The seven circles theorem revisited

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1. *Introduction*

The circles $C_1, \ldots, C_n$ form a *chain* of length n if C_i touches C_{i+1}, for $i = 1, \ldots, n-1$, and the chain is *closed* if also C_n touches C_1. A *cyclic* chain is a chain for which all the circles touch another circle S, the *base circle* of the chain. If C_i touches S at P_i, then $P_1, \ldots, P_n$ are the *base points* of the chain. Sometimes there may be coincidences among the base points; in particular, if $P_i = P_j$, then the line P_iP_j should be interpreted as the tangent to S at P_i.

The seven circles theorem first appeared in [1, §3.1], and some historical details of its genesis can be found in John Tyrrell's obituary [2]. The theorem concerns a closed cyclic chain of length 6, and says that, if a certain extra condition is satisfied, then the lines P_1P_4, P_2P_5, P_3P_6 joining opposite base points are concurrent. Here and throughout, 'concurrent' should be read as 'concurrent or all parallel', that is, the point of concurrency might be at infinity.

The extra condition is to do with the order in which the base points are distributed around the base circle. In [1] the condition is expressed in terms of *positive arcs*. An ordered triple (ABC) of points on S describes a *positive arc* if the route from A to C via B involves travelling anticlockwise around S. (This definition is the opposite of that given in [1], but this makes no difference in what follows.) The condition given in [1] under which the seven circles theorem is true is that, of the four arcs $(P_1P_2P_6)$, $(P_2P_3P_6)$, $(P_3P_4P_6)$, $(P_4P_5P_6)$, an even number are positive arcs. This is obviously satisfied, for instance, if the points P_1, P_2, P_3, P_4, P_5, P_6 are in this cyclic order on S. See Figure 1.

The method of proof in [1] was by inversion. A simpler proof, using complex numbers, appeared in [3]. Our approach will be similar to that in [3], except that we shall use the rational parametrisation $x = \dfrac{1-t^2}{1+t^2}$, $y = \dfrac{2t}{1+t^2}$ for S, which we take as the unit circle $x^2 + y^2 = 1$. Among other things, we shall obtain the original seven circles theorem and its converse, and we shall extend the converse to some cyclic chains of length other than 6, and we shall also investigate the 'failure' case of the seven circles theorem, where the joins of opposite base points are not concurrent.

The parameter t lies in $\mathbb{R} \cup \{\infty\}$, the value $t = \infty$ giving the point $(-1, 0)$; but since we are only concerned with a finite number of points on S, we shall assume throughout that the coordinate system has been chosen so that all the values of t that occur are finite.

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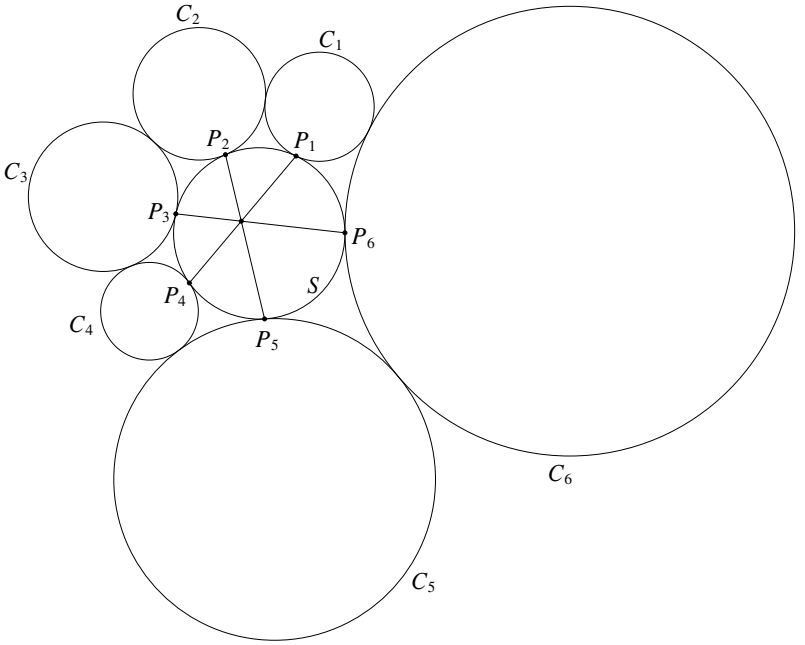


FIGURE 1

In [1], the seven circles theorem was followed, in §3.2, by a somewhat similar theorem about a chain of six circles alternately touching two parallel lines. We shall give a proof of this theorem and its converse, with an extension of the converse to more circles, in the final section of this paper.

2. *Concurrent chords*

Suppose the points $P_1, \ldots, P_6$ lie on the unit circle S: $x^2 + y^2 + 1$, centre O. Let P_i have parameter t_i, for $i = 1, \ldots, 6$.

Proposition 1: The lines P_1P_4, P_2P_5 and P_3P_6 are concurrent if, and only if,

$$(t_1 - t_2)(t_3 - t_4)(t_5 - t_6) + (t_2 - t_3)(t_4 - t_5)(t_6 - t_1) = 0. \qquad (1)$$

We note in passing that condition (1) is obviously unchanged by the cyclic permutation (1 2 3 4 5 6) of subscripts, but is also unchanged, though less obviously so, by the subscript permutations (1 4), (2 5) and (3 6). Note also that, in [2], the condition for concurrency is exactly our equation (1), *except* that in [2], $t_1, t_2, \ldots$ are not real numbers, parameters of the base points, but are complex numbers of modulus 1 representing them. We leave the reader to explore the coincidence.

Proof of Proposition 1: First we deal with the cases where some of the P_i, other than the opposite pairs (P_1, P_4), (P_2, P_5), (P_3, P_6), coincide; that is,

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where $P_i = P_{i+1}$ or $P_i = P_{i+2}$, some i, subscripts being interpreted modulo 6. If $P_1 = P_2$ and the three lines are concurrent, then either they meet at P_1, so that $P_1 = P_3$ or $P_1 = P_6$; or else $P_4 = P_5$, so that P_1P_4 and P_2P_5 are the same line. In all these cases, it is immediate that (1) holds. Then if $P_1 = P_3$ and the three lines are concurrent, either they meet at P_1, so that $P_1 = P_2$ or $P_1 = P_5$; or else $P_4 = P_6$, so that P_1P_4 and P_3P_6 are the same line. In all these cases too, it is easy to check that (1) holds.

Conversely, suppose (1) holds and $t_1 = t_2$. Then either $t_2 = t_3$, so that $P_1 = P_2 = P_3$, or $t_6 = t_1$, so that $P_1 = P_2 = P_6$, or $t_4 = t_5$, in which case P_1P_4 and P_2P_5 are the same line. Again, if (1) holds and $t_1 \neq t_2$ but $t_1 = t_3$, then (1) is equivalent to $(t_1 - t_5)(t_4 - t_6) = 0$, so that either $P_1 = P_3 = P_5$ or the lines P_1P_4 and P_3P_6 coincide.

Similar arguments hold if we start with the other possible equalities; so from now on we shall assume that $P_i \neq P_{i+1}$ and $P_i \neq P_{i+2}$, for all i. Equation (1) is then equivalent to

$$\frac{(t_1 - t_2)(t_3 - t_4)(t_5 - t_6)}{(t_2 - t_3)(t_4 - t_5)(t_6 - t_1)} = -1. \tag{2}$$

If the three lines are concurrent then there is an involution on S for which (P_1, P_4), (P_2, P_5), (P_3, P_6) are corresponding pairs. The involution takes the form $t \to f(t)$, where $f(t) = \dfrac{at + b}{ct - a}$, for all t, and $a^2 + bc \neq 0$. Note that, for any s, t, we have

$$\frac{s - f(t)}{f(s) - t} = \frac{cs - a}{a - ct}. \tag{3}$$

Then we must have

$$\frac{(t_1 - t_2)(t_3 - t_4)(t_5 - t_6)}{(t_2 - t_3)(t_4 - t_5)(t_6 - t_1)} = \frac{(t_1 - f(t_5))(t_3 - f(t_1))(t_5 - f(t_3))}{(f(t_5) - t_3)(f(t_1) - t_5)(f(t_3) - t_1)},$$

which, by three applications of (3), is equal to -1, as required.

