HILBERT TRANSFORMS AND ALMOST PERIODIC FUNCTIONS

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INTRODUCTION

1. The Hilbert transform, Hf, of a function† f is defined by $Hf = g$, where

$$g(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(t)}{t-x} dt, \tag{1·1}$$

P denoting the Cauchy principal value and the integral being assumed to exist in some sense. When f is suitably restricted, Hf exists and

$$H^2 f = H(Hf) = -f. \tag{1·2}$$

In the first part of Theorem 1 sufficient conditions are given for the validity of (1·2) rather more general than those of Wood ((6), p. 31). The present proof is based on the well-known condition of Riesz for the validity of (1·2), namely, that f is $L^p(-\infty, \infty)$ for some $p > 1$, and on the 'Parseval' relation (Lemma 3, (*b*)), which was used in a similar way by Hardy ((3), p. 110).

An inequality is also given in Theorem 1. In Part 2 the theorem is applied to obtain a result concerning almost periodic functions, which is described in §§ 2, 3.

2. Denote by L_0^p, where $p > 1$, the Banach space of functions f such that f is of period 2π, f is $L^p(0, 2\pi)$ and the mean value $M(f)$ is zero, the norm of f being defined by

$$\|f\| = \{M(|f|^p)\}^{1/p}, \tag{2·1}$$

where

$$M(f) = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx,$$

$$M(f) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(x)\, dx. \tag{2·2}$$

A function $f(x)$ of L_0^p has a Fourier series of the form

$$\sum_1^{\infty} (a_n \cos nx + b_n \sin nx).$$

Riesz proved ((4), pp. 218, 225–7; see also Zygmund (8), p. 147, or (9), p. 253) that the conjugate or allied series

$$\sum_1^{\infty} (b_n \cos nx - a_n \sin nx)$$

† In §§ 1, 2 and in Part 1 a function may be undefined in a set of measure zero; two functions h_1, h_2 equal almost everywhere are regarded as identical, and we write $h_1 = h_2$.

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is the Fourier series of a function, say $g(x)$, also of L_0^p, that

$$g(x) = \frac{1}{2\pi} P \int_0^{2\pi} f(t) \cot \frac{t-x}{2} dt, \tag{2·3}$$

and that the transformation T with domain L_0^p, defined by $Tf = g$, which is clearly additive and homogeneous, is a bounded linear transformation:

$$\|Tf\| \leqslant A_p \|f\|, \tag{2·4}$$

where A_p, the norm of T, is a finite number depending only on p.

Inspection of the conjugate of the Fourier series of Tf shows that $T^2 f = -f$. From this it follows that the range of T is the whole of L_0^p and that the equality $Tf_1 = Tf_2$ implies that $f_1 = f_2$. Thus T is a one-one bounded linear transformation of L_0^p onto itself.

Since the function f is periodic and has mean value zero, it has a bounded indefinite integral, and therefore by Lemma 1 (i) the integral (1·1) is convergent at $\pm\infty$. Hence, on account of the periodicity of f, (2·3) is the same as (1·1). Thus $T = H$.

3. The second object of this paper is to prove Theorem 2, which shows that H (or T) may be extended, without increase of norm, to a one-one bounded linear transformation of the wider space B_0^p onto itself, where B_0^p is the space of functions f such that f has mean value zero and is almost periodic in the sense of Besicovitch† with index p. Here the mean value is defined by (2·2) and the norm by (2·1). Formulae similar to (1·1) and (1·2) are also given in Theorem 2.

We have

$$U_0 \subset S_0^p \subset B_0^p,$$

where U_0 and S_0^p are the subsets of Bohr and Stepanoff functions. The operator H does not transform either U_0 or S_0^p into itself. In fact, Bochner ((2), p. 445) has given an example of a function f of U_0 the conjugate of whose Fourier series is not the Fourier series of a function of S_0^p for any p greater than or equal to 1.

PART 1

4. It is supposed that any given function is locally integrable, i.e. integrable in the sense of Lebesgue over every finite interval in which it is almost everywhere defined. The phrase 'locally L^p' is used in a similar sense. An integral in which one of the limits of integration is infinite is to be interpreted as the limit of an integral over a finite interval. The expressions

$$\int_{-\infty}^{\infty} \frac{h(t)}{t-x} dt, \quad \int_{-\infty}^{\infty} \frac{h(t)}{t-x} dt \quad (\mathrm{C}, 1)$$

will sometimes be used as abbreviations for

$$\lim_{\delta \to 0+} \lim_{T \to \infty} \left(\int_{-T}^{x-\delta} + \int_{x+\delta}^{T} \right) \left(1 - \frac{|t|}{T}\right)^r \frac{h(t)}{t-x} dt \quad (r = 0, 1).$$

We also write (cf. (2·2)) $$\overline{M}(h) = \overline{\lim_{T \to \infty}} \frac{1}{2T} \int_{-T}^{T} h(x)\, dx.$$

† Besicovitch (1), Chapter II; Zaanen (7), pp. 103–8.

