad \text{and} \quad D_n(\alpha;\alpha,\gamma,-\gamma)$$

are the numerator and denominator of the nth convergent of the continued fraction

$$(-1.8\alpha^2\gamma^2)+\frac{64\alpha^4\gamma^4(n^2-1^2)}{(-3.8\alpha^2\gamma^2)}+\frac{64\alpha^4\gamma^4(n^2-2^2)}{(-5.8\alpha^2\gamma^2)}+\ldots.$$

With the help of Lemma 2 it is easily seen that these are

$$\frac{(2n-1)!}{(n-1)!}(-8\alpha^2\gamma^2)^n, \quad \frac{(2n-1)!}{n!}(-8\alpha^2\gamma^2)^{n-1}$$

respectively.

Further

$$(-)^n\prod_{r=0}^{n-1}\{(2\alpha+n-2r-1)(-2\gamma+n-2r-1)\}^2=(-4\alpha^2\gamma^2)^n+\ldots,$$

$$\prod_{r=0}^{n-1}\{(2\alpha+2\gamma+n-2r-1)(2\alpha-2\gamma+n-2r-1)(n-2r-1)^2\}=c(4\alpha^2-4\gamma^2)^n+\ldots,$$

where $$c=\{(n-1)(n-3)(n-5)\ldots(1-n)\}^2.$$

Hence the terms of highest degree in $M_n(\alpha,\alpha,\gamma,-\gamma)$ and $C_n(\alpha,\alpha,\gamma,-\gamma)$ are

$$(-4\alpha^2\gamma^2)^n, \quad \frac{(-4\alpha^2\gamma^2)^{n-1}}{2n}$$

respectively.

By comparing coefficients of $(\alpha\gamma)^{4n-2}$ we therefore deduce that

$$-8\mu_n=\frac{(2n-1)!}{n!}(-8)^{n-1}(-4)^n-\frac{(2n-1)!}{(n-1)!}(-8)^n\frac{(-4)^{n-1}}{2n}$$
$$=0,$$

and the result that

$$D_n(\alpha,\beta,\gamma,\delta)\,M_n(\alpha,\beta,\gamma,\delta)-C_n(\alpha,\beta,\gamma,\delta)\,N_n(\alpha,\beta,\gamma,\delta)\equiv 0$$

is now proved on the hypothesis that it is true when n is replaced by m, where m may assume each of the values 1, 2, 3, ..., $n-1$.

Since there is no difficulty in proving that the result is true when $n=1$, the proof of Ramanujan's theorem by induction is now complete.

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V. Two limiting cases of Ramanujan's theorem

Suppose that, in Ramanujan's result, it is one of β, γ, δ that is an integer. Divide both sides of his formula by ϵ and then make $\epsilon \to 0$; this process presents no theoretical difficulties. We thus find that

$$\begin{aligned}
&\psi(\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta+\tfrac{1}{2}\gamma+\tfrac{1}{2}\delta+\tfrac{1}{2})+\psi(\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta-\tfrac{1}{2}\gamma-\tfrac{1}{2}\delta+\tfrac{1}{2})\\
&+\psi(\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta+\tfrac{1}{2}\gamma-\tfrac{1}{2}\delta+\tfrac{1}{2})+\psi(\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta-\tfrac{1}{2}\gamma+\tfrac{1}{2}\delta+\tfrac{1}{2})\\
&-\psi(\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta+\tfrac{1}{2}\gamma+\tfrac{1}{2}\delta+\tfrac{1}{2})-\psi(\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta-\tfrac{1}{2}\gamma+\tfrac{1}{2}\delta+\tfrac{1}{2})\\
&-\psi(\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta+\tfrac{1}{2}\gamma-\tfrac{1}{2}\delta+\tfrac{1}{2})-\psi(\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta-\tfrac{1}{2}\gamma-\tfrac{1}{2}\delta+\tfrac{1}{2})\\
=\ &\frac{16\alpha\beta\gamma\delta}{1\{2(\Sigma\alpha^4+1)-(\Sigma\alpha^2-1)^2-2^2\}}+\frac{-64\,.\,1^2\Pi(\alpha^2-1^2)}{3\{2(\Sigma\alpha^4+1)-(\Sigma\alpha^2-5)^2-6^2\}}\\
&+\frac{-64\,.\,2^2\Pi(\alpha^2-2^2)}{5\{2(\Sigma\alpha^4+1)-(\Sigma\alpha^2-13)^2-14^2\}}+\ldots,
\end{aligned}$$

where ψ denotes the logarithmic derivate of the Gamma-function, the sums and products on the right range over α, β, γ, δ, and it is supposed that one of β, γ, δ is a positive integer. This result was not stated by Ramanujan, though his note-book contains numerous special cases of it.

