J. Plasma Physics (2005), *vol.* 71, *part* 3, *pp.* 367–376. © 2005 Cambridge University Press
doi:10.1017/S0022377804003162 Printed in the United Kingdom

Self-fields in an Ion-channel Free-electron Laser

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(Received 29 October 2003 and accepted 28 April 2004)

Abstract. A theory of self-fields in a free-electron laser with a helical wiggler and ion-channel guiding is presented. The effects of self-electric and magnetic fields on electron trajectories are investigated. It is shown that, under the Budker condition, there is one group of orbits (group I) for which the wiggler-induced self-magnetic field is negative and acts as a diamagnetic correction to the wiggler magnetic field. The function Φ that determines the rate of change of axial velocity with energy in the presence of the self-fields is derived and studied numerically.

1. Introduction

Many free-electron lasers (FELs) currently in use employ a helical wiggler magnetic field generated by bifilar current winding. A uniform static axial magnetic field generated by current in a solenoid is often employed to collimate the relativistic electron beam in the transverse direction [1]. Ion-channel guiding of a relativistic electron beam in a FEL has been proposed as an alternative to guiding by a solenoid magnetic field [2]. Ion-channel guiding involves the formation of a positive ion core by the expulsion of an electron from a pre-ionized plasma channel. A plasma channel may be created, for example, by the passage of an UV-laser beam through a gas. The use of ion-channel guiding in a FEL offers a number of advantages. It is a less expensive alternative to solenoid magnetic guiding because of lower capital and operating costs [3]. Also, it permits beam currents higher than the vacuum limit [4]. A serious obstacle to intense electron beam transport with axial magnetic guiding is the transverse beam break-up instability. The instability may be effectively suppressed by ion-channel guiding [5]. In both types of guiding, the gain (growth rate) may be enhanced by exploiting the resonance between the frequency of the guiding device and the frequency of the wiggler field [1, 2, 6].

High-density, low-gamma, relativistic electron beams employed in FELs produce strong collective effects associated with the self-electric and magnetic fields of the beams. The self-fields in a FEL with a helical wiggler and axial magnetic field have been treated in both linear [7] and nonlinear [8, 9] theory.

Recently, theoretical studies of the electron trajectories and gain in a FEL with a helical wiggler and ion-channel guiding in the absence of the self-fields have been

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carried out by Jha and Kumar [2,5] and Esmaeilzadeh et al. [6]. The purpose of the present paper is to investigate self-fields in a FEL with a helical wiggler and ion-channel guiding, and to derive electron trajectories in the presence of the self-fields. In Sec. 2, self-electric and magnetic fields are derived from Poisson's equation and Ampère's law, respectively. A relativistic equation of motion for a single electron in the combined fields (including self-fields) is obtained and solved for the transverse coordinates and velocity components as a function of the axial coordinate. Next, a six-degree polynomial equation for axial velocity is obtained from the conservation of energy. An equation for the function Φ, which determines the rate of change of axial velocity with energy, is then derived. In Sec. 3, a numerical study based on the equations derived in Sec. 2 is presented. A summary and conclusions are given in Sec. 4.

2. Self-electric and self-magnetic fields

The self-fields are induced by the steady-state charge and current density of the non-neutral electron beam. For convenience a beam with uniform, azimuthally symmetric profiles in both the density and velocity is considered here. In such a case, the beam density is given by

$$n_{\mathrm{b}}(r) = \begin{cases} n_{\mathrm{b}} & \text{for } r \leqslant R_{\mathrm{b}}, \\ 0 & \text{otherwise}, \end{cases} \tag{1}$$

where R_{b} is the beam radius. The self-electric field may be obtained by Poisson's equation

$$\frac{1}{r}\frac{\partial}{\partial r}(rE_{\mathrm{s}}) = -4\pi e n_{\mathrm{b}}(r), \tag{2}$$

where $-e$ is the electron change. Solving (2), we obtain

$$\mathbf{E}_{\mathrm{s}} = -2\pi e n_{\mathrm{b}} r\hat{\mathbf{e}}_r = -2\pi e n_{\mathrm{b}}(x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y). \tag{3}$$