For the converse, suppose (2) holds, and choose P_6' so that P_1P_4, P_2P_5 and P_3P_6' are concurrent. So by the above we have

$$\frac{(t_1 - t_2)(t_3 - t_4)(t_5 - t_6')}{(t_2 - t_3)(t_4 - t_5)(t_6' - t_1)} = \frac{(t_1 - t_2)(t_3 - t_4)(t_5 - t_6)}{(t_2 - t_3)(t_4 - t_5)(t_6 - t_1)},$$

which reduces to $(t_5 - t_6')(t_6 - t_1) = (t_5 - t_6)(t_6' - t_1)$, or $(t_5 - t_1)(t_6' - t_6) = 0$. But $t_5 \neq t_1$, so $t_6' = t_6$, and we have finished.

It is interesting to try to reverse the involution argument to prove the last part above. Suppose (1) holds, where $P_i \neq P_{i+1}$ and $P_i \neq P_{i+2}$, for all i. Then there is a homography f on S with $f(t_1) = t_4$, $f(t_5) = t_2$ and $f(t_3) = t_6$. Thus (1) becomes

$$(t_1 - f(t_5))(t_3 - f(t_1))(t_5 - f(t_3)) + (f(t_5) - t_3)(f(t_1) - t_5)(f(t_3) - t_1) = 0. \tag{4}$$

Suppose $f(t) = \dfrac{at + b}{ct + d}$, where $ad - bc \neq 0$. By some heavy algebra

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(best done on a computer), the left-hand side of (4) reduces to

$$\frac{(t_1 - t_3)(t_3 - t_5)(t_1 - t_5)(ad - bc)(a + d)}{(ct_1 + d)(ct_3 + d)(ct_5 + d)}. \tag{5}$$

Since t_1, t_3, t_5 are distinct and $ad - bc \neq 0$, condition (4) implies that $a + d = 0$, which means that f is an involution, and so the lines P_1P_4, P_2P_5 and P_3P_6 meet at its vertex. (For the theory of involutions on a conic, see [4, Chapter VI, §4].) Note that some parts of the calculation that yields (5) are quite straightforward. We have a rational function in the seven variables $(t_1, t_3, t_5, a, b, c, d)$, and it is obvious that the denominator is $(ct_1 + d)(ct_3 + d)(ct_5 + d)$. As for the numerator, the left-hand side of (4) vanishes if we put $t_1 = t_3$ or $t_3 = t_5$ or $t_1 = t_5$. It also vanishes if $ad - bc = 0$, for then $f(t_1) = f(t_3) = f(t_5)$. And if $a + d = 0$ then f is an involution, so the left-hand side of (4) vanishes by the argument in the proof of Proposition 1. So the numerator contains

$$(t_1 - t_3)(t_3 - t_5)(t_1 - t_5)(ad - bc)(a + d)$$

as a factor; the problem is to see that it contains nothing else, as superficially it appears to be of higher degree than this in the seven variables.

The proof of Proposition 1 can be given quite easily in terms of coordinate geometry, with no mention of involutions, as the reader may wish to verify. The reason for using involutions, though, is that the method generalises very nicely to any odd number of chords (but in one direction only), as we now show:

Proposition 2: Let n be odd, $n \geqslant 3$, and let $P_1, \ldots, P_{2n}$ be distinct points on the unit circle, where P_i has parameter t_i, for $i = 1, \ldots, 2n$. If the chords $P_1P_{n+1}, P_2P_{n+2}, \ldots, P_nP_{2n}$ are concurrent, then

$$\prod_{i=1}^{n} \frac{t_{2i-1} - t_{2i}}{t_{2i} - t_{2i+1}} = -1$$

where the subscripts are interpreted modulo $2n$.

Proof: There is an involution f with $t_i = f(t_{i+n})$, for $i = 1, \ldots, 2n$. Then if $f(t) = \dfrac{at + b}{ct - a}$, for all t, we have

$$\prod_{i=1}^{n} \frac{t_{2i-1} - t_{2i}}{t_{2i} - t_{2i+1}} = \prod_{i=1}^{n} \frac{t_{2i-1} - f(t_{2i+n})}{f(t_{2i+n}) - t_{2i+1}} = \prod_{i=1}^{n} \frac{t_{2i-1} - f(t_{2i+n})}{f(t_{2i-1}) - t_{2i+n}} = \prod_{i=1}^{n} \frac{ct_{2i-1} - a}{a - ct_{2i+n}}$$

where the third expression is obtained from the second by rearranging the denominators, by means of the permutation $t_i \to t_j$, where $j \equiv i + \frac{1}{2}(n-1)$ modulo $2n$. The last expression is now equal to -1, since the denominators are a permutation of the negatives of the numerators, and n is odd.

Note that we cannot expect the converse of Proposition 2 to be true. For instance, to deduce that five chords are concurrent, we need to know that the

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first and second are concurrent with the third, with the fourth, and with the fifth, which is three separate conditions, not one.

3. *Adjacent circles*

We need a preliminary result about two adjacent circles of a cyclic chain. So let S be the base circle, the unit circle, centre O. Let the circle C_i, centre O_i, touch S at P_i, for $i = 1, 2$, and let C_1 and C_2 touch at Q. Let the common tangents at P_1, P_2 and Q meet at X, the radical centre of the three circles. Then there is a circle S', centre X, touching OO_1 at P_1, OO_2 at P_2, and O_1O_2 at Q. There are several possible configurations: no circle inside another (Figure 2); C_1 and C_2 both inside S (Figure 3); or S inside one of them — C_2, say — but not the other (Figure 4). There are also two limiting cases, where one of C_1, C_2 is a line (Figure 5) or where both are lines (Figure 6).

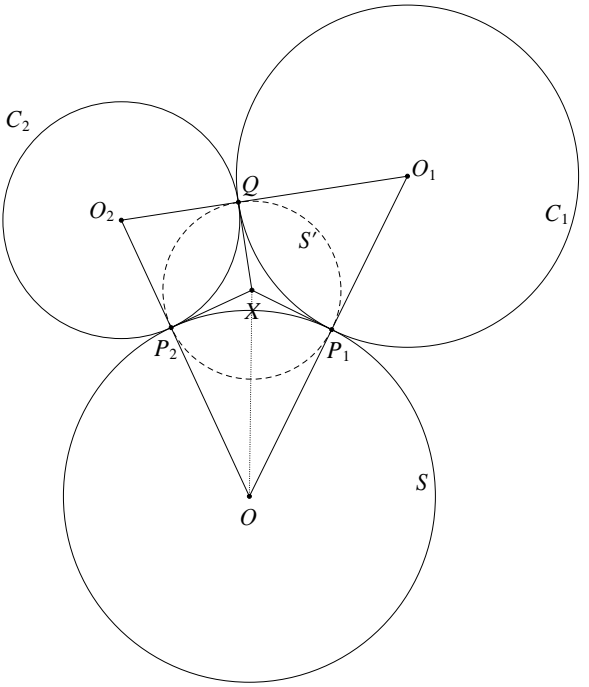


FIGURE 2

Let $\angle P_1OP_2 = 2\theta$, so that, since the radius of S is 1, the radius of S' is $\tan\theta$. Let the radius of C_i be r_i, $i = 1, 2$. To take account of the different configurations and avoid a multiplicity of formulæ, we define the signed radius ρ_i of C_i to be r_i if P_i is between O and O_i, and $-r_i$ otherwise. Clearly 0 must be a banned value of ρ_i, but so too must -1, since $C_i \neq S$. However, we allow $\rho_i = \infty$, when C_i becomes a line, the tangent to S at P_i, regarded as a circle of infinite radius.