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5. In Lemma 1 are collected a number of simple consequences of the second mean-value theorem of the integral calculus which may be stated thus:

If $d(t)$ is bounded, positive and decreasing in the finite interval $\alpha \leqslant t \leqslant \beta$, then

$$\int_\alpha^\beta d(t)\,h(t)\,dt = d(\alpha)\int_\alpha^\xi h(t)\,dt$$

for some ξ such that $\alpha \leqslant \xi \leqslant \beta$.

LEMMA 1. *Let*

$$I \equiv \int_c^\infty h(t)\,dt \quad (c>0).$$

(i) *If $m(t)$ is bounded and tends monotonically to zero as $t \to \infty$, and if I is bounded,† then*

$$\int_c^\infty m(t)\,h(t)\,dt$$

is convergent.

(ii) *If $m(t)$ is bounded and monotonic and if I is convergent, then the last integral is convergent.*

(iii) *If $d(t)$ is bounded, positive and decreasing, and if I is convergent, then*

$$\int_c^\infty d(t)\,h(t)\,dt = d(c)\int_c^\xi h(t)\,dt$$

for some ξ such that $c \leqslant \xi \leqslant \infty$.

(iv) *If $-c < a < b < c$ and if I is convergent then*

$$\int_c^\infty \frac{t^2 h(t)}{t^2-x^2}\,dt$$

is uniformly convergent in $a \leqslant x \leqslant b$.

(v) *If I is convergent then so is*

$$\int_c^\infty \frac{t h(t)}{t^2-x^2}\,dt \quad (0<x<c).$$

(vi) *If I is convergent, then, as $T \to \infty$,*

$$\int_T^\infty \frac{h(t)}{t}\,dt = o\left(\frac{1}{T}\right).$$

(vii) *If, as $T \to \infty$,*

$$\int_T^\infty h(t)\,dt = o\left(\frac{1}{T}\right),$$

then

$$\int_T^\infty \frac{t^2 h(t)}{t^2-x^2}\,dt = o\left(\frac{1}{T}\right).$$

(viii) *If*

$$\int_c^\infty \frac{h_1(t)}{t}\,dt, \quad \int_c^\infty \frac{h_2(t)}{t^2}\,dt$$

are convergent, then so are

$$\int_c^\infty \frac{h_1(t) \pm h_2(t)}{(t \mp x)^2}\,dt \quad (0<x<c).$$

† That is, if the integral over (c, x) is bounded for $x > c$.

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(ix) *If, as* $T \to \infty$, $\int_0^T h(t)\,dt = o(T)$, $\int_{-T}^0 h(t)\,dt = o(T)$,

then $\int_{x+1}^T \frac{th(t)}{t-x}dt = o(T)$, $\int_{-T}^{x-1} \frac{th(t)}{t-x}dt = o(T)$.

The proof of this lemma need not be given in detail. Parts (i) and (ii) are well known; (i) may be proved from the mean-value theorem stated above, with $d(t) = \pm m(t)$; (ii) may be deduced from (i) with $m(t) - m(\infty -)$ instead of $m(t)$; (iii) and (iv) follow from the mean-value theorem; (v) follows from (i) or (ii), (vi) and (vii) from (iii); for (viii) we may apply (ii) to

$$\int_c^\infty \frac{t^r}{(t \mp x)^2} \frac{h_r(t)}{t^r} dt \quad (r = 1, 2)$$

with $m(t) = t^r/(t \mp x)^2$; the first part of (ix) is obvious if we write

$$\int_{x+1}^T \frac{th(t)}{t-x}dt = \int_{x+1}^T h(t)\,dt + x\int_{x+1}^T \frac{h(t)}{t-x}dt$$

and apply (iii) to the last integral with $d(t) = 1/(t-x)$, and the second part is easily deducible from the first.

LEMMA 2. *As* $T \to \infty$,

(*a*) $$\int_0^T h(t)\,dt = o(T)$$

if and only if

(*b*) $$\int_T^\infty \frac{h(t)}{t^2}dt = o\left(\frac{1}{T}\right).$$

Proof. (i) Suppose (*b*) satisfied. Integrating by parts we have, if $T > 1$,

$$\int_1^T \frac{h(t)}{t^2}t^2 dt = -T^2\int_T^\infty \frac{h(u)}{u^2}du + \int_1^\infty \frac{h(u)}{u^2}du + 2\int_1^T \left(\int_t^\infty \frac{h(u)}{u^2}du\right) t\,dt.$$

From this (*a*) readily follows.

(ii) Suppose (*a*) satisfied. Integrating by parts we have, if $T' > T > 0$,

$$\int_T^{T'} \frac{h(t)}{t^2}dt = \frac{1}{T'^2}\int_0^{T'} h(u)\,du - \frac{1}{T^2}\int_0^T h(u)\,du + 2\int_T^{T'} \frac{1}{t^3}\left(\int_0^t h(u)\,du\right) dt.$$

The expression on the right tends to a finite limit as $T' \to \infty$, and we have

$$\int_T^\infty \frac{h(t)}{t^2}dt = -\frac{1}{T^2}\int_0^T h(u)\,du + 2\int_T^\infty \frac{1}{t^3}\left(\int_0^t h(u)\,du\right) dt.$$

From this (*b*) readily follows.