The self-magnetic field can be determined by Ampère's law. This yields (see the Appendix)

$$\mathbf{B}_{\mathrm{s}} = -2\pi e n_{\mathrm{b}}\beta_{\parallel}(y\hat{\mathbf{e}}_x - x\hat{\mathbf{e}}_y) + \frac{2\bar{\omega}_{\mathrm{b}}^2\beta_{\parallel}^2}{\bar{\omega}_{\mathrm{i}}^2 - \bar{\omega}_{\mathrm{b}}^2(1+\beta_{\parallel}^2) - \beta_{\parallel}^2}\mathbf{B}_{\mathrm{w}}, \tag{4}$$

where $\beta_{\parallel}(\equiv v_{\parallel}/c)$ is the normalized axial velocity, $\mathbf{B}_{\mathrm{w}}$ is the helical wiggler magnetic field, and $\bar{\omega}_{\mathrm{i}}$ and $\bar{\omega}_{\mathrm{b}}$ are the normalized ion-channel and electron-beam frequencies that are defined in the Appendix. The first term in (4) is due to the electron axial velocity; we call this term the axial velocity-induced self-magnetic field. The second term in (4) is (the first-order correction to) the self-magnetic field, which is due to the wiggler-induced transverse velocity; we call this term the wiggler-induced (or transverse velocity-induced) self-magnetic field.

An analysis of the relativistic motion of an electron with rest mass m in a FEL with a helical wiggler and ion-channel guiding in the presence of the self-electric and self-magnetic fields will be based on the equation of motion

$$\frac{d(\gamma m\mathbf{v})}{dt} = -e(E_{\mathrm{i}} + E_{\mathrm{s}}) + \frac{1}{c}\mathbf{v} \times (\mathbf{B}_{\mathrm{w}} + \mathbf{B}_{\mathrm{s}}), \tag{5}$$

where $\mathbf{E}_{\mathrm{i}}$ is the ion-channel electric field that is defined in the Appendix. Substituting (3) and (4) into (5) allows the equation of motion to be written in the scalar

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form:

$$\frac{d\beta_x}{dt} = ck_{\mathrm{w}}^2[\bar{\omega}_{\mathrm{i}}^2 - \bar{\omega}_{\mathrm{b}}^2(1-\beta_{\parallel}\beta_z)]x + \Lambda\beta_z \sin(k_{\mathrm{w}}z), \tag{6}$$

$$\frac{d\beta_y}{dt} = ck_{\mathrm{w}}^2[\bar{\omega}_{\mathrm{i}}^2 - \bar{\omega}_{\mathrm{b}}^2(1-\beta_{\parallel}\beta_z)]y - \Lambda\beta_z \cos(k_{\mathrm{w}}z), \tag{7}$$

$$\frac{d\beta_z}{dt} = ck_{\mathrm{w}}^2\beta_{\parallel}\bar{\omega}_{\mathrm{b}}^2(x\beta_x + y\beta_y)y + \Lambda(\beta_y \cos(k_{\mathrm{w}}z) - \beta_x \sin(k_{\mathrm{w}}z)), \tag{8}$$

where

$$\Lambda \equiv \frac{ck_{\mathrm{w}}\bar{\Omega}_{\mathrm{w}}\left(\bar{\omega}_{\mathrm{i}}^2 - \bar{\omega}_{\mathrm{b}}^2\gamma_{\parallel}^{-2} - \beta_{\parallel}^2\right)}{\bar{\omega}_{\mathrm{i}}^2 - \bar{\omega}_{\mathrm{b}}^2(1+\beta_{\parallel}^2) - \beta_{\parallel}^2}, \tag{9}$$