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Proposition 3: In all cases, $\tan^2\theta = \dfrac{\rho_1\rho_2}{1 + \rho_1 + \rho_2}$.

Proof: We use the standard notation for $\triangle ABC$, where the sides opposite A, B, C have lengths a, b, c respectively, s is the semi-perimeter, r the inradius, r_a the radius of the excircle opposite A, and so on. Recall that the tangents to the incircle from A, B, C have lengths $s - a$, $s - b$, $s - c$ respectively, and the tangents from A, B, C to the excircle opposite A have lengths s, $s - c$, $s - b$ respectively, and so on. Furthermore, $r^2 = s^{-1}(s-a)(s-b)(s-c)$, $r_a^2 = s(s-a)^{-1}(s-b)(s-c)$, and so on.

Suppose we relabel O as A, O_1 as B, and O_2 as C. Then in Figure 2, S' is the incircle, so we have $s = 1 + r_1 + r_2 = 1 + \rho_1 + \rho_2$, $s - a = 1$, $s - b = r_1 = \rho_1$, $s - c = r_2 = \rho_2$ and $r = \tan\theta$. The result follows.

In Figure 3, S' is the excircle opposite A, so we have $s = 1$, $s - a = 1 - r_1 - r_2 = 1 + \rho_1 + \rho_2$, $s - b = -\rho_2$, $s - c = -\rho_1$ and $r_a = \tan\theta$. The result follows, again.

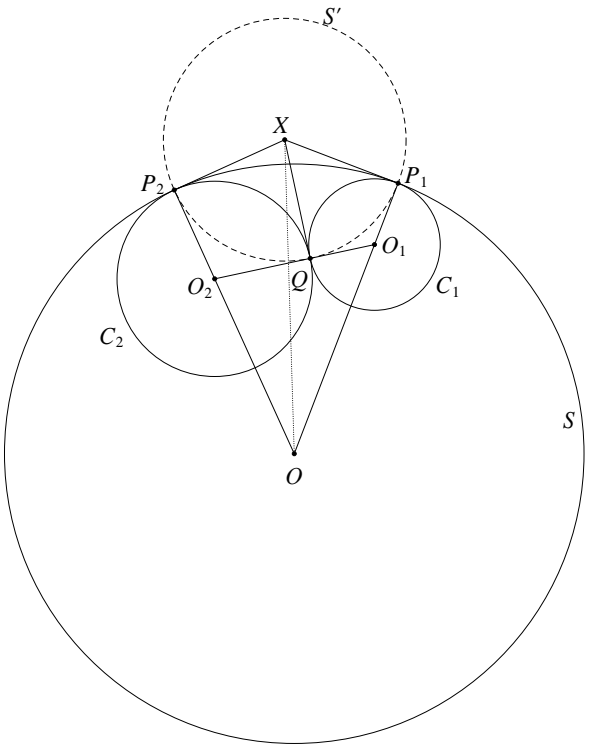


FIGURE 3

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In Figure 4, S' is the excircle opposite C, so we have $s = r_2 = -\rho_2$, $s - a = r_1 = \rho_1$, $s - b = 1$, $s - c = r_2 - r_1 - 1 = -(1 + \rho_1 + \rho_2)$ and $r_c = \tan\theta$. The result follows, once more. (And, by the symmetry, this would also hold if we chose to put S inside C_1 instead of inside C_2.)

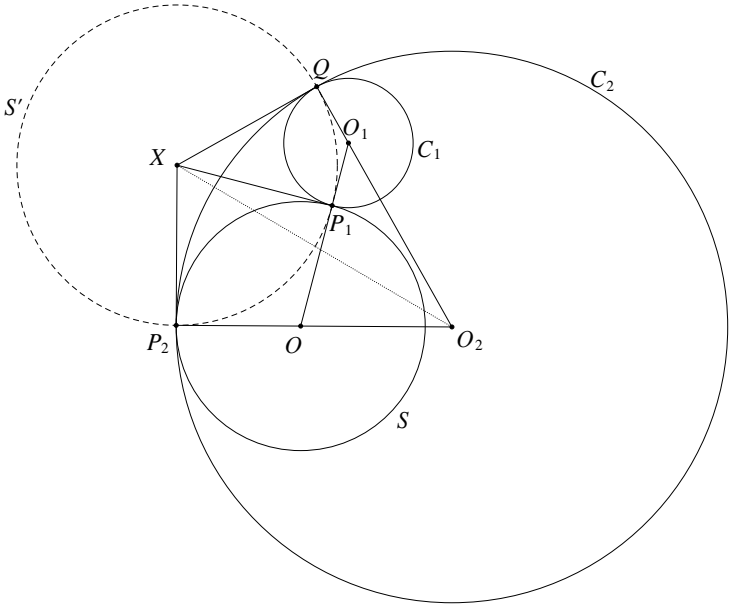


FIGURE 4

In Figure 5, we have allowed $\rho_1 \to \infty$, so C_1 becomes a line, the tangent to S at P_1, and O_1 is at infinity, so that $OP_1 \,//\, O_2Q$. Now OX bisects $\angle P_1OP_2$ and O_2X bisects $\angle P_2O_2Q$, from which it follows easily that

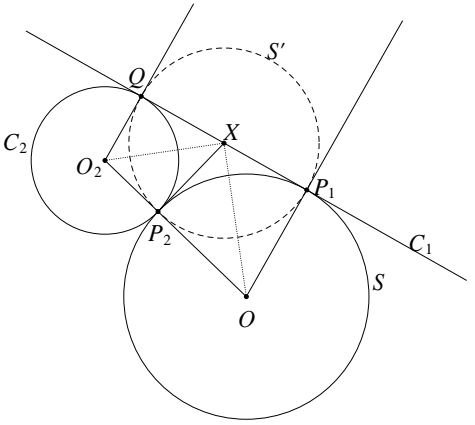


FIGURE 5

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$\angle O_2XO = \frac{\pi}{2}$, and so $\angle O_2XP_2 = \angle XOP_2 = \theta$. So

$$\lim_{\rho_1 \to \infty} = \frac{\rho_1\rho_2}{1+\rho_1+\rho_2} = \rho_2 = \frac{|O_2P_2|}{|P_2O|} = \left(\frac{|O_2P_2|}{|P_2X|}\right)\left(\frac{|XP_2|}{|P_2O|}\right) = (\tan\theta)(\tan\theta) = \tan^2\theta.$$

A similar calculation holds if instead $\rho_2 \to \infty$, so that C_2 is a line but C_1 is a circle.

In Figure 6, we have allowed both $\rho_1 \to \infty$ and $\rho_2 \to \infty$, so C_1 and C_2 are both lines, necessarily parallel since they touch at Q, which is now at infinity. Further, P_1, O and P_2 must be collinear, so $\theta = \frac{\pi}{2}$, and

$$\lim_{\substack{\rho_1 \to \infty \\ \rho_2 \to \infty}} \frac{\rho_1\rho_2}{1 + \rho_1 + \rho_2} = \infty = \tan^2\theta.$$

which completes the proof.

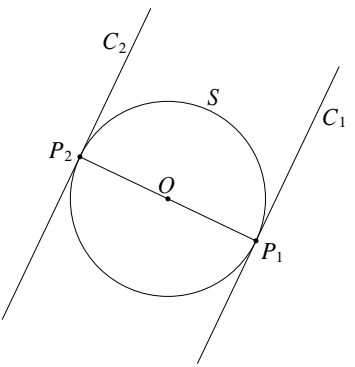


FIGURE 6

The equation of Proposition 2 can be rewritten as

$$\rho_2 = f(\rho_1) = \frac{\rho_1 + 1}{\rho_1 \cot^2\theta - 1}.$$

Here f is the Möbius transformation with matrix $\begin{pmatrix} 1 & 1 \\ \cot^2\theta & -1 \end{pmatrix}$. Note that f is an involution on $\mathbb{R} \cup \{\infty\}$, and interchanges the banned values 0 and -1, so it is an involution on the set of allowed values, $\mathbb{R} \cup \{\infty\} \setminus \{0, -1\}$.