LEMMA 3.† (*a*) *If* $p > 1$ *and* f *is* $L^p(-\infty, \infty)$, *then* $g \equiv Hf$ *exists and is* $L^p(-\infty, \infty)$, *the integral in* (1·1) *being absolutely convergent at* $\pm\infty$; *the reciprocal formula* (1·2) *is true; and*

$$\int_{-\infty}^\infty |g(x)|^p\,dx \leqslant A_p^p \int_{-\infty}^\infty |f(x)|^p\,dx,$$

where A_p *is the constant of* (2·4).

† For (*a*) see Riesz (4), pp. 234, 239, or (apart from the value of A_p) Titchmarsh (5), p. 132; and for (*b*) see Titchmarsh (5), p. 138 (cf. Riesz (4), p. 231).

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(*b*) *If, further,* λ *is* $L^q(-\infty,\infty)$, *where* $q = p/(p-1)$, *and* $\mu = H\lambda$, *then*

$$\int_{-\infty}^{\infty} g(x)\,\lambda(x)\,dx = -\int_{-\infty}^{\infty} f(x)\,\mu(x)\,dx,$$

these integrals being absolutely convergent.

LEMMA 4. *Suppose that*

$$L_\omega(u) = \log\left|\frac{\omega+u}{\omega-u}\right| + \frac{u}{\omega}\log\left|\frac{\omega^2-u^2}{u^2}\right|$$

and that (*a*) $\int_c^{\infty} \chi(u)\,du$ *is convergent,*

or (*b*) $\int_T^{\infty} \frac{\chi(u)}{u}\,du = o\left(\frac{1}{T}\right) \quad (T\to\infty).$

Then $I_\omega \equiv \int_c^{\infty} \chi(u)\,L_\omega(u)\,du \to 0 \quad as \quad \omega\to\infty.$

Proof. We may suppose that $\omega > 0$; and that (*b*) is satisfied, since (*a*) implies (*b*) by Lemma 1 (vi). The derivative

$$L'_\omega(u) = \frac{1}{\omega}\log\left|\frac{\omega^2-u^2}{u^2}\right|$$

exists and is continuous except at $u = 0,\ \pm\omega$, and is integrable. Since the discontinuities of $L_\omega(u)$ at $0,\ \pm\omega$ are removable, it follows that after their removal $L_\omega(u)$ is absolutely continuous. For each ω,

$$L_\omega(u) \to 0 \quad \text{as} \quad u\to\infty.$$

Also $$L_\omega(u) \to 0 \quad \text{as} \quad \omega\to\infty,$$

uniformly in each finite interval of u.

The integral

$$K \equiv \int_0^{\infty} \left|\frac{d}{dt}\{tL_1(t)\}\right| \frac{dt}{t}$$

is finite since the integrand is the modulus of

$$\frac{1}{t}\log\left|\frac{1+t}{1-t}\right| + 2\log\left|\frac{1-t^2}{t^2}\right|.$$

For each positive ϵ there is, since (*b*) is satisfied, a number $\rho = \rho_\epsilon > c$ such that

$$\left|\int_u^{\infty} \frac{\chi(v)}{v}\,dv\right| < \frac{\epsilon}{u} \quad (u \geqslant \rho).$$

Because of the absolute continuity of $uL_\omega(u)$ we may integrate by parts, obtaining, for $\tau > \sigma \geqslant \rho$,

$$\int_\sigma^\tau \frac{\chi(u)}{u}\,uL_\omega(u)\,du = \left[\left(-\int_u^{\infty}\frac{\chi(v)}{v}\,dv\right)uL_\omega(u)\right]_\sigma^\tau - \int_\sigma^\tau\left(-\int_u^{\infty}\frac{\chi(v)}{v}\,dv\right)\frac{d}{du}\{uL_\omega(u)\}\,du,$$

$$\left|\int_\sigma^\tau \chi(u)\,L_\omega(u)\,du\right| \leqslant \epsilon\,|L_\omega(\tau)| + \epsilon\,|L_\omega(\sigma)| + \epsilon\int_\sigma^\tau \frac{1}{u}\left|\frac{d}{du}\{uL_\omega(u)\}\right|du$$

$$\leqslant \epsilon|L_\omega(\tau)| + \epsilon\,|L_\omega(\sigma)| + \epsilon K.$$

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The last line is obtained by making the substitution $u = \omega t$ in the integral in the preceding line.