$k_{\mathrm{w}}(\equiv 2\pi/\lambda_{\mathrm{w}})$ is the wiggler wave number and $\bar{\Omega}_{\mathrm{w}}$ is the normalized wiggler magnetic field. The steady-state solutions of (7)–(9) with constant axial velocity, $\beta_z = \beta_{\parallel}$, in the presence of the self-fields can be written as

$$\beta_x \equiv \frac{\bar{\Omega}_{\mathrm{w}}\beta_{\parallel}^2}{\bar{\omega}_{\mathrm{i}}^2 - \bar{\omega}_{\mathrm{b}}^2(1+\beta_{\parallel}^2) - \beta_{\parallel}^2} \cos(k_{\mathrm{w}}z), \tag{10}$$

$$\beta_y \equiv \frac{\bar{\Omega}_{\mathrm{w}}\,\beta_{\parallel}^2}{\bar{\omega}_{\mathrm{i}}^2 - \bar{\omega}_{\mathrm{b}}^2(1+\beta_{\parallel}^2) - \beta_{\parallel}^2} \sin(k_{\mathrm{w}}z). \tag{11}$$

The steady-state electron displacements for constant axial velocity ($v_z = v_{\parallel} = c\beta_{\parallel}$; $z = c\beta_{\parallel}t$) are obtained by integrating (10) and (11). This yields

$$x = \frac{\bar{\Omega}_{\mathrm{w}}\,\beta_{\parallel}}{k_{\mathrm{w}}\left[\bar{\omega}_{\mathrm{i}}^2 - \bar{\omega}_{\mathrm{b}}^2(1+\beta_{\parallel}^2) - \beta_{\parallel}^2\right]} \sin(k_{\mathrm{w}}z), \tag{12}$$

$$y \equiv -\frac{\bar{\Omega}_{\mathrm{w}}\beta_{\parallel}}{k_{\mathrm{w}}\left[\bar{\omega}_{\mathrm{i}}^2 - \bar{\omega}_{\mathrm{b}}^2(1+\beta_{\parallel}^2) - \beta_{\parallel}^2\right]} \cos(k_{\mathrm{w}}z). \tag{13}$$

Hence, the electron trajectory is a perfect helix. Equations (10) and (11) show a resonant enhancement in the magnitude of the transverse electron velocity when

$$\bar{\omega}_{\mathrm{i}} = \left[\bar{\omega}_{\mathrm{b}}^2(1+\beta_{\parallel}^2) + \beta_{\parallel}^2\right]^{1/2}, \tag{14}$$

which separates the electron orbits into two groups. Group I and group II are defined by

$$\bar{\omega}_{\mathrm{i}} < \left[\bar{\omega}_{\mathrm{b}}^2(1+\beta_{\parallel}^2) + \beta_{\parallel}^2\right]^{1/2} \tag{15}$$

and

$$\bar{\omega}_{\mathrm{i}} > \left[\bar{\omega}_{\mathrm{b}}^2(1+\beta_{\parallel}^2) + \beta_{\parallel}^2\right]^{1/2}, \tag{16}$$

respectively. Note that, under the Budker condition $n_{\mathrm{b}} > n_{\mathrm{i}} \gg n_{\mathrm{b}}/\gamma^2$, there are only group I orbits; this condition eliminates the group II orbit since $\bar{\omega}_{\mathrm{b}} > \bar{\omega}_{\mathrm{i}}$. The foregoing analysis contains the following important case. If $\bar{\omega}_{\mathrm{b}}$ is set equal to zero (neglecting the self-fields), the transverse electron velocity and displacement

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components become

$$\beta_x \equiv \frac{\bar{\Omega}_w \beta_\parallel^2}{\bar{\omega}_i^2 - \beta_\parallel^2} \cos(k_w z), \tag{17}$$

$$\beta_y \equiv \frac{\bar{\Omega}_w \beta_\parallel^2}{\bar{\omega}_i^2 - \beta_\parallel^2} \sin(k_w z), \tag{18}$$

$$x = \frac{\bar{\Omega}_w \beta_\parallel}{k_w \left[\bar{\omega}_i^2 - \beta_\parallel^2\right]} \sin(k_w z), \tag{19}$$

$$y = -\frac{\bar{\Omega}_w \beta_\parallel}{k_w \left[\bar{\omega}_i^2 - \beta_\parallel^2\right]} \cos(k_w z), \tag{20}$$

respectively. Equations (17)–(20) agree with the steady-state transverse velocity and the displacement components given in Jha and Khumar [2].