We remark in passing that, given the circle S, centre O, points P_1 and P_2 on S, and the circle C_1, centre O_1, touching S at P_1, it is easy to construct the circle C_2 touching S at P_2 and also touching C_1. First construct tangents to S at P_1 and P_2, meeting at X. Next put in the circle S', centre X, through P_1 and P_2. Then S and C_1 meet at P_1 and again at Q; O_1Q meets OP_2 at O_2; and C_2 is the circle centre O_2 through Q and P_2.

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4. *Cyclic chains*

Let $C_1, \dots, C_n$ be a cyclic chain of length n, with base circle S, and base points $P_1, \dots, P_n$. From now on, we shall assume that $P_i \neq P_{i+1}$ and $P_i \neq P_{i+2}$, for $i = 1, \dots, n$, subscripts being interpreted modulo n. Let ρ_i be the signed radius of C_i, and let $\angle P_i O P_{i+1} = 2\theta_i$, for $i = 1, \dots, n$. So if f_i is the Möbius transformation with matrix $\begin{pmatrix} 1 & 1 \\ \cot^2\theta_i & -1 \end{pmatrix}$, then by the last section, we have $\rho_{i+1} = f_i(\rho_i)$, for $i = 1, \dots, n-1$. The condition for the chain to be closed is $\rho_1 = f_n(\rho_n)$, or $\rho_1 = f_n f_{n-1} \dots f_2 f_1(\rho_1)$.

We must calculate the matrix of $f_n f_{n-1} \dots f_2 f_1$. To this end, for any a, b, c, d, let

$$\mathbf{M}(a, b) = \begin{pmatrix} a & a \\ b-a & -a \end{pmatrix} \quad \text{and} \quad \mathbf{N}(c, d) = \begin{pmatrix} c & 0 \\ d-c & d \end{pmatrix}.$$

Proposition 4: For all a, b, c, d we have:

(i) $\mathbf{M}(a, b)\mathbf{M}(c, d) = \mathbf{N}(ad, bc)$;

(ii) $\mathbf{N}(a, b)\mathbf{N}(c, d) = \mathbf{N}(ac, bd)$;

(iii) $\mathbf{M}(a, b)\mathbf{N}(c, d) = \mathbf{M}(ad, bc)$.

Proof: Immediate.

Now, for $i = 1, \dots, n$, put $k_i = \operatorname{cosec}^2\theta_i$, so that $\cot^2\theta_i = k_i - 1$, and the matrix of f_i is $\mathbf{M}(1, k_i)$.

Proposition 5: The matrix of $f_n f_{n-1} \dots f_2 f_1$ is $\mathbf{N}(k_1 k_3 \dots k_{n-1}, k_2 k_4 \dots k_n)$ if n is even, or $\mathbf{M}(k_2 k_4 \dots k_{n-1}, k_1 k_3 \dots k_n)$ if n is odd.

Proof: Immediate, by Proposition 4.

Now let $P(1, 0)$ and let $\angle POP_i = 2\varphi_i$, for $i = 1, \dots, n$. So the rational parameter of P_i is $t_i = \tan\varphi_i$, for $i = 1, \dots, n$. Put $t_{n+1} = t_1$. Then $\theta_i = \varphi_{i+1} - \varphi_i$, for $i = 1, \dots, n$.

Proposition 6: Let $n = 2m$. Then the cyclic chain $C_1, \dots, C_n$ is closed if, and only if,

$$\prod_{i=1}^{m} (t_{2i-1} - t_{2i})^2 = \prod_{i=1}^{m} (t_{2i} - t_{2i+1})^2.$$

Proof: The condition we are after is that $\rho_1 = f(\rho_1)$, where $f = f_n f_{n-1} \dots f_2 f_1$. By Proposition 5, f has matrix $\mathbf{N}(h_1, h_2)$, where $h_1 = k_1 k_3 \dots k_{n-1}$ and $h_2 = k_2 k_4 \dots k_n$. So we need $\rho_1 = \dfrac{h_1\rho_1}{(h_2 - h_1)\rho_1 + h_2}$, which reduces to $(h_2 - h_1)\rho_1(\rho_1 + 1) = 0$. We reject the solution $\rho_1 = 0$, and also the solution $\rho_1 = -1$, since this would imply $C_1 = S$. So we are left

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with $h_1 = h_2$ as the required condition.

Now $\tan\theta_i = \tan(\varphi_{i+1} - \varphi_i) = \dfrac{t_{i+1} - t_i}{1 + t_{i+1}t_i}$,

so $$k_i = 1 + \cot^2\theta_i = \frac{(1 + t_i)^2(1 + t_{i+1})^2}{(t_{i+1} - t_i)^2},$$

for $i = 1, \ldots, n$. So the condition $h_1 = h_2$ is equivalent to

$$\prod_{i=1}^{m} \frac{(1 + t_{2i-1})^2(1 + t_{2i})^2}{(t_{2i-1} - t_{2i})^2} = \prod_{i=1}^{m} \frac{(1 + t_{2i})^2(1 + t_{2i+1})^2}{(t_{2i} - t_{2i+1})^2}.$$

Combining the fractions on each side, we find the numerators are identical, and equating the denominators yields the result.

Given $P_1, \ldots, P_n$ on S, with n even, choose the circle C_1 as any circle touching S at P_1, and we can then find C_i touching S at P_i, for $i = 2, \ldots, n$, so that $C_1, \ldots, C_n$ is a cyclic chain. Then from Proposition 6, *either* this chain is closed, for every choice of C_1 (that is, for every choice of ρ_1), *or* it is never closed, for any choice of C_1. Such a result, where there are either infinitely many solutions or else none at all, is known as a *porism*.

The situation is different for a closed cyclic chain of odd length, as we shall now see.

Proposition 7: Let n be odd. Then the cyclic chain $C_1, \ldots, C_n$ is closed if, and only if,

$$\rho_1 = \frac{h_2 \pm \sqrt{h_1 h_2}}{h_1 - h_2}$$

where $h_1 = k_1k_3\ldots k_n$ and $h_2 = k_2k_2\ldots k_{n-1}$.

Proof: Put $f = f_n f_{n-1}\ldots f_2 f_1$, as before. By Proposition 5, f has matrix $\mathbf{M}(h_2, h_1)$, so the condition $\rho_1 = f(\rho_1)$ reads $\rho_1 = \dfrac{h_2\rho_1 + h_2}{(h_1 - h_2)\rho_1 - h_2}$, or $(h_1 - h_2)\rho_1^2 - 2h_2\rho_1 - h_2 = 0$. The result follows.

So this time, given *any* base points $P_1, \ldots, P_n$, there are two closed cyclic chains with these base points. Furthermore, if ρ_1, ρ_1' are the two roots of the above quadratic equation, it is easy to see that $(1 + 2\rho_1)(1 + 2\rho_1') = 1$, so that the one chain can be obtained from the other by inversion in S, as one should expect.

5. *The five and seven circles theorems*

We now come to the main results of this article.

Proposition 8: (The five circles theorem.) Suppose we have a cyclic chain C_1, C_2, C_3, C_4, with base circle S and distinct base points P_1, P_2, P_3, P_4.

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Then the chain is closed if, and only if, the tangents to S at P_1 and P_3 meet on P_2P_4. See Figure 7.