To obtain another limiting case, put $\alpha = x+\epsilon$ in the general result and then make $\epsilon \to \infty$. We thus get Ramanujan's formula

$$\begin{aligned}
\frac{Q_1-P_1}{Q_1+P_1} = \ &\frac{2\beta\gamma\delta}{1(x^2-\beta^2-\gamma^2-\delta^2+1)}+\frac{4(\beta^2-1^2)(\gamma^2-1^2)(\delta^2-1^2)}{3(x^2-\beta^2-\gamma^2-\delta^2+5)}\\
&+\frac{4(\beta^2-2^2)(\gamma^2-2^2)(\delta^2-2^2)}{5(x^2-\beta^2-\gamma^2-\delta^2+13)}+\ldots,
\end{aligned}$$

where

$$P_1=\Gamma\left(\frac{x+\beta+\gamma+\delta+1}{2}\right)\Gamma\left(\frac{x+\beta-\gamma-\delta+1}{2}\right)\Gamma\left(\frac{x-\beta+\gamma-\delta+1}{2}\right)\Gamma\left(\frac{x-\beta-\gamma+\delta+1}{2}\right),$$

$$Q_1=\Gamma\left(\frac{x-\beta+\gamma+\delta+1}{2}\right)\Gamma\left(\frac{x+\beta-\gamma+\delta+1}{2}\right)\Gamma\left(\frac{x+\beta+\gamma-\delta+1}{2}\right)\Gamma\left(\frac{x-\beta-\gamma-\delta+1}{2}\right).$$

If it is one of β, γ, δ that is a positive integer, this passage to the limit presents no theoretical difficulties. If, however, none of them is a positive integer and ϵ is made to tend to infinity through positive integral values, the situation is more complicated. I shall not linger to discuss the validity of the passage to the limit; for any fixed values of β, γ, δ, the passage is presumably justifiable when x is subject to the restriction that its real part exceeds some definite value.

VI. The generalization of Ramanujan's continued fraction

It is possible to generalize Ramanujan's theorem in the way in which Heine generalized theorems about the ordinary hypergeometric series by means of his basic hypergeometric series.

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In order to avoid the introduction of numerous fractional indices, it is desirable to denote the base by q^2 instead of the more usual q. I accordingly define a function $G(x)$ by the infinite product

$$\frac{1}{G(x)} = \prod_{m=0}^{\infty} (1 - xq^{2m+1}). \quad (|q| < 1)$$

The generalized theorem then runs as follows:

Let

$$P = G(\alpha\beta\gamma\delta\epsilon)\, G\left(\frac{\alpha\beta\gamma}{\delta\epsilon}\right) G\left(\frac{\alpha\beta\delta}{\gamma\epsilon}\right) G\left(\frac{\alpha\gamma\delta}{\beta\epsilon}\right) G\left(\frac{\alpha\beta\epsilon}{\gamma\delta}\right) G\left(\frac{\alpha\gamma\epsilon}{\beta\delta}\right) G\left(\frac{\alpha\delta\epsilon}{\beta\gamma}\right) G\left(\frac{\alpha}{\beta\gamma\delta\epsilon}\right),$$

$$Q = G\left(\frac{\alpha\beta\gamma\delta}{\epsilon}\right) G\left(\frac{\alpha\beta\gamma\epsilon}{\delta}\right) G\left(\frac{\alpha\beta\delta\epsilon}{\gamma}\right) G\left(\frac{\alpha\gamma\delta\epsilon}{\beta}\right) G\left(\frac{\alpha\beta}{\gamma\delta\epsilon}\right) G\left(\frac{\alpha\gamma}{\beta\delta\epsilon}\right) G\left(\frac{\alpha\delta}{\beta\gamma\epsilon}\right) G\left(\frac{\alpha\epsilon}{\beta\gamma\delta}\right).$$