The normalized axial velocity for steady-state trajectories is obtained from the conservation of energy:

$$\gamma^{-2} = 1 - \beta_x^2 - \beta_y^2 - \beta_z^2. \tag{21}$$

Substituting (10) and (11) into (21) yields

$$\beta_\parallel^2 \left\{ 1 + \left[\frac{\bar{\Omega}_w \beta_\parallel}{\bar{\omega}_i^2 - \bar{\omega}_b^2 (1 + \beta_\parallel^2) - \beta_\parallel^2} \right] \right\} + \gamma^{-2} - 1 = 0. \tag{22}$$

Since this equation is symmetric in $\beta_\parallel$, it yields trajectories that are independent of the direction of electron beam propagation. There are in general six solutions for $\beta_\parallel$ for any set of parameters; only the three for which $\beta_\parallel > 0$ will be considered here.

3. Numerical results and discussion

A numerical study of the self-fields in a FEL with a helical wiggler and ion-channel guiding has been made. The wiggler-induced self-magnetic field has been computed using the second term of (4) and (22). The normalized wiggler magnetic field $\bar{\Omega}_w$ was taken to be 0.05.

Figure 1 shows the ratio B_{sw}/B_w of the wiggler-induced self-magnetic field to the wiggler magnetic field as a function of the normalized ion-channel frequency for different normalized beam frequencies. It is shown that the wiggler-induced self-magnetic field is negative and acts as a diamagnetic correction to the wiggler magnetic field. As Fig. 1 shows, the diamagnetic effect becomes stronger with increasing beam frequency $\bar{\omega}_b$.

Figure 2 shows the ratio of the wiggler-induced self-magnetic field to the wiggler magnetic field as a function of the normalized ion-channel frequency $\bar{\omega}_i$ for different electron energies (or different Lorentz relativistic factors). As this figure shows the magnitude of the wiggler-induced self-magnetic field increases with increasing Lorentz relativistic factor.

4. Summary and conclusion

In this paper the self-fields associated with the charge density and current of the electron beam are evaluated in a FEL with a helical wiggler and ion-channel

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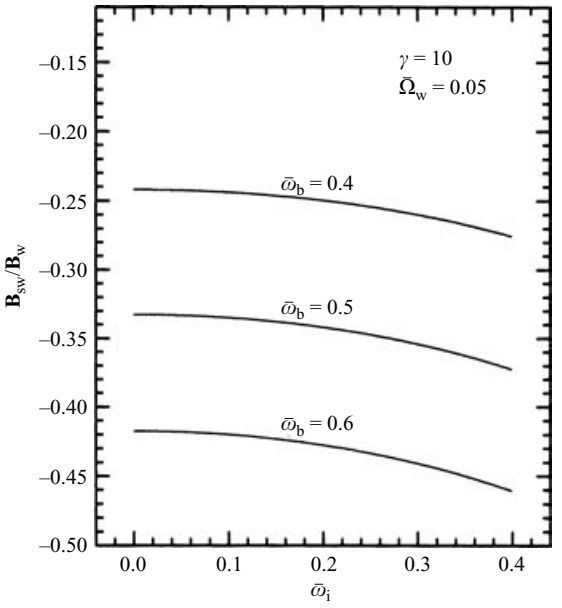


Figure 1. Graph of the ratio of the wiggler-induced self-magnetic field to the wiggler magnetic field as a function of ion-channel frequency for different normalized beam frequencies. Other parameters are shown in the figure.

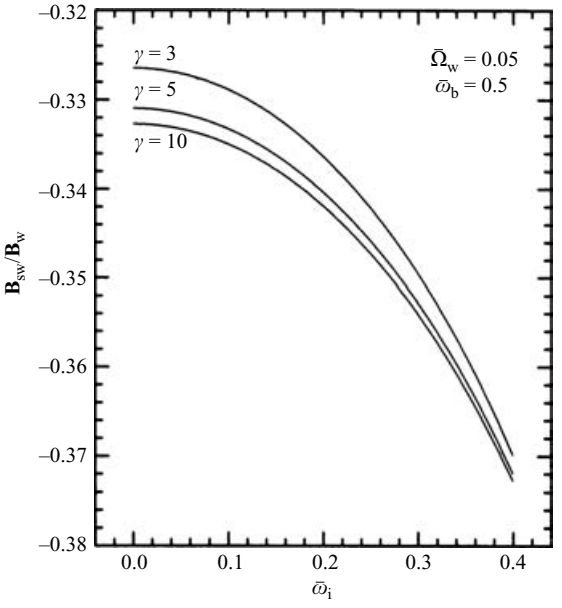


Figure 2. Graph of the ratio of the wiggler-induced self-magnetic field to the wiggler magnetic field for different Lorentz relativistic factors. Other parameters are shown in the figure.