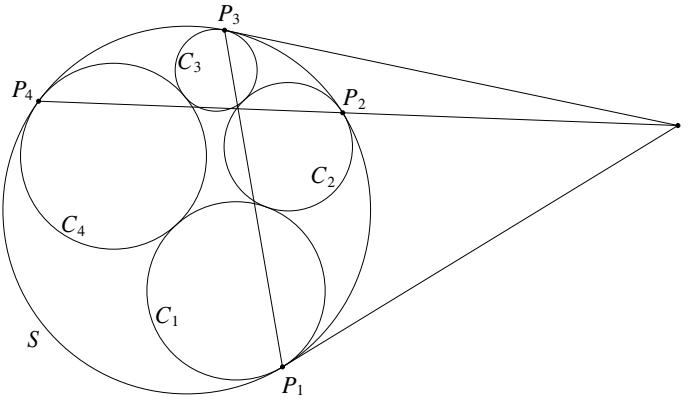


FIGURE 7

Proof: Applying Proposition 1 to the degenerate hexagon $P_1P_2P_3P_1P_4P_3$, we see that the tangents at P_1 and P_3 meet on P_2P_4 if, and only if,

$$\frac{(t_1 - t_2)(t_3 - t_1)(t_4 - t_3)}{(t_2 - t_3)(t_1 - t_4)(t_3 - t_1)} = -1,$$

that is

$$\frac{(t_1 - t_2)(t_3 - t_4)}{(t_2 - t_3)(t_4 - t_1)} = -1.$$

Now we cannot have $(t_1 - t_2)(t_3 - t_4) = (t_2 - t_3)(t_4 - t_1)$, or else $(t_1 - t_3)(t_2 - t_4) = 0$, which is impossible since the P_i are distinct. So

$$\frac{(t_1 - t_2)(t_3 - t_4)}{(t_2 - t_3)(t_4 - t_1)} = -1$$

if, and only if,

$$(t_1 - t_2)^2(t_3 - t_4)^2 = (t_2 - t_3)^2(t_4 - t_1)^2,$$

which, by Proposition 6, is the condition for the chain to be closed.

Another way of stating the above condition is that the pole of P_1P_3 with respect to S lies on P_2P_4 (and vice versa), that is, the lines P_1P_3 and P_2P_4 are conjugate with respect to S. This also follows directly from the equation $\frac{(t_1 - t_2)(t_3 - t_4)}{(t_2 - t_3)(t_4 - t_1)} = -1$, which just says that P_1, P_3 separate P_2, P_4 harmonically on S.

Proposition 9: (The seven circles theorem.) Suppose we have a cyclic chain $C_1, \ldots, C_6$, with base circle S and base points $P_1, \ldots, P_6$. Let P_6' be the harmonic conjugate of P_6 with respect to P_1, P_5 on S, that is, the lines P_1P_5

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and P_6P_6' are conjugate with respect to S. Then the chain is closed if, and only if, either (i) P_1P_4, P_2P_5 and P_3P_6 are concurrent, or (ii) P_1P_4, P_2P_5 and P_3P_6' are concurrent.

Proof: The chain is closed if, and only if,

$$(t_1 - t_2)(t_3 - t_4)(t_5 - t_6) = \pm(t_2 - t_3)(t_4 - t_5)(t_6 - t_1),$$

by Proposition 6.

Case (i): P_1P_4, P_2P_5 and P_3P_6 are concurrent if, and only if,

$$(t_1 - t_2)(t_3 - t_4)(t_5 - t_6) = -(t_2 - t_3)(t_4 - t_5)(t_6 - t_1),$$

by Proposition 1. See Figure 1.

Case (ii): $(t_5 - t_6)(t_6' - t_1) = -(t_6 - t_1)(t_5 - t_6')$, since P_1P_5 and P_6P_6' are conjugate. So P_1P_4, P_2P_5 and P_3P_6' are concurrent if, and only if,

$$(t_2 - t_3)(t_4 - t_5)(t_6' - t_1) = -(t_1 - t_2)(t_3 - t_4)(t_5 - t_6'),$$

and by the above this holds if, and only if,

$$(t_1 - t_2)(t_3 - t_4)(t_5 - t_6) = (t_2 - t_3)(t_4 - t_5)(t_6 - t_1),$$

which completes the proof. See Figure 8.

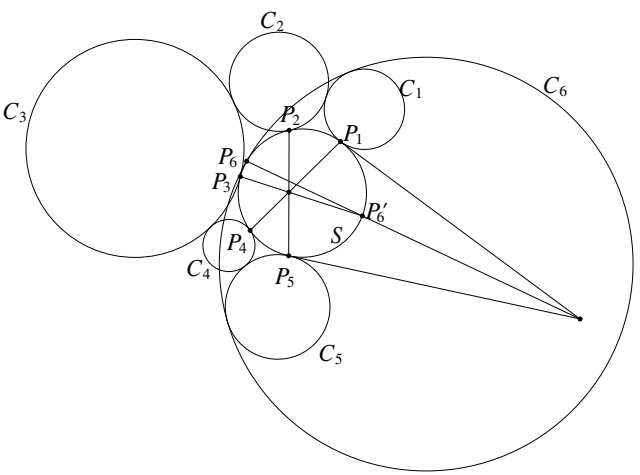


FIGURE 8

Proposition 9 is not quite the seven circles theorem as it appeared in [1]. There, they start with the cyclic chain $C_1, \ldots, C_5$, as above, and are concerned with where to place C_6 in order to make a closed cyclic chain such that P_1P_4, P_2P_5 and P_3P_6 are concurrent. There are two circles touching C_1, C_5 and S, either of which will close the chain, but only one of which gives the concurrency, so, to quote [1, p.31], 'One is tempted to say that the

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theorem has a 50% chance of being true!' Now, if we call the two possible candidates C_6 and C_6', with base points P_6 and P_6', then C_5, C_6, C_1, C_6' is a closed cyclic chain of length 4, so that P_6, P_6' separate P_1, P_5 harmonically on S, by Proposition 8. Thus in our version of the seven circles theorem, case (i) is the 50% success, and case (ii) is the 50% failure of the original version.

As mentioned in the introduction, the condition for (i) to happen, rather than (ii), can be given in terms of directed arcs. The arc $P_iP_jP_k$ is positive if the route from P_i to P_k around S, via P_j, involves travelling anticlockwise. In terms of the parameters, this says that either $t_i < t_j < t_k$ or $t_j < t_k < t_i$ or $t_k < t_i < t_j$; and this in turn is equivalent to requiring that $(t_i - t_j)(t_j - t_k)(t_k - t_i) > 0$, or $\tau_{ij}\tau_{jk}\tau_{ki} = +1$, where τ_{ij} is $+1$ or -1 according as $(t_i - t_j)$ is positive or negative, and similarly for the others.

So now suppose an even number of the arcs $(P_1P_2P_6)$, $(P_2P_3P_6)$, $(P_3P_4P_6)$, $(P_4P_5P_6)$ are positive. Then

$$(\tau_{12}\tau_{26}\tau_{61})(\tau_{23}\tau_{36}\tau_{62})(\tau_{34}\tau_{46}\tau_{63})(\tau_{45}\tau_{56}\tau_{64}) = +1$$

which is equivalent to

$$\tau_{12}\tau_{34}\tau_{56} = -\tau_{23}\tau_{45}\tau_{61}$$

and this in turn means we are in case (i) of Proposition 9, not case (ii).

If we are given a cyclic chain $C_1, \ldots, C_5$, then Proposition 9 can be used to construct C_6 to close the chain, that is, to touch C_1, C_5 and S. All that is needed is to put in P_6 on S so that the lines P_1P_4, P_2P_5 and P_3P_6 are concurrent, and then use the procedure described at the end of Section 3 to construct P_6 touching C_5 and touching S at P_6, and this circle will necessarily touch P_1 as well.

One of the most famous problems in plane geometry is *Apollonius' circle problem*, to construct a circle or circles touching three given circles. See [5]. In general, the problem has eight solutions, but when two of the given circles touch the third, it has only four solutions. In this case, we can use Proposition 9 to find two of them. (This may not be the simplest method, but it is a nice application of Proposition 9.)