Since $L_\omega(u) \to 0$ as $u \to \infty$, it is now clear that the integral I_ω is convergent; and also that, for each $\epsilon > 0$,

$$|I_\omega| \leqslant \left| \int_c^\rho \chi(u) L_\omega(u)\, du \right| + \epsilon |L_\omega(\rho)| + \epsilon K,$$

where $\rho = \rho_\epsilon$, as above. As $\omega \to \infty$, $L_\omega(u) \to 0$ uniformly in $c \leqslant u \leqslant \rho$, and so

$$\overline{\lim_{\omega \to \infty}} |I_\omega| \leqslant K\epsilon.$$

Hence the result.

THEOREM 1. *Suppose that $p > 1$, that f is locally L^p, that*

$$\int_1^\infty \frac{f(t) - f(-t)}{t}\, dt \tag{5·1}$$

is convergent, and that†

$$\int_T^\infty \frac{f(t) + f(-t)}{t^2}\, dt = o\left(\frac{1}{T}\right) \quad (T \to \infty). \tag{5·2}$$

Then‡

$$g(x) \equiv \frac{1}{\pi} \int_{-\infty}^\infty \frac{f(t)}{t - x}\, dt \tag{5·3}$$

exists and is locally L^p,

$$f(x) = -\frac{1}{\pi} \int_{-\infty}^\infty \frac{g(t)}{t - x}\, dt \quad (\mathrm{C}, 1) \tag{5·4}$$

and

$$\bar{M}(|g|^p) \leqslant A_p^p \bar{M}(|f|^p), \tag{5·5}$$

where A_p is the constant of (2·4).

6. Theorem 1 is proved in §§6–9. In the present section it is shown that g exists and is L^p over an arbitrary finite interval (a, b). Let

$$\phi(t) = \frac{f(t) - f(-t)}{t}, \quad \psi(t) = \frac{f(t) + f(-t)}{t^2},$$

$$c > 0, \quad -c < a < b < c,$$

$$f_c(t) = f(t) \quad (-c \leqslant t \leqslant c), \qquad f_c(t) = 0 \quad (|t| > c),$$

$$f = f_c + F_c,$$

$$g_c(x) = \frac{1}{\pi} P \int_{-c}^c \frac{f(t)}{t - x}\, dt = \frac{1}{\pi} \int_{-\infty}^\infty \frac{f_c(t)}{t - x}\, dt.$$

Since f_c is $L^p(-\infty, \infty)$, g_c exists and is $L^p(-\infty, \infty)$ by Lemma 3, and so g_c is $L^p(a, b)$. If $a \leqslant x \leqslant b$ and $T > c$,

$$\left(\int_{-T}^{-c} + \int_c^T \right) \frac{f(t)}{t - x}\, dt = \int_c^T \frac{t^2}{t^2 - x^2} \{\phi(t) + x\psi(t)\}\, dt.$$

Since the integrals of ϕ and ψ over (c, ∞) are convergent, it follows from Lemma 1 (iv) that as $T \to \infty$ this tends uniformly to a continuous limit, say $\pi G_c(x)$, in $a \leqslant x \leqslant b$. It is now clear that $g\,(= g_c + G_c)$ exists in (a, b) and is $L^p(a, b)$.

† By Lemma 2, (5·2) is equivalent to $M(f) = 0$, where the mean value is defined by (2·2).

‡ See §4 and the footnote to §1 for the interpretation of (5·3), (5·4) and (5·5).

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7. With a view to establishing the inversion formula (5·4) we show in this section that if λ vanishes outside a finite interval (a, b) and is L^q, where $q = p/(p-1)$, and if $\mu = H\lambda$, then

$$\int_{-\infty}^{\infty} g(x)\,\lambda(x)\,dx = -\lim_{T\to\infty}\int_{-T}^{T} f(t)\,\mu(t)\,dt. \tag{7·1}$$

To prove this, let $c > 0$, $-c < a < b < c$, as in §6. Then since f_c is $L^p(-\infty, \infty)$ we have by Lemma 3

$$\int_{-\infty}^{\infty} g_c(x)\,\lambda(x)\,dx = -\int_{-\infty}^{\infty} f_c(t)\,\mu(t)\,dt. \tag{7·2}$$

Also, since

$$G_c(x) = \frac{1}{\pi}\lim_{T\to\infty}\int_{-T}^{T}\frac{F_c(t)}{t-x}\,dt \quad (a \leqslant x \leqslant b),$$

the limit existing uniformly with respect to x, we have

$$\int_a^b G_c(x)\,\lambda(x)\,dx = \frac{1}{\pi}\lim_{T\to\infty}\int_{-T}^{T} F_c(t)\,dt\int_a^b\frac{\lambda(x)}{t-x}\,dx,$$

the inversion of the repeated integral being justified since $F_c(t) = 0$ in $(-c, c)$, so that, effectively, $1/(t-x)$ is bounded. Hence

$$\int_{-\infty}^{\infty} G_c(x)\,\lambda(x)\,dx = -\lim_{T\to\infty}\int_{-T}^{T} F_c(t)\,\mu(t)\,dt. \tag{7·3}$$

The result (7·1) follows from (7·2) and (7·3) by addition.