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guiding. The self-electric field is derived from Poisson's equation and a more detailed analytical model for the self-magnetic field is obtained from Ampère's law. The mutual influence of electron velocity and self-magnetic field is considered here. It is shown that the wiggler-induced self-magnetic field leads to a decrease in the total magnetic field diamagnetic field (the diamagnetic effect).

Appendix. Calculation of self-magnetic field

The self-magnetic field may be determined by use of Ampère's law in the form

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} = -\frac{4\pi}{c} e n_{\mathrm{b}} \mathbf{v}, \tag{A 1}$$

where c is the speed of light in vacuo, $\mathbf{J}$ is the electron current density and $\mathbf{v}$ is the electron velocity. The lowest order representation for the self-magnetic field is determined under the assumption that the beam propagates paraxially with constant velocity ($\mathbf{v} = v_z \hat{\mathbf{e}}_z$). In this case (A 1) can be written in the form

$$\frac{1}{r}\frac{\partial}{\partial r}(r B_{\mathrm{s}\theta}) = -4\pi e n_{\mathrm{b}} \beta_z, \tag{A 2}$$

which has the solution

$$\mathbf{B}_{\mathrm{s}} = -2\pi e n_{\mathrm{b}} \beta_z r \hat{\mathbf{e}}_\theta = 2\pi e n_{\mathrm{b}} \beta_z (y\hat{\mathbf{e}}_x - x\hat{\mathbf{e}}_y), \tag{A 3}$$

where $\beta_z \equiv v_z/c$. Equation (A 3) represents the self-magnetic field of a paraxially propagating electron beam. When an electron beam passes through a FEL, the wiggler magnetic field affects the electron trajectories (induces transverse velocity) and consequently alters the self-magnetic field. An analysis of the relativistic motion of an electron with rest mass m in a FEL with helical wiggler and ion-channel guiding will be based on the equation of motion,

$$\frac{d(\gamma m \mathbf{v})}{dt} = -e\left(E + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right). \tag{A 4}$$

The electrostatic field due to an ion channel will be taken as [2]

$$\mathbf{E}_{\mathrm{i}} = 2\pi e n_{\mathrm{i}} (x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y), \tag{A 5}$$

where n_{i} is the density of positive ions with charge $+e$. In the idealized one-dimensional approximation, a helical wiggler magnetic field will be described by [1]

$$\mathbf{B}_{\mathrm{w}} = B_{\mathrm{w}}[\hat{\mathbf{e}}_x \cos(k_{\mathrm{w}} z) + \hat{\mathbf{e}}_y \sin(k_{\mathrm{w}} z)], \tag{A 6}$$

where $k_{\mathrm{w}} \equiv 2\pi/\lambda_{\mathrm{w}}$ is the wiggler wave number. Substituting (A 5) and (A 6) into (A 4) leads to

$$\frac{d(\gamma m v_x)}{dt} = -2\pi e^2 n_{\mathrm{i}} x + \frac{e v_z B_{\mathrm{w}}}{c} \sin(k_{\mathrm{w}} z), \tag{A 7}$$

$$\frac{d(\gamma m v_y)}{dt} = -2\pi e^2 n_{\mathrm{i}} y - \frac{e v_z B_{\mathrm{w}}}{c} \cos(k_{\mathrm{w}} z), \tag{A 8}$$

$$\frac{d(\gamma m v_z)}{dt} = \frac{e B_{\mathrm{w}}}{c}[v_y \cos(k_{\mathrm{w}} z) - v_x \sin(k_{\mathrm{w}} z)]. \tag{A 9}$$