Assume, then, that we are given S and two circles C_1 and C_5, touching S at P_1 and P_5, and assume they are external to one another, as in Figure 9. Put in a common tangent to S and C_1, label it C_2 and let it touch S at P_2. Likewise put in a common tangent to S and C_5, label it C_4 and let it touch S at P_4. A circle C_3 (not shown) touching S, C_2 and C_4, must touch S at a point P_3 midway between P_2 and P_4, that is, at one end or other of the diameter of S through the meet K of C_2 and C_4. With P_i in place, for $i = 1, \ldots, 5$, we can now construct P_6, and then C_6 touching C_1, C_5 and S, usimg the construction descibed in the paragraph before last. The two choices for P_3 yield the two possible circles C_6; in Figure 9 the second choices are labelled P_3', P_6' and C_6'. We leave it to the reader to verify that making the other choice of common tangent to S and C_1, or the other choice of common tangent to S and C_5, or both, will give the same two circles C_6, C_6'.

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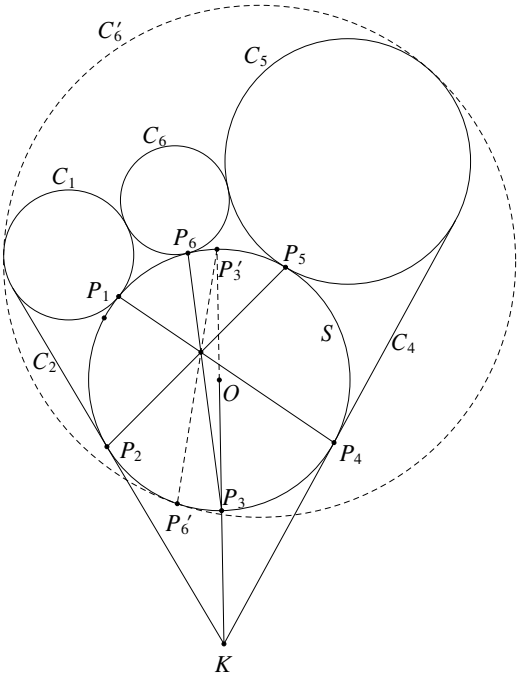


FIGURE 9

The method can be adapted to cover cases where (say) S is inside C_1 or vice versa, so that they have no common tangent (apart from the one at P_1). In that case, choose C_2 arbitrarily to touch C_1 and S, and then it is not hard to construct C_4 touching C_5 and S, and of the same radius as C_2. Then once again P_3 is midway between P_2 and P_4, and the construction can be completed.

The remaining two solutions of Apollonius' problem are the circle touching S and C_1 at P_1 and also touching C_2; and the circle touching S and C_2 at P_2 and also touching C_1. Both of these are easily constructed by the procedure described at the end of Section 3.

6. *The* $4m + 3$ *circles theorems*

We now prove two results about chains of length $4m + 2$, $m \geqslant 1$, the first of which extends the converse of the original seven circles theorem.

Proposition 10: Put $n = 2m + 1$, and suppose $C_1, \ldots, C_{2n}$ is a cyclic chain, with base circle S and base points $P_1, \ldots, P_{2n}$. If the lines P_1P_{n+1}, $P_2P_{n+2}, \ldots, P_nP_{2n}$ are concurrent, then the chain is closed.

Proof: Immediate from Proposition 2 and Proposition 6.

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An example of the case $n = 7$, involving fifteen circles, is shown in Figure 10.

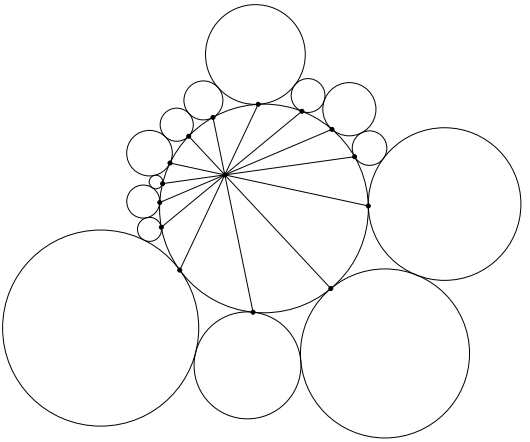


FIGURE 10

For the second result, we let the base points coincide in pairs in a certain way; and no further condition is needed:

Proposition 11: Put $n = 2m + 1$, and suppose $C_1, \ldots, C_{2n}$ is a cyclic chain, with base circle S and base points $P_1, \ldots, P_{2n}$, where $P_i = P_{n+i}$ for $i = 1, \ldots, n$. Then the chain is closed.

Proof: We have $t_i = t_{n+i}$ for $i = 1, \ldots, n$. Put $t_{2n+1} = t_1$. Then the quantities $t_{2i-1} - t_{2i}$, for $i = 1, \ldots, n$, are the same as the quantities $t_{2i} - t_{2i+1}$, for $i = 1, \ldots, n$, but in a different order. For example, $t_1 - t_2 = t_{n+1} - t_{n+2}$, and $t_n - t_{n+1} = t_{2n} - t_1$. So

$$\prod_{i=1}^{n} (t_{2i-1} - t_{2i}) = \prod_{i=1}^{n} (t_{2i} - t_{2i+1}),$$

and the result follows, by Proposition 6.

Referring back to Proposition 7, the map f is an involution, so that for different choices of ρ_1, the points O_1, O_{n+1} are point pairs of an involution on OP_1, of which the points O, P_1 are another pair (corresponding to the banned values -1, 0 of ρ_1). If we take ρ_1 as one of the fixed points of this involution, then we have $C_i = C_{n+1}$, for $i = 1, \ldots, n$, and our closed chain of length $2n$ is actually a closed chain of length n, repeated, in fact the chain we obtained in Proposition 7. To construct this chain, first choose any position for O_1 and construct O_{n+1}, by repeating the construction at the end of Section 3. Then construct the fixed points of the involution determined by the pairs O_1, O_{n+1} and O, P_1. This can be done, for example, by putting in circles with diameters OP_1 and O_1O_{n+1}. We need the point-circles of the

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coaxial system they determine, so construct one of their common tangents and put in the circle on this as diameter; this will cut OP_1 at the required points, the first centre for a closed cyclic chain of length n, rather than $2n$.

Note that, when $n = 3$, Proposition 11 is a special case of the nine circles theorem from [1], when three of the nine circles move into coincidence (see [1, p. 68]), and it is also an example of case (ii) of the seven circles theorem, the 'failure' case, since the lines P_1P_4, P_2P_5 and P_3P_6 become the tangents to S at P_1, P_2 and P_3, which are not concurrent; see Figure 11. Here, just for once, case (ii) gives a prettier diagram than the 'success' case, Figure 12. Then Figure 13 shows the case $n = 3$ of Proposition 11 when C_2, C_4 and C_6 are lines, in which case they are sides of an equilateral triangle with incircle S and excircles C_1, C_3 and C_5.