8. *Proof of the inversion formula* (5·4). The function $\bar{f}(t) \equiv f(-t)$ satisfies the conditions imposed on $f(t)$ in Theorem 1, and the Hilbert transform $\bar{g}(x)$ of $\bar{f}(t)$, which therefore exists by §6, is $-g(-x)$ by (5·3). Thus if (5·4) holds for a certain value of x with $\bar{f}, \bar{g}$ instead of f, g, then it holds (for f, g) with $-x$ instead of x.

It follows that it will be enough to establish the inversion formula (5·4) for (almost all) values of x in $(0, c)$, where c is an arbitrary positive number. Further, we may suppose $f = 0$ in $(-c, c)$. To show this, write $f = f_c + F_c$ as in §6. Both f_c and F_c satisfy the conditions imposed on f in Theorem 1, so that their transforms exist and are locally L^p by §6. If the inversion formula is true in $(0, c)$ for f_c and F_c, it is clearly true for their sum f. Now it is true for f_c by Lemma 3 since f_c is $L^p(-\infty, \infty)$ and convergence implies summability (C, 1). It will therefore be enough to prove it for F_c. This function vanishes in $(-c, c)$.

We may, then, in proving (5·4), suppose that $f = 0$ in $(-c, c)$; and we need only prove that then

$$\lim_{\delta\to 0+}\lim_{\omega\to\infty}\left(\int_{-\omega}^{x-\delta} + \int_{x+\delta}^{\omega}\right)\left(1-\frac{|t|}{\omega}\right)\frac{g(t)}{t-x}\,dt = 0 \quad (0 < x < c). \tag{8·1}$$

Suppose that $0 < x < c$, x and c being now fixed until the end of §8. Let

$$0 < \delta < x, \quad \omega > x+\delta,$$

so that

$$0 < x-\delta < x+\delta < \omega.$$

Let (cf. Hardy (3), p. 110)

$$\lambda(t) = \begin{cases} 0 & \text{in } x-\delta \leqslant t \leqslant x+\delta \text{ and in } |t| \geqslant \omega, \\ \left(1-\dfrac{|t|}{\omega}\right)\dfrac{1}{t-x} & \text{elsewhere.} \end{cases}$$

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Then $\mu \equiv H\lambda$ is given by

$$\pi\mu(u) = \left(\int_{-\omega}^{x-\delta} + \int_{x+\delta}^{\omega}\right)\left(1-\frac{|t|}{\omega}\right)\frac{1}{t-x}\frac{1}{t-u}\,dt,$$

this being a principal value with respect to u if this point is in the interior of the range of integration. When $u \neq x,\ x\pm\delta,\ \pm\omega,\ 0$,

$$\begin{aligned}\pi\mu(u) &= \int_{-\omega}^{0} + \int_{0}^{x-\delta} + \int_{x+\delta}^{\omega} \\ &= \int_{-\omega}^{0} \frac{1}{x-u}\left(\frac{1+x/\omega}{t-x} - \frac{1+u/\omega}{t-u}\right)dt + \left(\int_{0}^{x-\delta} + \int_{x+\delta}^{\omega}\right)\frac{1}{x-u}\left(\frac{1-x/\omega}{t-x} - \frac{1-u/\omega}{t-u}\right)dt.\end{aligned}$$

By using the fact that $$\int_{\alpha}^{\beta}\frac{dt}{t-\gamma} = \log\left|\frac{\beta-\gamma}{\alpha-\gamma}\right| \quad (\gamma \neq \alpha, \beta),$$

this being a principal value if γ lies between α and β, we find that, apart from a finite number of values of u,

$$\pi\mu(u) = \{\theta_1(u)+\theta_2(u)+\theta_3(u)+\theta_4(u)\}/(x-u),$$

where, with the notation of Lemma 4,

$$\theta_1(u) = -L_\omega(x), \qquad \theta_2(u) = L_\omega(u),$$

$$\theta_3(u) = \log\left|\frac{x-u+\delta}{x-u-\delta}\right|, \quad \theta_4(u) = -\frac{u}{\omega}\log\left|\frac{x-u+\delta}{x-u-\delta}\right|.$$

Hence by (7·1), $$\left(\int_{-\omega}^{x-\delta} + \int_{x+\delta}^{\omega}\right)\left(1-\frac{|t|}{\omega}\right)\frac{g(t)}{t-x}\,dt = \lim_{T\to\infty}\frac{1}{\pi}\sum_{r=1}^{4} J_r, \tag{8·2}$$

where, since $f = 0$ in $(-c, c)$,

$$J_r = \left(\int_{-T}^{-c} + \int_{c}^{T}\right)\frac{f(u)}{u-x}\theta_r(u)\,du.$$

Referring to (8·1) we see that it will be enough to prove that

$$\lim_{\delta\to 0+}\lim_{\omega\to\infty}\lim_{T\to\infty} J_r = 0 \quad (r = 1, 2, 3, 4).$$