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Steady-state solutions of (A 7)–(A 9) have been found to be [2, 6]

$$\beta_x = \beta_\perp \cos(k_w z), \tag{A 10}$$

$$\beta_y = \beta_\perp \sin(k_w z), \tag{A 11}$$

$$\beta_z = \beta_\parallel = \text{constant}, \tag{A 12}$$

where

$$\beta_\perp \equiv \frac{\bar{\Omega}_w \beta_\parallel^2}{\bar{\omega}_i^2 - \beta_\parallel^2} \tag{A 13}$$

is the magnitude of the normalized electron transverse velocity ($v_\perp/c$) induced by the wiggler magnetic field, $\bar{\Omega}_w \equiv eB_w/(\gamma mc^2 k_w)$, $\bar{\omega}_i^2 \equiv 2\pi e^2 n_i/(\gamma mc^2 k_w^2)$, $\beta_x = v_x/c$, $\beta_y = v_y/c$ and $\beta_\parallel = v_\parallel/c$. Note that $\bar{\omega}_i$ is defined in terms of the electron mass, not the ion mass. A more detailed analytical model for the self-magnetic field than is given in (A 3) can be determined by Ampère's law (A 1), using the velocity components of the electron in the presence of the wiggler and ion-channel fields. Using (A 10)–(A 12), Ampère's law can be written as

$$\frac{1}{r}\frac{\partial}{\partial\theta}B_{sz} - \frac{\partial}{\partial z}B_{s\theta} = -4\pi e n_b \beta_\perp \cos(k_w z - \theta), \tag{A 14}$$

$$\frac{\partial}{\partial z}B_{sr} - \frac{\partial}{\partial r}B_{sz} = -4\pi e n_b \beta_\perp \sin(k_w z - \theta), \tag{A 15}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_{s\theta}) - \frac{1}{r}\frac{\partial}{\partial\theta}B_{sr} = -4\pi e n_b \beta_\parallel. \tag{A 16}$$

The solution of (A 14)–(A 16) may be obtained by using the method used by Freund et al. [9]. This yields

$$\mathbf{B}_s^{(1)} = \mathbf{B}_{s\parallel} + \mathbf{B}_{sw}^{(1)}, \tag{A 17}$$

where

$$\mathbf{B}_{s\parallel} = -2\pi e n_b \beta_\parallel r \hat{\mathbf{e}}_\theta, \tag{A 18}$$

$$\begin{aligned}\mathbf{B}_{sw}^{(1)} &= \frac{4\pi e n_b}{k_w}\beta_\perp[\hat{\mathbf{e}}_r \cos(k_w z - \theta) + \hat{\mathbf{e}}_\theta \sin(k_w z - \theta)], \\ &= \frac{2\bar{\omega}_b^2 \beta_\parallel^2}{\bar{\omega}_i^2 - \beta_\parallel^2}\mathbf{B}_w,\end{aligned} \tag{A 19}$$

where $\bar{\omega}_b^2 \equiv 2\pi e^2 n_b/(\gamma mc^2 k_w^2)$ is the square of the normalized beam frequency. The first term in (A 17) is due to axial velocity and is exactly the same as the self-magnetic field previously derived in (A 3). The second term in (A 17) is the first correction to the self-magnetic field, which is due to the wiggler-induced transverse velocity ((A 10) and (A 11)), and we call this term the wiggler-induced self-magnetic field. It is interesting to note that for group I orbits ($\bar{\omega}_i < \beta_\parallel$) [6], the wiggler induced self-magnetic field is negative and acts as a diamagnetic correction to the wiggler magnetic field, while for group II orbits ($\bar{\omega}_i > \beta_\parallel$) [6], it is positive and acts as a paramagnetic correction to the wiggler magnetic field.