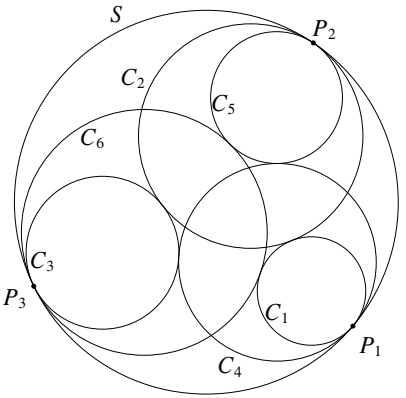


FIGURE 11

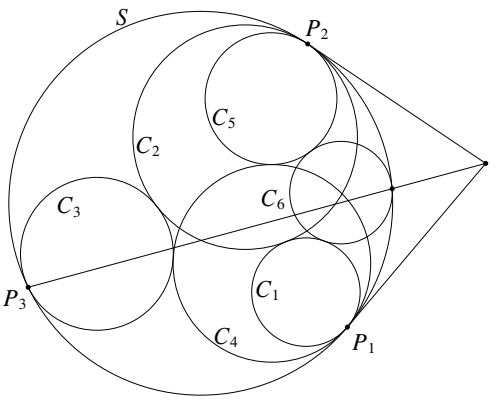


FIGURE 12

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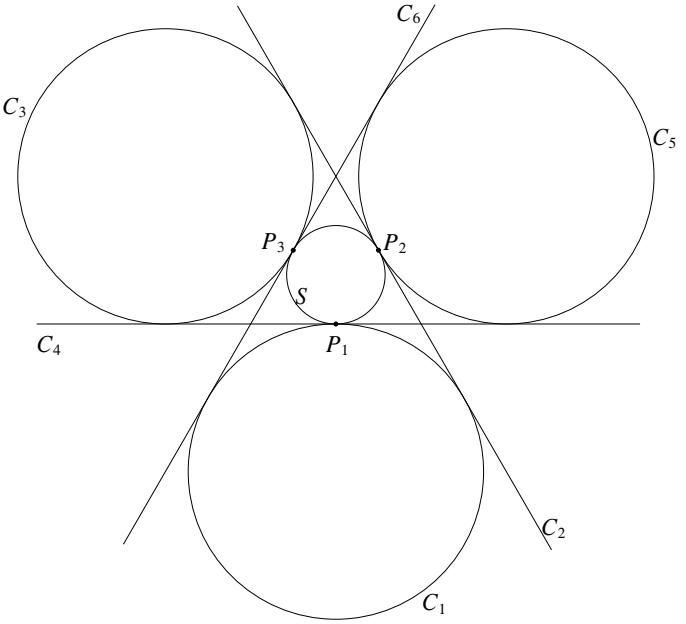


FIGURE 13

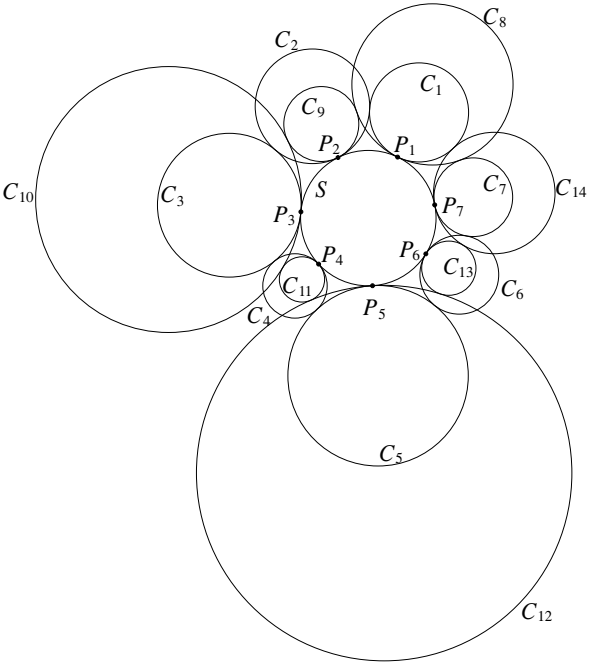


FIGURE 14

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An example of the case $n = 7$ of Proposition 11, involving fifteen circles, is shown in Figure 14; cf. Figure 10. Proposition 11 involves two circuits of the base points $P_1, \ldots, P_n$, each base point being used twice. There is no equivalent configuration for n even, since (in the notation of the proof of Proposition 6) closure at the end of the second circuit would involve $h_1^2 = h_2^2$; but since the h_i are positive, this reduces to $h_1 = h_2$, so that the chain would actually close after the first circuit of the base points, giving a closed chain of length n, repeated.

7. *Alternating chains*

As we remarked in the introduction, in [1] the seven circles theorem is followed, in [1, §3.2], by a somewhat similar result about six circles and two parallel lines. In this section we give a proof of this theorem, together with a partial extension to more circles; see Proposition 14, below. Specifically, let us take $S_1 : y = 0$ and $S_2 : y = 1$ as *base lines*, and define the chain of circles $C_1, C_2, \ldots$ to be *alternating* if it is of even length and, for i odd (respectively, even), C_i touches S_1 (respectively, S_2) at the base point P_i.

For i odd, let P_i be the point $(t_i, 0)$, and let the centre O_i of C_i be the point (t_i, ρ_i). For i even, let P_i be the point $(t_i, 1)$, and let the centre O_i of C_i be the point $(1 - \rho_i, 1)$. So for all i, the radius of C_i is $|\rho_i|$, the sign of ρ_i determining on which side of the relevant base line the circle lies: a positive value of ρ_i means that the circle C_i lies above S_1 (i.e. in $y \geqslant 0$) if i is odd or below S_2 (i.e. in $y \leqslant 1$) if i is even.

A new feature here is that, given P_1, C_1 and P_2, there are *two* choices of ρ_2, that is, of C_2, to continue the chain. For one of these, which we shall call type 1 contact, we have $|O_1O_2| = |\rho_1 + \rho_2|$, and for the other, which we call type 2 contact, we have $|O_1O_2| = |\rho_1 - \rho_2|$. If ρ_1 and ρ_2 are both positive, the circles are external to each other for type 1 contact, but one is inside the other for type 2 contact. If ρ_1 and ρ_2 have opposite signs, it is the other way around. (ρ_1 and ρ_2 cannot both be negative.) Another way to distinguish the two types, as we leave the reader to verify, is that for type 1 contact, C_1 and C_2 touch at a point on P_1P_2, whereas for type 2 contact, they touch at a point on $P_1'P_2$, where P_1' is the point of C_1 diametrically opposite P_1.

From now on, we shall assume that all the contacts are of the same type, including the last-to-first contact of a closed chain. See Figure 15 for two examples of type 1 contact, and Figure 16 for two examples of type 2 contact.

Proposition 12: Let $n = 4m + 2$. Then the type 1 alternating chain $C_1, \ldots, C_n$ is closed if, and only if,

$$\sum_{i=1}^{n} (-1)^{i+1} t_i t_{i+1} = 0$$

where $t_{n+1} = t_1$.

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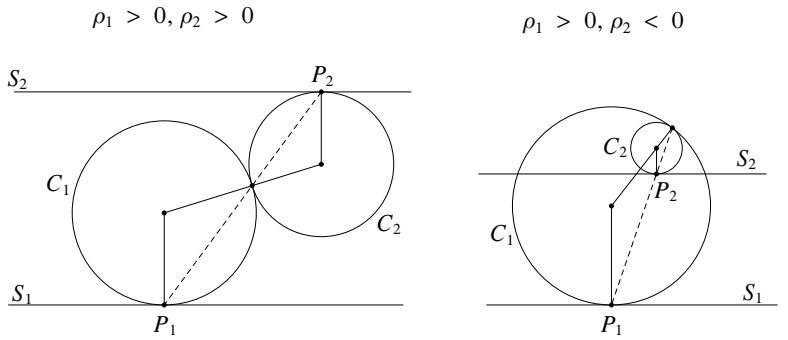


FIGURE 15

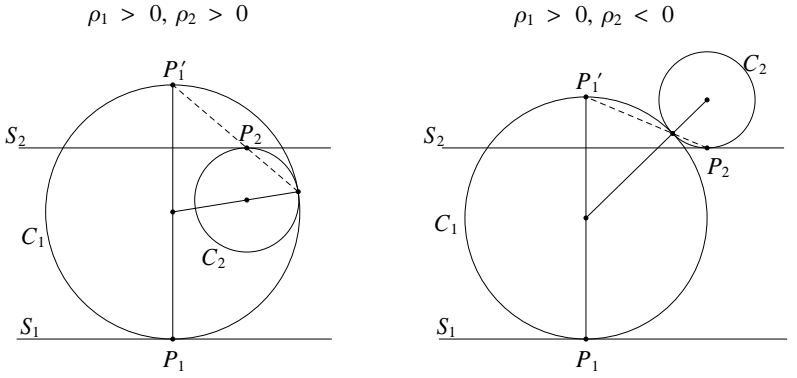


FIGURE 16

Proof: Let C_{n+1} be the circle with signed radius ρ_{n+1} and base point $P_{n+1} = P_1$ (so that $t_{n+1} = t_1$), having type 1 contact with C_1. So our chain is closed if, and only if, $C_{n+1} = C_1$, that is if, and only if, $\rho_{n+1} = \rho_1$.