Now $\theta_1(u)$ is independent of u. Thus $\lim_{T\to\infty} J_1$ exists (see § 6); and it tends to zero as $\omega \to \infty$ since $\theta_1(u)$ does so (see the definition of L_ω in Lemma 4). Secondly,

$$J_2 = \int_{c}^{T}\{\chi_1(u)+\chi_2(u)\}\,L_\omega(u)\,du,$$

where $$\chi_1(u) = \frac{x}{u^2-x^2}\{f(u)-f(-u)\}, \quad \chi_2(u) = \frac{u}{u^2-x^2}\{f(u)+f(-u)\};$$

and so $$\lim_{\omega\to\infty}\lim_{T\to\infty} J_2 = 0$$

by Lemma 4, since the hypothesis (a) of that lemma is satisfied with $\chi = \chi_1$ on account of (5·1) and Lemma 1 (v), and (b) is satisfied with $\chi = \chi_2$ on account of (5·2) and Lemma 1 (vii).

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For the case $r = 3$ it is enough to prove that the integral

$$P \equiv \int_c^\infty \frac{f(u)}{(u-x)^2}(u-x)\log\left|\frac{1+\delta/(u-x)}{1-\delta/(u-x)}\right| du,$$

and the integral Q of the same function over $(-\infty, -c)$, exist and tend to zero as $\delta \to 0+$. Now by (5·1), (5·2) and Lemma 1 (viii),

$$\int_c^\infty \frac{f(u)}{(u-x)^2} du, \quad \int_c^\infty \frac{f(-u)}{(u+x)^2} du \tag{8·3}$$

are convergent. The coefficient of $f(u)/(u-x)^2$ in the integrand of P is, when

$$0 < \delta < c-x,$$

positive and decreasing in $u \geqslant c$, as may be seen by expanding in negative powers of $u-x$; and so by Lemma 1 (iii) P exists and

$$P = (c-x)\log\frac{c-x+\delta}{c-x-\delta}\int_c^\xi \frac{f(u)}{(u-x)^2} du,$$

where $c \leqslant \xi \leqslant \infty$. Hence $P \to 0$ as $\delta \to 0+$. A simple change of variable shows that Q may be obtained from P by replacing $f(u)$ by $f(-u)$ and $-x$ by x. An argument similar to that for P, depending on the convergence of the second of the integrals (8·3), then shows that $Q \to 0$ as $\delta \to 0+$.

The existence of $\lim_{T\to\infty} J_4$ now follows from (8·2). A glance at the definition of θ_4 is then enough to show that

$$\lim_{\omega\to\infty} \lim_{T\to\infty} J_4 = 0.$$

This completes the proof of (5·4).

9. *Proof of the inequality* (5·5). Let $0 < k < 1$, $c > 0$ and, with the notation used in §6 and later, write $f = f_c + F_c$. Then

$$\int_{-kc}^{kc} |g_c|^p\, dx \leqslant \int_{-\infty}^{\infty} |g_c|^p\, dx \leqslant A_p^p \int_{-\infty}^{\infty} |f_c|^p\, dx = A_p^p \int_{-c}^{c} |f|^p\, dx, \tag{9·1}$$

the second inequality following from Lemma 3. Also, when $-kc \leqslant x \leqslant kc$,

$$\begin{aligned}\pi G_c(x) &= \int_c^\infty \frac{t^2}{t^2-x^2}\{\phi(t)+x\psi(t)\}\, dt \\ &= \frac{c^2}{c^2-x^2}\int_c^\xi \{\phi(t)+x\psi(t)\}\, dt,\end{aligned}$$

where $c \leqslant \xi \leqslant \infty$, by Lemma 1 (iii); and so

$$\begin{aligned}\pi|G_c(x)| &\leqslant \frac{1}{1-k^2}\left\{\left|\int_c^\xi \phi(t)\, dt\right| + kc\left|\int_c^\xi \psi(t)\, dt\right|\right\} \\ &= \frac{1}{1-k^2}\left\{o(1) + kc\,.\,o\left(\frac{1}{c}\right)\right\}\end{aligned}$$

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for large c, by (5·1) and (5·2), the o's being uniform with respect to x. Hence

$$|G_c(x)| \leqslant U(c) \quad (-kc \leqslant x \leqslant kc), \tag{9·2}$$

where, for fixed k, $\qquad U(c) \to 0 \quad \text{as} \quad c \to \infty.$

Since $g = g_c + G_c$ we have, by Minkowski's inequality and (9·1), (9·2),

$$\left(\frac{1}{2kc}\int_{-kc}^{kc}|g|^p\,dx\right)^{1/p} \leqslant \left(\frac{1}{2kc}\int_{-kc}^{kc}|g_c|^p\,dx\right)^{1/p} + \left(\frac{1}{2kc}\int_{-kc}^{kc}|G_c|^p\,dx\right)^{1/p}$$
$$\leqslant \left(\frac{A_p^p}{2kc}\int_{-c}^{c}|f|^p\,dx\right)^{1/p} + U(c).$$

Letting $c \to \infty$, we have

$$\{\bar{M}(|g|^p)\}^{1/p} \leqslant \{k^{-1}A_p^p\bar{M}(|f|^p)\}^{1/p}$$

for every positive k less than 1; and (5·5) follows.