With the inclusion of these self-fields, the total electric and magnetic fields may be written as

$$\mathbf{E} = \mathbf{E}_i + \mathbf{E}_s = 2\pi e(n_i - n_b)(x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y), \tag{A 20}$$

$$\mathbf{B}^{(1)} = \mathbf{B}_w + \mathbf{B}_s^{(1)} = 2\pi e n_b \beta_\parallel (y\hat{\mathbf{e}}_x - x\hat{\mathbf{e}}_y) + \lambda_b^{(1)}\mathbf{B}_w, \tag{A 21}$$

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respectively, where

$$\lambda_{\rm b}^{(1)} = 1 + \frac{2\bar{\omega}_{\rm b}^2\beta_\parallel^2}{\bar{\omega}_{\rm i}^2 - \beta_\parallel^2} \tag{A 22}$$

is the first-order correction factor for the wiggler magnetic field. Using (A 20) and (A 21), the equation of electron motion (A 4), in presence of the self-fields, may be written in the scalar form:

$$\frac{d\beta_x}{dt} = ck_{\rm w}^2\left[\bar{\omega}_{\rm i}^2 - \bar{\omega}_{\rm b}^2(1-\beta_\parallel\beta_z)\right]x + \lambda_{\rm b}^{(1)}\bar{\Omega}_{\rm w}ck_{\rm w}\beta_z\sin(k_{\rm w}z), \tag{A 23}$$

$$\frac{d\beta_y}{dt} = ck_{\rm w}^2\left[\bar{\omega}_{\rm i}^2 - \bar{\omega}_{\rm b}^2(1-\beta_\parallel\beta_z)\right]y - \lambda_{\rm b}^{(1)}\bar{\Omega}_{\rm w}ck_{\rm w}\beta_z\cos(k_{\rm w}z), \tag{A 24}$$

$$\frac{d\beta_z}{dt} = ck_{\rm w}^2\beta_\parallel\bar{\omega}_{\rm b}^2(x\beta_x + y\beta_y) + \lambda_{\rm b}^{(1)}\bar{\Omega}_{\rm w}ck_{\rm w}[\beta_y\cos(k_{\rm w}z) - \beta_x\sin(k_{\rm w}z)]. \tag{A 25}$$

Assuming a low current density of the electron beam and neglecting the chaotic effects of the self-fields (if any), the steady-state solution of (A 23) and (A 24) may be obtained as

$$\beta_x^{(1)} = \frac{\lambda_b^{(1)}\bar{\Omega}_{\rm w}\beta_\parallel^2}{\bar{\omega}_{\rm i}^2 - \bar{\omega}_{\rm b}^2\gamma_\parallel^{-2} - \beta_\parallel^2}\cos(k_{\rm w}z), \tag{A 26}$$

$$\beta_y^{(1)} = \frac{\lambda_b^{(1)}\bar{\Omega}_{\rm w}\beta_\parallel^2}{\bar{\omega}_{\rm i}^2 - \bar{\omega}_{\rm b}^2\gamma_\parallel^{-2} - \beta_\parallel^2}\sin(k_{\rm w}z), \tag{A 27}$$

$$\beta_z^{(1)} = \beta_\parallel = \text{constant}. \tag{A 28}$$

Equations (A 26)–(A 28) can describe the normalized electron velocity components in the presence of the self-electric field and first-order (correction to the) self-magnetic field. Substituting new electron velocity components ((A 26)–(A 28)) and again using the method used by Freund et al. [9], we obtain the wiggler-induced self-magnetic field as

$$\mathbf{B}_{\rm sw}^{(2)} = K\lambda_{\rm b}^{(1)}\mathbf{B}_{\rm w}, \tag{A 29}$$

where

$$K \equiv \frac{2\bar{\omega}_{\rm b}^2\beta_\parallel^2}{\bar{\omega}_{\rm i}^2 - \bar{\omega}_{\rm b}^2\gamma_\parallel^{-2} - \beta_\parallel^2}. \tag{A 30}$$

Using (A 29), the total magnetic field, up to a second-order correction, is given by

$$\mathbf{B}_{\rm sw}^{(2)} = 2\pi e n_{\rm b}\beta_\parallel(y\hat{\mathbf{e}}_x - x\hat{\mathbf{e}}_y) + \lambda_{\rm b}^{(2)}\mathbf{B}_{\rm w}, \tag{A 31}$$

where

$$\lambda_{\rm b}^{(2)} = 1 + K\lambda_{\rm b}^{(1)}. \tag{A 32}$$

Equation (A 31) is the same as (A 21) with $\lambda_{\rm b}^{(1)}$ replaced by $\lambda_{\rm b}^{(2)}$. Thus the scalar form of the electron equation of motion is the same as (A 23)–(A 25) with $\lambda_{\rm b}^{(1)}$ replaced by $\lambda_{\rm b}^{(2)}$. Therefore, the steady-state solution for the normalized electron velocity up to a second-order correction can be obtained by replacing $\lambda_{\rm b}^{(1)}$ by $\lambda_{\rm b}^{(2)}$