We have

$$(\rho_j + \rho_{j+1})^2 = (t_j - t_{j+1})^2 + (1 - \rho_j - \rho_{j+1})^2, \tag{6}$$

so that

$$\rho_j + \rho_{j+1} = \tfrac{1}{2}\big((t_j - t_{j+1})^2 + 1\big) = \tfrac{1}{2}\big(t_j^2 + t_{j+1}^2 + 1\big) - t_j t_{j+1},$$

for $j = 1, \ldots, n$, and thus

$$\rho_{j+2} - \rho_j = \tfrac{1}{2}\big(t_{j+2}^2 - t_j^2\big) + t_j t_{j+1} - t_{j+1} t_{j+2} \tag{7}$$

for $j = 1, \ldots, n-1$. Summing (7) over odd values of j from 1 to $n-1$ gives

$$\rho_{n+1} - \rho_1 = \sum_{i=1}^{n} (-1)^{i+1} t_i t_{i+1}$$

and the result follows.

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Note that, since the above result depends only on the positions of the P_j, we have another porism: either the chain is never closed, whatever the value of ρ_1, or it is closed for every value of ρ_1.

Proposition 13: Let $n = 4m + 2 = 2\ell$. For $i = 1, \dots, n$, let $P_i = (t_i, 0)$ if i is odd, or $(t_i, 1)$ if i is even, the P_i being distinct. Suppose the lines $P_1P_{1+\ell}, \dots, P_\ell P_{2\ell}$, are concurrent. Then

$$\sum_{i=1}^{n} (-1)^{i+1} t_i t_{i+1} = 0 \tag{8}$$

where $t_{n+1} = t_1$. Conversely, if $n = 6$ and (8) holds, then the lines P_1P_4, P_2P_5 and P_3P_6 are concurrent.

Proof: If the lines are concurrent, then there is a perspectivity f from S_2 to S_1, with $f(t_{2i}) = t_{2i+\ell}$, for $i = 1, \dots, \ell$, subscripts being interpreted modulo n. Since $S_1 \,/\!/\, S_2$, we have $f(\infty) = \infty$, and therefore the perspectivity takes the form $f(t) = at + b$, for all t, for some constants a and b, with $a \neq 0$. (For the theory of homographic ranges in perspective, see [4, Chapter IV, §7].) So now

$$\begin{aligned}
\sum_{i=1}^{n} (-1)^{i+1} t_i t_{i+1} &= \sum_{i=1}^{\ell} t_{2i-1} t_{2i} - \sum_{i=1}^{\ell} t_{2i} t_{2i+1} \\
&= \sum_{i=1}^{\ell} t_{2i-1} f(t_{2i+\ell}) - \sum_{i=1}^{\ell} f(t_{2i+\ell}) t_{2i+1} \\
&= \sum_{i=1}^{\ell} t_{2i-1} (a t_{2i+\ell} + b) - \sum_{i=1}^{\ell} (a t_{2i+\ell} + b) t_{2i+1} = 0
\end{aligned}$$

which completes the first part. Then, if $n = 6$ and (8) holds, let P_6' be on S_2 such that P_1P_4, P_2P_5 and P_3P_6' are concurrent, and we have

$$t_1t_2 - t_2t_3 + t_3t_4 - t_4t_5 + t_5t_6 - t_6t_1 = t_1t_2 - t_2t_3 + t_3t_4 - t_4t_5 + t_5t_6' - t_6't_1$$

from which it is immediate that $t_6 = t_6'$, or $P_6 = P_6'$, and we have finished.

Proposition 14: Let $n = 4m + 2 = 2\ell$ and let $C_1, \dots, C_n$, be a type 1 alternating chain, with base points $P_1, \dots, P_n$. Then if the lines $P_1P_{1+\ell}, \dots, P_\ell P_{2\ell}$, are concurrent, the chain is closed. Further, when $n = 6$, the chain is closed if, and only if, P_1P_4, P_2P_5 and P_3P_6 are concurrent. (See Figure 17.)

Proof: Immediate from Propositions 12 and 13.

Propositions 9 and 10 on the one hand, and Proposition 14 on the other have, to quote [1, p. 42], 'certain affinities'; the authors of [1] tried, and failed, 'to find a more general theorem of which both are special cases'.

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We now look at type 2 alternating chains. Here we have $|O_jO_{j+1}| = |\rho_j - \rho_{j+1}|$, so that (6) is replaced by

$$(\rho_j - \rho_{j+1})^2 = (t_j - t_{j+1})^2 + (1 - \rho_j - \rho_{j+1})^2,$$

from which

$$(1 - 2\rho_j)(1 - 2\rho_{j+1}) = (t_j - t_{j+1})^2.$$

It follows that

$$\frac{1 - 2\rho_j}{1 - 2\rho_{j+2}} = \frac{(t_j - t_{j+1})^2}{(t_{j+1} - t_{j+2})^2}. \tag{9}$$

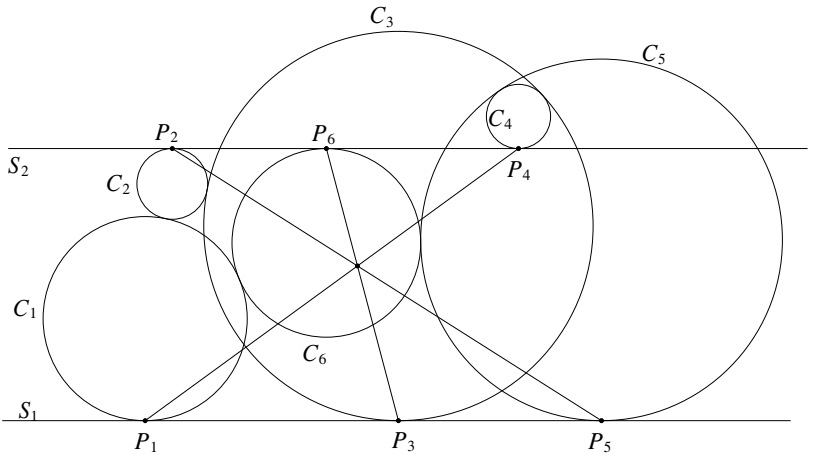


FIGURE 17

Proposition 15: Let $n = 2\ell$. Then the type 2 alternating chain $C_1, \ldots, C_n$ is closed if, and only if,

$$\prod_{i=1}^{\ell} (t_{2i-1} - t_{2i})^2 = \prod_{i=1}^{\ell} (t_{2i} - t_{2i+1})^2$$

where $t_{n+1} = t_1$.

Proof: Let C_{n+1} be the circle with signed radius ρ_{n+1} and base point $P_{n+1} = P_1$ (so that $t_{n+1} = t_1$), having type 2 contact with C_1. So our chain is closed if, and only if, $C_{n+1} = C_1$, that is if, and only if, $\rho_{n+1} = \rho_1$. Then from (9),

$$\frac{1 - 2\rho_1}{1 - 2\rho_{n+1}} = \prod_{i=1}^{\ell} \frac{1 - 2\rho_{2i-1}}{1 - 2\rho_{2i+1}} = \prod_{i=1}^{\ell} \frac{(t_{2i-1} - t_{2i})^2}{(t_{2i} - t_{2i+1})^2},$$

and the result follows.

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The reader will have noted the similarity between the statements of Proposition 15 and Proposition 6. So there is an algebraic affinity between cyclic chains and type 2 alternating chains; it would be nice to find some geometric affinities here as well.

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