The proof of Theorem 1 is now complete.

PART 2

10. The application of Theorem 1 to almost periodic functions, alluded to in §3, will now be considered. Functions h_1, h_2 of B_0^p for which $\|h_1 - h_2\| = 0$ will be said to be equivalent (B^p). A necessary and sufficient† condition for this equivalence is that the functions should have the same Fourier series. Equivalent functions are associated with the same point of the space.

THEOREM 2. *Suppose that $p > 1$ and that the Fourier series of a function f of B_0^p ($\|f\| \neq 0$) is written in the form*

$$\Sigma(a_n \cos\lambda_n x + b_n \sin\lambda_n x) \quad (\lambda_n > 0).$$

Then

$$\Sigma(b_n \cos\lambda_n x - a_n \sin\lambda_n x)$$

is the Fourier series of a function g of B_0^p, and

$$\|g\| \leqslant A_p \|f\|,$$

where A_p is the constant of (2·4).

Moreover, f and g are equivalent (B^p) to functions f^* and g^* of B_0^p which satisfy for almost all values of x the relations (*see* §4)

$$g^*(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f^*(t)}{t-x}\,dt, \quad f^*(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g^*(t)}{t-x}\,dt.$$

11. Theorem 2 is proved in §§11, 12.

The set of exponents λ_n is not null since $\|f\| \neq 0$. By the approximation theorem (Besicovitch (1), Ch. II, §8, and Ch. I, §9), there is a sequence of trigonometric polynomials

$$s_k(x) \equiv \sum_{n=1}^{n_k}(a_{nk}\cos\lambda_n x + b_{nk}\sin\lambda_n x) \quad (k = 1, 2, \ldots)$$

† Besicovitch (1), p. 109. The Fourier series may be 'equal to nought' (cf. Besicovitch (1), p. 21, §4).

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involving all the exponents λ_n and such that

$$n_1 \leqslant n_2 \leqslant \ldots,$$

$$a_{nk} \to a_n, \quad b_{nk} \to b_n \quad \text{as} \quad k \to \infty, \tag{11·1}$$

$$\|f - s_k\| \to 0 \quad \text{as} \quad k \to \infty. \tag{11·2}$$

Let $$\bar{s}_k(x) = \sum_{n=1}^{n_k} (b_{nk} \cos \lambda_n x - a_{nk} \sin \lambda_n x).$$

It follows readily from the equality

$$\int_{-\infty}^{\infty} \frac{\sin \lambda t}{t} dt = \pi \quad (\lambda > 0)$$

that (see (1·1)) $H \cos \lambda x = -\sin \lambda x, \qquad H \sin \lambda x = \cos \lambda x \quad (\lambda > 0),$

and therefore that $$\bar{s}_k(x) - \bar{s}_l(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{s_k(t) - s_l(t)}{t - x} dt.$$

Hence, since any trigonometric polynomial satisfies the conditions imposed on f in Theorem 1,

$$\|\bar{s}_k - \bar{s}_l\| \leqslant A_p \|s_k - s_l\|. \tag{11·3}$$

As $k, l \to \infty$, the right side tends to zero, by (11·2). So therefore does the left side; and so, by the completeness of the space B^p (cf. Zaanen (7), p. 105) and since trigonometric polynomials belong to B^p, there is a function g of B^p such that

$$\|g - \bar{s}_k\| \to 0 \quad \text{as} \quad k \to \infty. \tag{11·4}$$

An argument of a familiar nature enables us to find the Fourier coefficients of g. By Hölder's inequality,

$$|M(\{g(x) - \bar{s}_k(x)\} \cos \lambda x)| \leqslant \|g - \bar{s}_k\|,$$

and so $$M(g(x) \cos \lambda x) = \lim_{k \to \infty} M(\bar{s}_k(x) \cos \lambda x). \tag{11·5}$$

From such obvious equalities as

$$M(\cos \mu x \cos \lambda x) = 0 \quad (\lambda \neq \mu), \qquad M(\cos^2 \lambda x) = \tfrac{1}{2} \quad (\lambda \neq 0),$$

it is clear that the mean value on the right side of (11·5) is zero when λ is different from all the exponents λ_n, and in particular when $\lambda = 0$; and equal to $\frac{1}{2} b_{nk}$ when $\lambda = \lambda_n$ and k is so great that $n_k \geqslant n$. Hence by (11·1)

$$M(g(x) \cos \lambda x) = \begin{cases} \frac{1}{2} b_n & \lambda = \lambda_n \quad (n \geqslant 1), \\ 0 & \text{otherwise.} \end{cases}$$

From this and a similar result we see that the Fourier series of $g(x)$ is

$$\Sigma(b_n \cos \lambda_n x - a_n \sin \lambda_n x);$$

and that $M(g) = 0$, so that g belongs to B_0^p.