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in (A 26)–(A 28). This yields

$$\beta_x^{(2)} \equiv \frac{\lambda_{\rm b}^{(2)}\bar{\Omega}_{\rm w}\beta_\parallel^2}{\bar{\omega}_{\rm i}^2 - \bar{\omega}_{\rm b}^2\gamma_\parallel^{-2} - \beta_\parallel^2}\cos(k_{\rm w}z), \tag{A 33}$$

$$\beta_y^{(2)} = \frac{\lambda_{\rm b}^{(2)}\bar{\Omega}_{\rm w}\beta_\parallel^2}{\bar{\omega}_{\rm i}^2 - \bar{\omega}_{\rm b}^2\gamma_\parallel^{-2} - \beta_\parallel^2}\sin(k_{\rm w}z), \tag{A 34}$$

$$\beta_z^{(2)} = \beta_\parallel = \text{constant}. \tag{A 35}$$

Continuing this procedure, we obtain

$$\left.\begin{aligned} \mathbf{B}_{\rm sw}^{(3)} &= K\lambda_{\rm b}^{(2)}\mathbf{B}_{\rm w}, \\ \mathbf{B}_{\rm sw}^{(4)} &= K\lambda_{\rm b}^{(3)}\mathbf{B}_{\rm w}, \\ &\vdots \\ \mathbf{B}_{\rm sw}^{(n)} &= K\lambda_{\rm b}^{(n-1)}\mathbf{B}_{\rm w}, \\ &\vdots \end{aligned}\right\} \tag{A 36}$$

where

$$\left.\begin{aligned} \lambda^{(3)} &= 1 + K\lambda^{(2)} \\ \lambda^{(4)} &= 1 + K\lambda^{(3)} \\ &\vdots \\ \lambda^{(n)} &= 1 + K\lambda^{(n-1)} \\ &\vdots \end{aligned}\right\}. \tag{A 37}$$

Using (A 36), (A 37), (A 30) and (A 22), the total wiggler-induced magnetic field becomes

$$\mathbf{B}_{\rm sw} = \lim_{n\to\infty}\mathbf{B}_{\rm sw}^{(n)} = K\lambda\mathbf{B}_{\rm w}, \tag{A 38}$$

where

$$\lambda = \lim_{n\to\infty}\lambda^{(n)} = \lim_{n\to\infty}\sum_{\ell=0}^{n-1}K^\ell + \lim_{n\to\infty}\frac{2\bar{\omega}_{\rm b}^2\beta_\parallel^2}{\bar{\omega}_{\rm i}^2 - \beta_\parallel^2}K^{n-1}. \tag{A 39}$$

If the absolute value of K (A 30) is less than one, then the series in (A 39) converges and the last term on the right-hand side goes to zero. In this case, (A 39) can be written as

$$\lambda = \frac{1}{1-K} = \frac{\bar{\omega}_{\rm i}^2 - \bar{\omega}_{\rm b}^2\gamma_\parallel^{-2} - \beta_\parallel^2}{\bar{\omega}_{\rm i}^2 - \bar{\omega}_{\rm b}^2(1+\beta_\parallel^2) - \beta_\parallel^2}. \tag{A 40}$$

Substituting (A 40) into (A 38), and using (A 30), the total wiggler-induced self-magnetic field is given by

$$\mathbf{B}_{\rm sw} = \frac{2\bar{\omega}_{\rm b}^2\beta_\parallel^2}{\bar{\omega}_{\rm i}^2 - \bar{\omega}_{\rm b}^2(1+\beta_\parallel^2) - \beta_\parallel^2}\mathbf{B}_{\rm w}. \tag{A 41}$$

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