Then, provided that one of the numbers $\beta, \gamma, \delta, \epsilon$ is of the form $q^{\pm n}$ $(n = 1, 2, 3, \ldots)$,

$$\frac{P-Q}{P+Q} = \frac{A_0}{b_0} + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \ldots,$$

where

$$A_0 = (q + q^{-1}) \prod (\alpha - \alpha^{-1}),$$

$$a_m = (q^{m+1} + q^{-m-1})(q^{m-1} + q^{1-m}) \prod (\alpha^2 + \alpha^{-2} - q^{2m} - q^{-2m}),$$

$$b_m = (q^{2m+1} - q^{-2m-1})\{(q^m + q^{-m})(q^{m+1} + q^{-m-1})(\Sigma\alpha^2 + \Sigma\alpha^{-2} + 2) - \prod (\alpha + \alpha^{-1}) - (q + q^{-1})(q^m + q^{-m})(q^{m+1} + q^{-m-1})(q^{2m+1} + q^{-2m-1})\}$$

the products and sums ranging over the numbers $\alpha, \beta, \gamma, \delta, \epsilon$.

It is to be remarked that, when, say, ϵ is of the form $q^{\pm n}$, the quotient

$$G(\alpha\beta\gamma\delta\epsilon)/G\left(\frac{\alpha\beta\gamma\delta}{\epsilon}\right)$$

is expressible as a rational function of q, and the same is true of other appropriately chosen quotients of G-functions occurring in the quotient P/Q. The theorem can therefore be written in such a way as to assert the equality of two rational functions of q, and in such an assertion the limitation that $|q| < 1$ is clearly superfluous.

It seems unnecessary to write out a formal proof of the generalized theorem. The reader should have no difficulty in constructing a proof on the lines adopted for proving Ramanujan's theorem; I therefore confine myself to a statement of the appropriate generalizations of Lemmas 1 and 2; they are as follows.

LEMMA 3. *In the continued fraction*

$$d_0 + \frac{c_1}{d_1} + \frac{c_2}{d_2} + \ldots$$

let $c_m = -(q^{m-1} + q^{1-m})(q^{m+1} + q^{-m-1})(q^m - q^{-m})^2 (x^2 + x^{-2} - q^{2m} - q^{-2m})^2,$

$$d_m = (q^{2m+1} - q^{-2m-1})\{(q + q^{-1})(q^m + q^{-m})(q^{m+1} + q^{-m-1}) - 2(x + x^{-1})^2\}.$$

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Then the numerator of the nth convergent of the continued fraction is

$$(-)^n \prod_{m=1}^{n} \{(q^{m-1}+q^{1-m})(q^m-q^{-m})(x^2+x^{-2}-q^{2m}-q^{-2m})\}.$$

This is easily deduced by induction from the recurrence formula which connects the numerators of three consecutive convergents.

LEMMA 4. *In the continued fraction*

$$\frac{C_{0,n}}{d_{0,n}}+\frac{c_{1,n}}{d_{1,n}}+\frac{c_{2,n}}{d_{2,n}}+\ldots+\frac{c_{n-1,n}}{d_{n-1,n}}$$

let

$$C_{0,n}=(q+q^{-1})(q^n-q^{-n}),$$

$$c_{m,n}=(q^{m-1}+q^{1-m})(q^{m+1}+q^{-m-1})(q^{2n}+q^{-2n}-q^{2m}-q^{-2m}),$$

$$d_{m,n}=-(q^{2m+1}-q^{-2m-1})(q^n+q^{-n}).$$

Then the numerator and denominator of the last convergent of the continued fraction are equal to

$$(-)^{n-1}\prod_{r=1}^{n}(q^r+q^{-r}).\prod_{r=n}^{2n-1}(q^r-q^{-r})$$

and

$$(-)^{n}\prod_{r=1}^{n}(q^r+q^{-r}).\prod_{r=n}^{2n-1}(q^r-q^{-r})$$

respectively.

This lemma may be established by proving by induction that the numerator and denominator of the last convergent of the continued fraction

$$\frac{c_{m,n}}{d_{m,n}}+\frac{c_{m+1,n}}{d_{m+1,n}}+\ldots+\frac{c_{n-1,n}}{d_{n-1,n}}$$

are equal to

$$(-)^{n-m+1}(q^{m-1}+q^{1-m})(q^{n-m}-q^{m-n})\prod_{r=m+1}^{n}(q^r+q^{-r}).\prod_{r=n+m}^{2n-1}(q^r-q^{-r})$$

and

$$(-)^{n-m}\prod_{r=m+1}^{n}(q^r+q^{-r}).\prod_{r=n+m}^{2n-1}(q^r-q^{-r})$$

respectively.

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