Not only have we $M(g) = 0$, but, by a similar argument,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T g(t)\, dt = 0, \quad \lim_{T \to \infty} \frac{1}{T} \int_{-T}^0 g(t)\, dt = 0. \tag{11·6}$$

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The equalities like (11·6) for *any* f of B_0^p follow in a similar way from (11·2) if $\|f\| \neq 0$, and are trivial if the norm vanishes.

The inequality
$$\|g\| \leqslant A_p \|f\|$$
follows from
$$\|\bar{s}_k\| \leqslant A_p \|s_k\|$$
(cf. (11·3)); for (11·2) and (11·4) imply that, as $k \to \infty$,
$$\|s_k\| \to \|f\|, \quad \|\bar{s}_k\| \to \|g\|.$$

12. The existence of the functions f^* and g^* will now be established. The function of B_0^p
$$F(x) \equiv \tfrac{1}{2}\{f(x)+f(-x)\}, \tag{12·1}$$
satisfies the conditions imposed on f in Theorem 1. As regards (5·2) this follows, by a footnote to the enunciation of Theorem 1, from the fact that, since F is in B_0^p, $M(F) = 0$. Hence by Theorem 1 the function
$$\overline{F}(x) \equiv \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{F(t)}{t-x}\,dt \tag{12·2}$$
exists and
$$F(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\overline{F}(t)}{t-x}\,dt, \tag{12·3}$$
where the last integral is taken in the (C, 1) sense, although it will be shown presently that it is convergent in the sense of §4. Let
$$S_k(x) = \sum_{n=1}^{n_k} a_{nk}\cos\lambda_n x, \quad \bar{S}_k(x) = \sum_{n=1}^{n_k} -a_{nk}\sin\lambda_n x.$$
Then by (11·2) and (12·1)
$$\|F - S_k\| \to 0 \quad \text{as} \quad k \to \infty.$$
Also, $F - S_k$ satisfies the conditions imposed on f in Theorem 1, $\overline{F} - \bar{S}_k = H(F - S_k)$ and, by (5·5),
$$\overline{M}(|\overline{F} - \bar{S}_k|^p) \leqslant A_p^p \|F - S_k\|^p.$$
Hence as $k \to \infty$ the left side tends to zero, so that $\overline{F}$ is a B^p-limit ((1), p. 73, 3°) of the sequence $\{\bar{S}_k\}$, and therefore belongs to B^p, which is ((1), p. 100) the B^p-closure of the set of trigonometric polynomials. Thus $\|\overline{F} - \bar{S}_k\|$ exists and tends to zero as $k \to \infty$. It follows (cf. §11) that $\overline{F}(x)$ has the Fourier series† $\Sigma - a_n \sin\lambda_n x$ and is in B_0^p. But the function $\frac{1}{2}\{g(x)-g(-x)\}$ of B_0^p clearly has the same Fourier series. Hence $\overline{F}(x)$ and $\frac{1}{2}\{g(x)-g(-x)\}$ are equivalent (B^p).

Since $\overline{F}$ is in B_0^p,
$$\int_{-T}^{0} \overline{F}(t)\,dt = o(T), \quad \int_{0}^{T} \overline{F}(t)\,dt = o(T)$$
as $T \to \infty$, by the remark following (11·6). Hence by Lemma 1 (ix)
$$\int_{-T}^{x-1} \frac{t}{T}\frac{\overline{F}(t)}{t-x}\,dt \to 0, \quad \int_{x+1}^{T} \frac{t}{T}\frac{\overline{F}(t)}{t-x}\,dt \to 0,$$
and so for almost all values of x the integral in (12·3) is not only summable (C, 1), but convergent in the sense of §4.

† All a_n may vanish. See the footnote to §10.

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It may be proved similarly that, if

$$\dot{G}(x) \equiv \tfrac{1}{2}\{g(x)+g(-x)\},$$

the function
$$\bar{G}(x) \equiv \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{G(t)}{t-x}\,dt \tag{12·4}$$

exists and is such that
$$G(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\bar{G}(t)}{t-x}\,dt; \tag{12·5}$$

and that the functions $\bar{G}(x)$ and $\frac{1}{2}\{f(-x)-f(x)\}$ (which, as well as G, are in B_0^p) are equivalent (B^p).

Now let
$$f^* = F-\bar{G}, \quad g^* = \bar{F}+G.$$

These two functions belong to B_0^p. Also

$$f^*(x)-f(x) = \tfrac{1}{2}\{f(-x)-f(x)\}-\bar{G}(x),$$

so that f^* and f are equivalent (B^p). Similarly for g^* and g.

Finally, from (12·2) and (12·5) by addition, and from (12·3) and (12·4) by subtraction, we have

$$g^* = Hf^*, \quad f^* = -Hg^*.$$

The proof of Theorem 2 is now complete.

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